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ON THE EXISTENCE OF SIMPLE LIAPUNOV FUNCTIONS FOR LINEAR RETARDED DIFFERENCE-DIFFERENTIAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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Abstract. Liapunov functions of simple form have been used for the study of stability properties of difference-differential equations. In this paper we provide necessary and sufficient conditions for the existence of such functions.

I. Introduction. In this paper we consider the existence of two simple types of Liapunov functions for retarded linear autonomous difference-differential equations. The study of the stability properties of these equations was pioneered by Yoshizawa [11, 12], Razumikhin [9], Krasovskii [8], Driver [2], and Hale [3].

Restriction to Liapunov functions of simple form leads to stability criteria which are overly sufficient; on the other hand, such functions are particularly desirable in specific applications where ease of computation is of primary importance. Here we provide necessary and sufficient conditions for the existence of useful Liapunov functions having standard simple forms for certain difference-differential equations.

Consider the difference-differential equation

$$\dot{y}(t) = Ay(t) + \sum_{k=1}^{m} B_k y(t - \tau_k) , \qquad t \geqq 0 , \tag{1.1}$$

with initial data

$$y(\theta) = v_0(\theta) , \qquad -\tau \leqq \theta \leqq 0 ,$$

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where $\tau_k \geqq 0$, $\tau \equiv \max_k \tau_k$, and the $n \times n$ complex matrices A, B_k are given, as is the initial function $v_0 : [-\tau, 0] \to \boldsymbol{C}^n$.

Let us replace the above equation by an abstract evolution equation on a Banach space $\mathcal{X}$,

$$(1.2) \qquad \begin{aligned} \dot{x}(t) &= Fx(t)\ , \quad t \geqq 0\ , \\ x(0) &= x_0 \equiv (v_0(0),\ v_0) \in \mathscr{D}(F) \subset \mathcal{X}\ , \end{aligned}$$

where $\mathcal{X} \equiv \{x = (y, v) \in \boldsymbol{C}^n \times \mathscr{C}([-\tau, 0]; \boldsymbol{C}^n) \mid y = v(0)\}$ with $\|x\|_{\mathcal{X}} \equiv \sup_{-\tau \leqq \theta \leqq 0} \|v(\theta)\|_{\boldsymbol{C}^n}$; the operator F has domain $\mathscr{D}(F) \equiv \{x = (y, v) \in \mathcal{X} \mid v' \in \mathscr{C}([-\tau, 0]; \boldsymbol{C}^n),\ v'(0) = Ay + \sum_{k=1}^m B_k v(-\tau_k)\}$, where $v'(\theta) \equiv (d/d\theta)v(\theta)$, and is defined by $Fx \equiv (Ay + \sum_{k=1}^m B_k v(-\tau_k),\ v')$ for all $x = (y, v) \in \mathscr{D}(F)$. It it well known [7] that F is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t \geqq 0}$ of bounded linear operators $S(t): \mathcal{X} \to \mathcal{X}$; in particular, for every $x_0 \in \mathscr{D}(F)$, $S(\cdot)x_0$ is the unique strong solution of (1.2).

We are interested in obtaining stability results for (1.2) (hence for (1.1)) through the use of Liapunov functions.

DEFINITION 1.1. A continuous function $V: \mathcal{X} \to \boldsymbol{R}$ is said to be a continuous Liapunov function for $\{S(t)\}_{t \geqq 0}$ on $\mathcal{X}$ if $\dot{V}(x) \leqq 0$ for all $x \in \mathcal{X}$, where $\dot{V} : \mathcal{X} \to \boldsymbol{R}$ is given by

$$\dot{V}(x) \equiv \liminf_{t \searrow 0}\, [V(S(t)x) - V(x)]/t\ , \qquad x \in \mathcal{X}\ .$$

It is well known that if V is a Liapunov function on $\mathcal{X}$, then $V(S(t)x) \leqq V(x)$ for all $t \geqq 0$, $x \in \mathcal{X}$. The major difficulty encountered in using such a conclusion to derive stability results is the construction of a suitable Liapunov function.

In [1, 7] very complicated Liapunov functions have been shown to yield necessary and sufficient conditions for stability and asymptotic stability for (1.2), hence for (1.1). On the other hand, most attention [3, 4, 8] has been centered on the use of functions having very simple structure; in particular, functions of the form

$$(1.3) \qquad V(x) = y^* R y + \sum_{k=1}^m \int_{-\tau_k}^0 v^*(\theta) Q_k v(\theta) d\theta\ , \qquad x = (y, v) \in \mathcal{X}\ ,$$

or of the form

$$(1.4) \qquad W(x) = \sup_{-\tau \leqq \theta \leqq 0} v^*(\theta) R v(\theta)\ , \qquad x = (y, v) \in \mathcal{X}\ ,$$

where y^* denotes the conjugate transpose of the column vector y, and R, Q_k, are $n \times n$ Hermitian matrices. Computing $\dot{V} : \mathcal{X} \to \boldsymbol{R}$ and $\dot{W}: \mathcal{X} \to \boldsymbol{R}$ according to (1.2), we obtain

$$\dot{V}(x) = y^*\Big(RA + A^*R + \sum_{k=1}^{m} Q_k\Big)y + 2\,\mathrm{Re}\sum_{k=1}^{m} y^*RB_k v(-\tau_k) - \sum_{k=1}^{m} v^*(-\tau_k)Q_k v(-\tau_k) \tag{1.5}$$

