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ON A CLASS OF ASYMPTOTICALLY STABLE LINEAR DIFFERENTIAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

ROBERTO CONTI

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1. We consider a linear ordinary differential equation

$$\text{(A)} \qquad \dot{x} - A(t)x = 0$$

where $A: t \mapsto A(t)$ is a continuous, complex $n \times n$ matrix valued function of $t \in J =]\alpha, +\infty[$, $-\infty \leqq \alpha$.

We denote by $E: (t, s) \mapsto E(t, s)$ the evolution operator generated by A, defined for $(t, s) \in J^2$ by

$$E(t, s) = \lim_k \left[I + \int_s^t A(t_1)dt_1 + \cdots + \int_s^t \cdots \int_s^{t_{k-1}} A(t_1) \cdots A(t_k)dt_k \cdots dt_1 \right],$$

where I is the $n \times n$ identity matrix.

Given $p \geqq 1$, we say that A belongs to the class L^pS if for each $\theta \in J$ there exist $k_p(\theta) > 0$ such that

$$\text{(1.1)} \qquad \alpha < \theta \leqq t \Rightarrow \left(\int_\theta^t |E(t, s)|^p ds \right)^{1/p} \leqq k_p(\theta) .$$

The class $L^\infty S$, defined by

$$\text{(1.2)} \qquad \alpha < \theta \leqq t \Rightarrow \sup\{|E(t, s)| : \theta \leqq s \leqq t\} \leqq k_\infty(\theta)$$

for some $k_\infty(\theta) > 0$, coincides with the class of A such that

$$\alpha < \theta \leqq s \leqq t \Rightarrow |E(t, s)| < \gamma(\theta)$$

for some $\gamma(\theta) > 0$, and will be denoted by US. It corresponds, in fact, to equations (A) whose solutions are uniformly stable for $t \to +\infty$.

The family L^pS, depending on the parameter p, $1 \leqq p \leqq +\infty$, has been considered (Coppel [1], [2]; Conti [3]) in connection with boundedness properties of the solutions of

$$\dot{x} - A(t)x = b(t) .$$

Later on other properties of L^pS were established (Conti [4], [5]). We want to add here a few remarks and put one question.

2. We shall first prove by an example the validity of

$$L^pS\setminus L^{p+\varepsilon}S \neq \varnothing\,, \qquad p \geqq 1\,,\quad \varepsilon > 0\,. \tag{2.1}$$

Let $n = 1$, $A(t) = \dot{f}(t)/f(t)$, $f(t) = e^{-t}[1 + g(t)]^{-1}$ where g is defined as follows. Let $\{t_k\}$, $\{\delta_k\}$, $\{\gamma_k\}$ be three sequences such that $t_k \uparrow +\infty$, $\delta_k > 0$, $\gamma_k > 0$, $t_k + \delta_k < t_{k+1} - \delta_{k+1}$, so that no two intervals $J_k = [t_k - \delta_k, t_k + \delta_k]$, $(k = 1, 2, \cdots)$, overlap. Then let $g(t) = 0$ for $t \in \boldsymbol{R}\setminus\bigcup J_k$, let $g(t_k) = \gamma_k$ and let the graph of g on J_k be the union of the two segments with endpoints $(t_k - \delta_k, 0)$, (t_k, γ_k) and (t_k, γ_k), $(t_k + \delta_k, 0)$. Since $E(t, s) = f(t)/f(s)$, using the inequality $(\alpha + \beta)^p \leqq 2^{p-1}(\alpha^p + \beta^p)$ for $p \geqq 1$, we have

$$\int_0^t |E(t, s)|^p ds = e^{-pt}[1 + g(t)]^{-p}\int_0^t e^{ps}[1 + g(s)]^p ds \leqq e^{-pt}\int_0^t e^{ps}[1 + g(s)]^p ds$$
$$\leqq 2^{p-1}e^{-pt}\int_0^t e^{ps}[1 + g^p(s)]ds \leqq 2^{p-1}\Big[p^{-1} + \sum_1^\infty{}_k \int_{J_k} g^p(s)ds\Big]\,.$$

On the other hand, since $g(s) = \lambda s + \mu$ on $[t_k - \delta_k, t_k]$, g^p is a convex function increasing from 0 to γ_k^p and the integral of g^p over $[t_k - \delta_k, t_k]$ is less than the area $\delta_k\gamma_k^p/2$ of the triangle with vertices at $(t_k - \delta_k, 0)$, (t_k, γ_k^p), $(t_k, 0)$. Similarly, the integral of g^p over $[t_k, t_k + \delta_k]$ is less than $\delta_k\gamma_k^p/2$. Therefore we have

$$\int_0^t |E(t, s)|^p ds \leqq 2^{p-1}\Big[p^{-1} + \sum_1^\infty{}_k \delta_k\gamma_k^p\Big]\,. \tag{2.2}$$

We also have by Hölder's inequality

$$\Big(\int_{t_k}^{t_k+\delta_k} |E(t_k + \delta_k, s)|^p ds\Big)^{1/p} = e^{-t_k-\delta_k}\Big(\int_{t_k}^{t_k+\delta_k} e^{ps}[1 + g(s)]^p ds\Big)^{1/p}$$
$$\geqq e^{-t_k-\delta_k}\delta_k^{1/p-1}\int_{t_k}^{t_k+\delta_k} e^s[1 + g(s)]ds \geqq e^{-\delta_k}\delta_k^{1/p-1}\int_{t_k}^{t_k+\delta_k} g(s)ds$$

and finally

$$\Big(\int_{t_k}^{t_k+\delta_k} |E(t_k + \delta_k, s)|^p ds\Big)^{1/p} \geqq 2^{-1}e^{-\delta_k}\delta_k^{1/p}\gamma_k\,. \tag{2.3}$$

Since δ_k, γ_k can be chosen so that $\delta_k \to 0$, $\sum_{k=1}^\infty \delta_k\gamma_k^p < +\infty$, whereas $\delta_k^{1/(p+\varepsilon)}\gamma_k \to +\infty$ (for instance, $\delta_k = a^k$, $\gamma_k = 2^k$, $2^{-(p+\varepsilon)} < a < 2^{-p}$) we see from (2.2) and (2.3) that there are $A \in L^pS\setminus L^{p+\varepsilon}S$.

The relation (2.1) suggests the following

QUESTION. *Does*

$$L^{p+\varepsilon}S \subset L^pS\,, \qquad 1 \leqq p\,,\quad \varepsilon > 0 \tag{2.4}$$

hold?

3. We say that A belongs to the class AS if

$$\lim_{t\to+\infty} E(t, \tau) = 0\ , \quad \tau \in J\ ,$$

which amounts to asymptotic stability of the solutions of (A) as $t \to +\infty$.

It can be shown (Coppel [2], Conti [5]) that

$$L^p S \subset \mathrm{AS}\ , \quad 1 \leqq p$$

and that the inclusion is strict.

We say that A belongs to the class ES if for each $\theta \in J$ there exist $\gamma(\theta) > 0$, $\mu(\theta) > 0$, such that

$$\alpha < \theta \leqq s \leqq t \Rightarrow |E(t, s)| \leqq \gamma(\theta) e^{-\mu(\theta)(t-s)}\ . \tag{3.1}$$

It is readily seen that if $A \in \mathrm{ES}$ then

$$\alpha < \theta \leqq t \Rightarrow \left