and

$$\dot{W}(x) \leqq \begin{cases} 0 & \text{if} \quad y^*Ry < W(x)\,, \\ \max\Big\{0,\, 2\,\mathrm{Re}\, y^*\Big(Ay + \sum_{k=1}^{m} B_k v(-\tau_k)\Big)\Big\} & \text{if} \quad y^*Ry = W(x)\,, \end{cases} \tag{1.6}$$

for all $x = (y, v) \in \mathcal{X}$; hence, V is a Liapunov function on $\mathcal{X}$ if and only if

$$\begin{bmatrix} RA + A^*R + \sum_{k=1}^{m} Q_k & RB_1 & RB_2 \cdots RB_m \\ B_1^*R & -Q_1 & 0 \cdots 0 \\ B_2^*R & 0 & -Q_2 \cdots 0 \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ B_m^*R & 0 & 0 \cdots -Q_m \end{bmatrix} \leqq 0 \tag{1.7}$$

(i.e., this matrix is negative-semidefinite), while W is a Liapunov function on $\mathcal{X}$, if and only if

$$\mathrm{Re}\, z_0^*R\Big(Az_0 + \sum_{k=1}^{m} B_k z_k\Big) \leqq 0 \tag{1.8}$$

for all $z_0, z_1, \cdots, z_m \in C^n$ such that $z_0^*Rz_0 \geqq z_k^*Rz_k$ for all $k = 1, 2, \cdots, m$. We remark that under condition (1.8) the Liapunov function $W(x)$ corresponds to a "Razumikhin function" [3, 7] of the form y^*Ry.

It is noted that the existence of a Liapunov function of either of the foregoing forms is independent of the delays $\{\tau_k\}_{k=1}^m$; hence so will be any stability conclusions drawn from the use of such functions. This remark suggests that the class of equations of the form (1.1) for which such functions exist is narrow. In the sequel we give a characterization of this class; moreover, we show by counterexample that there do exist equations of the form (1.1) that yield asymptotic stability for all delays $\{\tau_k\}_{k=1}^m$ and yet there exists no simple Liapunov function, having either of the foregoing forms, that can be used to establish this fact.

II. The single delay case. In this section we consider the case where equation (1.2) involves only a single delay ($m = 1, \quad \tau \equiv \tau_1$).

Essential to our arguments is the following result.

LEMMA 2.1. *Given $n \times n$ complex matrices M, N, P, with M and N Hermitian, the following two statements are equivalent:*

(i) $M + N + e^{i\phi}P + e^{-i\phi}P^* \leqq 0$ *for all* $\phi \in \boldsymbol{R}$.

(ii) *There exists a Hermitian matrix* Q *such that*

$$\begin{bmatrix} M+Q & P \\ P^* & N-Q \end{bmatrix} \leqq 0 .$$

PROOF. Noting that (ii) implies

$$[I, e^{-i\phi}I]\begin{bmatrix} M+Q & P \\ P^* & N-Q \end{bmatrix}\begin{bmatrix} I \\ e^{i\phi}I \end{bmatrix} = M + N + e^{i\phi}P + e^{-i\phi}P^* \leqq 0$$

for all $\phi \in \boldsymbol{R}$, we see that (ii) implies (i).

To show the converse, note from (i) that if $\hat{u}(i\phi)$ denotes the Fourier transform of an arbitrary $u(t)$, $u \in \mathscr{L}_2(\boldsymbol{R}; \boldsymbol{C}^n)$, then $\int_{-\infty}^{\infty} \hat{u}^*(i\phi)[M + N + e^{i\phi}P + e^{-i\phi}P^*]\hat{u}(i\phi)d\phi \leqq 0$; hence, by Parseval's equation,

$$\int_{-\infty}^{\infty} [u^*(t)(M + N)u(t) + 2\,\mathrm{Re}\, u^*(t)Pu(t-1)]dt \leqq 0 .$$

Let $u(t) = 0$ for $t < 0$ and $u(t) = u_k$ for $k \leqq t < k+1$, $k = 0, 1, 2, \cdots$; it then follows that $\sum_{k=0}^{\infty} [u_k^*(M + N)u_k + 2\,\mathrm{Re}\, u_k^* P u_{k-1}] \leqq 0$ for all $u_k \in \boldsymbol{C}^n$ such that $\{u_k\}_{k=0}^{\infty} \in l_2^n$, $u_{-1} \equiv 0$.

Define the functional

$$J(u_0) \equiv \sup_{\{u_k\}_{k=1}^{\infty}} \sum_{k=1}^{\infty} [u_k^*(M + N)u_k + 2\,\mathrm{Re}\, u_k^* P u_{k-1}] , \qquad u_0 \in \boldsymbol{C}^n .$$

We note that $-u_0^*(M + N)u_0 \geqq J(u_0) \geqq 0$ and, therefore, $J(0) = 0$ and $J(u_0)$ is continuous at $u_0 = 0$; furthermore, it is easily verified that $J(\gamma u_0) = |\gamma|^2 J(u_0)$ for all $\gamma \in \boldsymbol{C}$.

We first wish to show that $J^{1/2}(\cdot)$ is a seminorm on $\boldsymbol{C}^n$; it only remains to be shown that $J^{1/2}(\cdot)$ satisfies the triangle inequality. To this end, consider positive real numbers α, β, such that $\alpha^2 + \beta^2 = 1$, and notice that, for arbitrary $w, z \in \boldsymbol{C}^n$,

$$\begin{aligned} J(w + z) = {} & \sup_{\{u_k\}_{k=1}^{\infty}} \{u_1^*(M + N)u_1 + 2\,\mathrm{Re}\, u_1^* P(w + z) \\ & \qquad + \sum_{k=2}^{\infty} [u_k^*(N + M)u_k + 2\,\mathrm{Re}\, u_k^* P u_{k-1}]\} \\ \leqq {} & \sup_{\{u_k\}_{k=1}^{\infty}} \{(\alpha u_1)^*(M + N)(\alpha u_1) + 2\,\mathrm{Re}\,(\alpha u_1)^* P(\alpha^{-1}w) \\ & \qquad + \sum_{k=2}^{\infty} [(\alpha u_k)^*(M + N)(\alpha u_k) + 2\,\mathrm{Re}\,(\alpha u_k)^* P(\alpha u_{k-1})]\} \\ & + \sup_{\{u_k\}_{k=1}^{\infty}} \{(\beta u_1)^*(M + N)(\beta u_1) + 2\,\mathrm{Re}\,(\beta u_1)^* P(\beta^{-1}z) \end{aligned}$$

$$+ \sum_{k=2}^{\infty} [(\beta u_k)^*(M + N)(\beta u_k) + 2 \operatorname{Re} (\beta u_k)^* P(\beta u_{k-1})]\}$$
$$= J(\alpha^{-1}w) + J(\beta^{-1}z) = \alpha^{-2}J(w) + \beta^{-2}J(z) .$$

If $J(w)J(z) = 0$, say $J(w) = 0$, it follows that $J(w + z) \leqq J(z)$; if not, let $\alpha^2 \equiv J^{1/2}(w)/[J^{1/2}(w) + J^{1/2}(z)]$, $\beta^2 \equiv 1 - \alpha^2$, and note that we obtain $J(w + z) \leqq J(w) + 2J^{1/2}(w)J^{1/2}(z) + J(z)$. Therefore, $J^{1/2}(\cdot)$ is a seminorm on C^n.

We now wish to show that this seminorm satisfies the parallelogram law. Let $w_0, z_0 \in C^n$, $\varepsilon > 0$, and note that there exist sequences $\{w_k\}_{k=1}^{\infty}$, $\{z_k\}_{k=1}^{\infty} \in l_2^n$ such that

$$J(w_0) \leqq \varepsilon + \sum_{k=1}^{\infty} [w_k^*(M + N)w_k + 2 \operatorname{Re} w_k^* P w_{k-1}] ,$$
$$J(z_0) \leqq \varepsilon + \sum_{k=1}^{\infty} [z_k^*(M + N)z_k + 2 \operatorname{Re} z_k^* P z_{k-1}] ;$$

consequently,

$$J(w_0) + J(z_0) \leqq 2\varepsilon + \sum_{k=1}^{\infty} [(w_k + z_k)^*(M + N)(w_k + z_k)$$
$$+ 2 \operatorname{Re}(w_k + z_k)^* P(w_{k-1} + z_{k-1})]/2 + \sum_{k=1}^{\infty} [(w_k - z_k)^*(M + N)(w_k - z_k)$$
$$+ 2 \operatorname{Re}(w_k - z_k)^* P(w_{k-1} - z_{k-1})]/2$$
$$\leqq 2\varepsilon + (1/2)J(w_0 + z_0) + (1/2)J(w_0 - z_0) .$$

As $\varepsilon > 0$ is arbitrary, $2J(w_0) + 2J(z_0) \leqq J(w_0 + z_0) + J(w_0 - z_0)$; by an obvious change of variables and quadratic homogeneity, the reverse inequality holds as well. Hence $J^{1/2}(\cdot)$ satisfies the parallelogram law.

Define $q : C^n \times C^n \to C$ by

$$q(w, z) \equiv [J(w + z) - J(w - z)]/4 + [J(w + iz) - J(w - iz)]i/4 , \quad w, z \in C^n .$$

We note that q is continuous with $q(w, w) = J(w) \geqq 0$, $q(w, z) = \overline{q(z, w)}$, and a standard argument [10] shows that

$$q(\alpha w + \hat{w}, z) = \alpha q(w, z) + q(\hat{w}, z) \quad \text{for all} \quad w, \hat{w}, z \in C^n, \alpha \in C .$$

Consequently, Riesz' representation theorem implies the existence of a unique Hermitian matrix Q such that $z^*Qw = q(w, z) + z^*Nw, z, w \in C^n$.

Finally, we note that, for every $u_1 \in C^n$,

$$q(u_1, u_1) = J(u_1) = \sup_{\{u_k\}_{k=2}^{\infty}} \sum_{k=2}^{\infty} [u_k^*(M + N)u_k + 2 \operatorname{Re} u_k^* P u_{k-1}]$$

and therefore, for every $u_0, u_1 \in C^n$,

$$q(u_1, u_1) + u_1^*(M + N)u_1 + 2\,\mathrm{Re}\, u_1^* P u_0 \leqq J(u_0) = q(u_0, u_0)\ ;$$

consequently,

$$u_1^*(Q - N)u_1 + u_1^*(M + N)u_1 + 2\,\mathrm{Re}\, u_1^* P u_0 \leqq u_0^*(Q - N)u_0\ ,$$

and we conclude that (i) implies (ii). ■

Through the use of this lemma we can provide existence criteria for Liapunov functions of the form (1.3) or (1.4) when equation (1.1) involves only one delay; i.e.,

$$\dot{y}(t) = Ay(t) + By(t - \tau)\ , \qquad t \geqq 0\ , \tag{2.1}$$

with $m = 1$, $\tau_1 = \tau$, $B_1 \equiv B$. In this case the function (1.3) has the form

$$V(x) = y^*Ry + \int_{-\tau}^{0} v^*(\theta)Qv(\theta)d\theta\ , \qquad x = (y, v) \in \chi\ , \tag{2.2}$$

for some $n \times n$ Hermitian matrices R and Q. The following theorem gives necessary and sufficient conditions for (2.2) to be a Liapunov function on χ for (2.1).

THEOREM 2.1. *Given arbitrary $n \times n$ positive-semidefinite Hermitian matrices C, D, the following two statements are equivalent*:

(i) *There exist $R = R^*$, $Q = Q^*$, such that V given by* (2.2) *is a nontrivial Liapunov function for* (2.1) *on χ and*

$$\dot{V}(x) \leqq -y^*Cy - v^*(-\tau)Dv(-\tau)\ , \qquad x = (y, v) \in \chi\ .$$

(ii) *There exists $R = R^* \not\equiv 0$ such that*

$$RA + A^*R + C + D + e^{i\phi}RB + e^{-i\phi}B^*R \leqq 0 \quad \textit{for all} \quad \phi \in \mathbf{R}\ .$$

PROOF. By direct computation of $\dot{V}$ (see (1.7)), we find that $\dot{V}(x)$ satisfies the estimate in (i) if and only if the $2n \times 2n$ Hermitian matrix

$$\begin{bmatrix} RA + A^*R + Q + C & BR \\ R^*B & D - Q \end{bmatrix}$$

is negative semidefinite; hence, Lemma 2.1 shows that (i) is equivalent to (ii). ■

This theorem leads to the following criteria for stability, asymptotic stability and instability of the trivial solution (hence, by linearity, of all solutions) of equation (2.1).

THEOREM 2.2. *Let there exist $n \times n$ Hermitian matrices C, R, such that*

(2.3) $RA + A^*R + C + e^{i\phi}RB + e^{-i\phi}B^*R \leqq 0$ *for all* $\phi \in \boldsymbol{R}$.

Then, for all $\tau \geqq 0$, *we have*:

(i) *If* $C \geqq 0$ *and* $R > 0$, *the trivial solution of* (2.1) *is stable.*

(ii) *If* $C > 0$ *and* $R > 0$, *the trivial solution of* (2.1) *is asymptotically stable.*

(iii) *If* $C > 0$ *and* $y_0^* R y_0 < 0$ *for some* $y_0 \in \boldsymbol{C}^n$, *the trivial solution of* (2.1) *is unstable.*

PROOF. By Lemma 2.1, condition (2.3) implies the existence of $Q = Q^*$ such that

$$\begin{bmatrix} RA + A^*R + Q + C & RB \\ B^*R & -Q \end{bmatrix} \leqq 0 ;$$

hence, $Q \geqq 0$, $V(x) \geqq y^*Ry$, and $\dot{V}(x) \leqq -y^*Cy$ for the function $V(x) = y^*Ry + \int_{-\tau}^{0} v^*(\phi)Qv(\phi)d\phi$, $x = (y, v) \in \chi$.

Using the semigroup notation of Section I, we recall that $C \geqq 0$ implies $V(S(t)x) \leqq V(x)$, $t \geqq 0$, $x \in \chi$; hence, $V(x_0) \geqq V(S(t)x_0) \geqq y^*(t)Ry(t)$, $t \geqq 0$. Consequently, if $R > 0$ and $\|x_0\|_\chi \leqq \delta$, there exists $c_0 \geqq 1$ such that $\|y(t)\|_{C^n}^2 \leqq c_0\delta^2$, $t \geqq 0$; this implies that $\|S(t)x_0\|_\chi^2 \leqq c_0\delta^2$ for all $t \geqq 0$. It follows that (i) has been proved; moreover, for $R > 0$, every motion $S(\cdot)x_0 \colon \boldsymbol{R}^+ \to \chi$ has bounded positive orbit $\gamma(x_0) \equiv \bigcup_{t \geqq 0} S(t)x_0$. It is well known that, for (1.2), bounded positive orbits are precompact [7]; hence, making the stronger assumption that $C > 0$ and noting that the largest positive invariant set [11] in $\{x \in \chi \mid \dot{V}(x) = 0\}$ is $\{0\}$, we see that the Invariance Principle [11] yields the conclusion that $S(t)x_0 \to 0$ as $t \to \infty$, for every $x_0 \in \chi$, which proves (ii).

Now consider (iii), with $C > 0$ and $y_0^*Ry_0 < 0$ for some $y_0 \in \boldsymbol{C}^n$. The Invariance Principle shows that $\gamma(x_0)$ is not precompact (hence, not bounded) for some x_0 in each neighborhood of $0 \in \chi$; hence, $x = 0$ is unstable and the proof is complete. ∎

We remark that if only the hypotheses of (i) are satisfied, rather than the stronger hypotheses of (ii), it may still be possible to employ the Invariance Principle [5] to conclude asymptotic stability rather than mere stability; one needs only to be able to show that the largest invariant set in $\{x \in \chi \mid \dot{V}(x) = 0\}$ is $\{0\}$.

Let us now consider the function W of (1.4),

(2.4) $$W(x) = \sup_{-\tau \leqq \theta \leqq 0} v^*(\theta)Rv(\theta) , \qquad x = (y, v) \in \chi .$$

Recall that, for this function to be a Liapunov function for equation (2.1), it is required that

(2.5) $$\operatorname{Re} z_0^* R(Az_0 + Bz_1) \leqq 0$$

for all $z_0, z_1 \in C^n$ such that $z_0^* R z_0 \geqq z_1^* R z_1$. By making the particular choice $z_1 = e^{i\phi} z_0$ we note that a consequence of condition (2.5) is that condition (2.3) must hold for $C = 0$. Hence the existence of a Liapunov function of the form (2.4) implies the existence of a Liapunov function of the form (2.2), but not conversely unless, in (2.2), we have that $Q = \alpha R$ for some $\alpha > 0$. It follows that the class of equations of the form (2.1) which admit "Razumikhin-type" Liapunov functions is a subset of the class that admits Liapunov functions of the form (2.2).

III. The multi-delay case. In this section we generalize the results presented in the previous section so as to encompass the case in which several delays occur, as in equation (1.1).

For this purpose, we first present a generalization of Lemma 2.1.

LEMMA 3.1. *Given $n \times n$ complex matrices M, N_k, P_k, with M and N_k Hermitian, $k = 1, 2, \cdots, m$, the following two statements are equivalent:*

(i) *There exist Hermitian matrices M_k, $k = 1, 2, \cdots, m$, such that $M \leqq \sum_{k=1}^m M_k$ and, for each $k = 1, 2, \cdots, m$, $M_k + N_k + e^{i\phi} P_k + e^{-i\phi} P_k^* \leqq 0$ for all $\phi \in \boldsymbol{R}$.*

(ii) *There exist Hermitian matrices Q_k, $k = 1, 2, \cdots, m$, such that*

$$\begin{bmatrix} M + \sum_{k=1}^m Q_k & P_1 & P_2 \cdots\cdots P_m \\ P_1^* & N_1 - Q_1 & 0 \cdots\cdots 0 \\ P_2^* & 0 & N_2 - Q_2 \cdots 0 \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ P_m^* & 0 & 0 \cdots N_m - Q_m \end{bmatrix} \leqq 0 .$$

PROOF. It is clear that (ii) holds if and only if there exist Hermitian matrices H_k, $k = 1, 2, \cdots, m$, such that $\sum_{k=1}^m H_k \geqq M + \sum_{k=1}^m Q_k$ and

$$\begin{bmatrix} H_k & P_k \\ P_k^* & N_k - Q_k \end{bmatrix} \leqq 0 , \qquad k = 1, 2, \cdots, m .$$

Defining $M_k \equiv H_k - Q_k$ and applying Lemma 1 for each $k = 1, 2, \cdots, m$, we see that (i) is equivalent to (ii). ∎

As in the previous section, we will use this lemma to provide existence criteria for Liapunov functions of the form (1.3) or (1.4) for equation (1.1),

$$\dot{y}(t) = Ay(t) + \sum_{k=1}^{m} B_k y(t - \tau_k) , \qquad t \geqq 0 . \tag{3.1}$$

We recall that we are interested in a function V having the from (1.3),

$$V(x) = y^*Ry + \sum_{k=1}^{m} \int_{-\tau_k}^{0} v^*(\theta)Q_k v(\theta)d\theta , \qquad x = (y, v) \in \chi , \tag{3.2}$$

for some $n \times n$ Hermitian matrices R and Q_k, $k = 1, \cdots, m$.

THEOREM 3.1. *Given arbitrary $n \times n$ positive-semidefinite Hermitian matrices C, D_k, $k = 1, 2, \cdots, m$, the following two statements are equivalent*:

(i) *There exist $R = R^*$, $Q_k = Q_k^*$, such that V given by* (3.2) *is a nontrivial Liapunov function for* (3.1) *on χ and*

$$\dot{V}(x) \leqq -y^*Cy - \sum_{k=1}^{m} v^*(-\tau_k)D_k v(-\tau_k) , \qquad x = (y, v) \in \chi .$$

(ii) *There exist $R = R^* \not\equiv 0$, $M_k = M_k^*$, such that $RA + A^*R + C \leqq \sum_{k=1}^{m} M_k$ and, for each $k = 1, 2, \cdots, m$,*

$$M_k + D_k + e^{i\phi}RB_k + e^{-i\phi}B_k^*R \leqq 0 \quad \textit{for all} \quad \phi \in \boldsymbol{R} .$$

The proof of this theorem follows immediately from application of Lemma 3.1, by the same arguments as in the proof of Theorem 2.1. This result immediately leads to

THEOREM 3.2. *Let there exist $n \times n$ Hermitian matrices C, M_k, R such that $RA + A^*R + C \leqq \sum_{k=1}^{m} M_k$ and, for each $k = 1, 2, \cdots, m$,*

$$M_k + e^{i\phi}RB_k + e^{-i\phi}B_k^*R \leqq 0 \quad \textit{for all} \quad \phi \in \boldsymbol{R} .$$

Then, for all $\tau_k \geqq 0$, $k = 1, 2, \cdots, m$, we have:

(i) *If $C \geqq 0$ and $R > 0$, the trivial solution of* (3.1) *is stable.*

(ii) *If $C > 0$ and $R > 0$, the trivial solution of* (3.1) *is asymptotically stable.*

(iii) *If $C > 0$ and $y_0^*Ry_0 < 0$ for some $y_0 \in \boldsymbol{C}^n$, the trivial solution of* (3.1) *is unstable.*

We omit the proof of this theorem since it parallels that of Theorem 2.2. As for Theorem 2.2, we remark that it may be possible to conclude asymptotic stability in certain applications satisfying the hypotheses of (i) but not those of (ii); this merely requires a more detailed application of the Invariance Principle [5].

The remarks made in Section II regarding the existence of a "Razumikhin-type" Liapunov function (i.e., of the form (1.4)) apply here as well; existence of this type of Liapunov function implies the existence of a Liapunov function of the form (3.2), but the converse need not hold unless $Q_k = \alpha_k R$, $k = 1, 2, \cdots, m$, for some real $\alpha_k > 0$.

IV. A converse question on asymptotic stability. If the hypotheses of conclusion (ii) of Theorem 3.2 are satisfied, then asymptotic stability of all solutions of

$$\dot{y}(t) = Ay(t) + \sum_{k=1}^{m} B_k y(t - \tau_k) , \qquad t \geqq 0 , \tag{4.1}$$

irrespective of the values of the delays τ_k, $k = 1, \cdots, m$ is guaranteed. It follows that no function V of the form (1.3) will be a Liapunov function with $R > 0$, $\dot{V}(x) \leqq -y^* Cy$, $C > 0$, unless equation (4.1) is asymptotically stable for all delays. Given this observation, and recalling the power of the Invariance Principle [5], it is natural to ask the following question: given that the trivial solution of (4.1) is asymptotically stable, irrespective of the delays τ_k, $k = 1, \cdots, m$, does there necessarily exist a Liapunov function of the form (1.3) with $R > 0$? This is essentially the question posed by Hale (see [4, p. 108]).

To investigate this question, let us recall a recent result [12] that characterizes those $(m + 1)$-tuples of real matrices $(A, B_1, \cdots, B_m)$ such that the trivial solution of (4.1) will be asymptotically stable irrespective of the delays.

THEOREM 4.1 [12]. *The trivial solution of equation* (4.1) *is asymptotically stable for any set of* $\tau_k \geqq 0$ *if and only if*

(i) *the real parts of the eigenvalues of* $A + \sum_{k=1}^{m} B_k$ *are negative, and*

(ii) *for all* $\phi_k \in \boldsymbol{R}$, $A + \sum_{k=1}^{m} e^{i\phi_k} B_k$ *has no nonzero imaginary eigenvalue.*

Thus, in the case of real matrices $A, B_1, \cdots, B_m$, our question is reduced to the following: do the hypotheses of Theorem 4.1 imply the existence of Hermitian matrices $R > 0$, $C \geqq 0$, $D_k \geqq 0$ and M_k, $k = 1, \cdots, m$, such that statement (ii) in Theorem 3.1 is true?

This question has an affirmative answer in the scalar case (i.e., $A = a \in \boldsymbol{R}$, $B_k = b_k \in \boldsymbol{R}$). Indeed the conditions of Theorem 4.1 become, in this case

$$a + \sum_{k=1}^{m} b_k < 0 , \tag{4.2}$$

$$a + \sum_{k=1}^{m} |b_k| \leqq 0 . \tag{4.3}$$

Hence, letting $R=1$, $C=0$, $D_k=0$, and $M_k=-2|b_k|$, it is immediately seen from (4.3) that all conditions of statement (ii) in Theorem 3.1 are satisfied, and, therefore, there exists a Liapunov function of the form

$$V(x)=|y|^2+\sum_{k=1}^{m}\int_{-\tau_k}^{0}Q_k|v(\theta)|^2d\theta\,, \qquad x=(y,v)\in\mathcal{X}\,, \tag{4.4}$$

for some $Q_1, Q_2, \cdots, Q_m\in\boldsymbol{R}$, such that $\dot{V}(x)\leqq 0$. From the proof of Lemma 3.1, it is easily seen that the Q_k must satisfy

$$\begin{bmatrix}-2|b_k|+Q_k & b_k\\ b_k & -Q_k\end{bmatrix}\leqq 0\,, \qquad k=1,2,\cdots,m\,;$$

hence, $Q_k=|b_k|$ is the only possible choice, and we obtain

$$\dot{V}(x)=2\Big(a+\sum_{k=1}^{m}|b_k|\Big)|y|^2-\sum_{k=1}^{m}|b_k|[y-(\operatorname{sgn}b_k)v(-\tau_k)]^2$$

for all $x=(y,v)\in\mathcal{X}$.

Although $C\leqq 0$ and result (ii) of Theorem 3.2 may not apply, the Invariance Principle [5] can be employed directly to show that, under condition (4.2), this Liapunov function does establish asymptotic stability.

The above question, however, has a negative answer at the vector level, as we can show through a counterexample. Indeed, in (4.1) let $m=1$ and

$$A=\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}, \qquad B_1=\begin{bmatrix}\alpha & 0\\ 0 & -\alpha\end{bmatrix},$$

for some $\alpha\in\boldsymbol{R}$. It is easily verified that the conditions of Theorem 4.1 are satisfied if and only if $\alpha^2<2$. On the other hand, it is not possible to satisfy condition (2.3) with any $R=R^*>0$, $C=C^*\geqq 0$, when $\alpha^2>1$. Indeed, suppose $1<\alpha^2<2$ and

$$R(A+e^{i\phi}B_1)+(A+e^{-i\phi}B_1)^*R\leqq -C \quad \text{for all} \quad \phi\in\boldsymbol{R}\,, \tag{4.5}$$

where $C=C^*\geqq 0$ and

$$R=\begin{bmatrix}1 & \beta\\ \beta^* & \delta\end{bmatrix}, \qquad \beta\in\boldsymbol{C}\,,\quad \delta\in\boldsymbol{R}\,,\quad \delta>|\beta|^2\,.$$

Satisfaction of (4.5) implies that, for all $\phi\in\boldsymbol{R}$,

$$\begin{bmatrix}-2+2\alpha\cos\phi+2\operatorname{Re}\beta & -1-2\beta+\delta+2i\alpha\beta\sin\phi\\ -1-2\bar{\beta}+\delta-2i\alpha\bar{\beta}\sin\phi & -2\operatorname{Re}\beta-2\delta-2\delta\alpha\cos\phi\end{bmatrix}\leqq 0\,.$$

For this to be so, it is necessary that

$$-2\delta(1+\alpha\cos\phi)\leqq 2\operatorname{Re}\beta\leqq 2(1-\alpha\cos\phi)$$

for all $\varphi \in \boldsymbol{R}$, and this is impossible for $1 < \alpha^2$. Hence, for $1 < \alpha^2 < 2$, we have an example displaying asymptotic stability for all $\tau \geqq 0$, but having no nontrivial Liapunov function of the form (1.3).

V. Additional remarks. In this section we present two additional results based on the use of a simple Liapunov function of the form (1.3). The first of these provides a sufficient condition for, and an estimate on, exponential decay of solutions. The second provides sufficient conditions for stability and asymptotic stability that are simpler than, but not as "necessary" as, those of Theorem 3.2.

In the result (ii) of Theorem 3.2, and also (ii) of Theorem 2.2, we have concluded only asymptotic stability; without additional assumptions, this conclusion can be strengthened to exponential stability by employing yet another function, closely related to the function V of (1.3),

$$(5.1) \qquad V_\delta(x) \equiv y^*Ry + \sum_{k=1}^{m} \int_{-\tau_k}^{0} e^{\delta(\theta+\tau_k)} v^*(\theta) Q_k v(\theta) d\theta \,, \qquad x = (y, v) \in \chi \,,$$

where $R = R^*$, $Q_k = Q_k^*$, and δ is a positive real number. Computing $\dot{V}_\delta : \chi \to \boldsymbol{R}$ according to (1.2), we obtain

$$(5.2) \qquad \begin{aligned} \dot{V}_\delta(x) = {} & y^*\Big(RA + A^*R + \sum_{k=1}^{m} e^{\delta\tau_k} Q_k\Big)y + 2\,\mathrm{Re} \sum_{k=1}^{m} y^*RB_k v(-\tau_k) \\ & - \sum_{k=1}^{m} v^*(-\tau_k) Q_k v(-\tau_k) - \delta \sum_{k=1}^{m} \int_{-\tau_k}^{0} e^{\delta(\theta+\tau_k)} v^*(\theta) Q_k v(\theta) d\theta \,, \\ & x = (y, v) \in \chi \,. \end{aligned}$$

Under the hypotheses of (ii) in Theorem 3.2 there exist $C = C^* > 0$, $R = R^* > 0$, $Q_k = Q_k^*$ such that the Hermitian matrix

$$\begin{bmatrix} RA + A^*R + C + \sum_{k=1}^{m} Q_k & RB_1 & RB_2 \cdots RB_m \\ B_1^*R & -Q_1 & 0 \ \cdots \ 0 \\ B_2^*R & 0 & -Q_2 \cdots 0 \\ \vdots & \vdots & \vdots \qquad \vdots \\ B_m^*R & 0 & 0 \ \cdots -Q_m \end{bmatrix}$$

is negative-semidefinite. Consequently, $Q_k \geqq 0$ and the use of these matrices in V_δ leads to

$$\begin{aligned} \dot{V}_\delta(x) \leqq {} & -\delta V_\delta(x) + y^*\Big(RA + A^*R + \delta R + \sum_{k=1}^{m} e^{+\delta\tau_k} Q_k\Big)y \\ & + 2\,\mathrm{Re} \sum_{k=1}^{m} y^*RB_k v(-\tau_k) - \sum_{k=1}^{m} v^*(-\tau_k) Q_k v(-\tau_k) \end{aligned}$$

$$\leqq -\delta V_\delta(x) - y^*\Big[C - \delta R - \sum_{k=1}^{m}(e^{\delta\tau_k} - 1)Q_k\Big]y\ .$$

Since $C > 0$, we may now choose a $\delta > 0$ so small that $\dot{V}_\delta(x) \leqq -\delta V_\delta(x)$, which leads to the estimate

$$V_\delta(S(t)x) \leqq e^{-\delta t}V_\delta(x) \quad \text{for all} \quad t \geqq 0\ , \quad x \in \chi\ . \tag{5.3}$$

Consequently,

$$y^*(t)Ry(t) \leqq e^{-\delta t}\Big\{v_0^*(0)Rv_0(0) + \sum_{k=1}^{m} e^{\delta\tau_k}\int_{-\tau_k}^{0} v_0^*(\theta)Q_k v(\theta)d\theta\Big\} \tag{5.4}$$

for all $t \geqq 0$, implying the existence of $c_0 \geqq 1$ such that $||S(t)x_0||_\chi^2 \leqq c_0e^{-\delta t}||x_0||_\chi^2$ for all $t \geqq 0$, $x_0 \in \chi$. It follows that the hypotheses of (ii) in Theorem 3.2 (hence, of (ii) in Theorem 2.2) actually imply exponential stability of the trivial solution, with exponent $-\delta t/2$ such that $\delta > 0$ and $C \geqq \delta R + \sum_{k=1}^{m}(e^{\delta\tau_k} - 1)Q_k$; hence, δ depends on the delays τ_k, $k = 1, 2, \cdots, m$.

Turning to the second remark we wish to make, we refer again to the function V of (1.3), recalling that V is a Liapunov function for (1.2) if and only if (1.7) holds; i.e.,

$$\begin{bmatrix} RA + A^*R + \sum_{k=1}^{m} Q_k & RB_1 & RB_2 \cdots RB_m \\ B_1^*R & -Q_1 & 0 \ \cdots \ 0 \\ B_2^*R & 0 & -Q_2 \cdots \ 0 \\ \vdots & \vdots & \vdots \qquad \vdots \\ B_m^*R & 0 & 0 \ \cdots -Q_m \end{bmatrix} \leqq 0 \tag{5.5}$$

for some $R = R^*$, $Q_k = Q_k^*$, $k = 1, 2, \cdots, m$. Theorem 3.1 shows this condition to be satisfied if and only if the statement (ii) of that theorem is true for $C \equiv 0$, $D_k \equiv 0$, $k = 1, 2, \cdots, m$. Unfortunately, this statement may be difficult to verify and does not suggest an explicit construction for the matrices Q_k. In certain applications it may be desirable instead to simply "guess" a plausible form for $Q_k = Q_k^*$, and then search for some $R = R^*$ such that (5.5) holds. We will now describe a simple technique of this type.

Condition (5.5) implies that $Q_k \geqq 0$ with null space no larger than that of RB_k. Hence, let us suppose Q_k to be of the form $Q_k = B_k^*RZ_kRB_k$ for some $Z_k = Z_k^* > 0$, $k = 1, 2, \cdots, m$. Then (5.5) will be satisfied if there exists $R = R^*$, $Z_k = Z_k^* > 0$, such that

$$RA + A^*R \leqq -\sum_{k=1}^{m}(B_k^*RZ_kRB_k + Y_k) \tag{5.6}$$

where Y_k is a Hermitian matrix that inverts Z_k on the range of RB_k ($Y_k = Z_k^{-1}$ if $\det|RB_k| \neq 0$). In fact, a careful review of the proofs of Lemmas 2.1 and 3.1 reveals that if (5.5) can be satisfied by *some* $R = R^* \not\equiv 0$, $Q_k = Q_k^*$, then it can be satisfied by *some* $R = R^* \not\equiv 0$, $Q_k = Q_k^*$, with Q_k of the form we have suggested. Hence, satisfaction of (5.6) by some $R = R^* \not\equiv 0$, $Z_k = Z_k^* > 0$, is necessary and sufficient for the existence of any non-trivial Liapunov function of the form (1.3) for equation (1.2).

Condition (5.6) certainly will be satisfied if there exists $R = R^* \geqq 0$ and real numbers $\beta_k > 0$ such that

$$RA + A^*R + \sum_{k=1}^{m} (\beta_k B_k^* R B_k + \beta_k^{-1} R) \leqq 0 \,. \tag{5.7}$$

Although much more restrictive than our necessary and sufficient conditions, the comparatively simple condition (5.7) is certainly sufficient to ensure that V is a Liapunov function on χ for (1.3) with $Q_k \equiv \beta_k B_k^* R B_k$, and that the trivial solution is stable if $R > 0$; moreover, $\dot{V}$ is given by

$$\dot{V}(x) = y^*\left[RA + A^*R + \sum_{k=1}^{m} (\beta_k B_k^* R B_k + \beta_k^{-1} R)\right]y$$
$$- \beta_k^{-1}[y - \beta_k B_k v(-\tau_k)]^* R[y - \beta_k B_k v(-\tau_k)] \,, \quad x = (y, v) \in \chi \,.$$

Hence, under (5.7), if $R > 0$, it is possible to use the Invariance Principle [5] to provide conditions sufficient for asymptotic stability; for example, it is easily found that the trivial solution is asymptotically stable if $R = R^* > 0$ and $\beta_k > 0$, $k = 1, 2, \cdots, m$, are such that inequality (5.7) holds strictly.

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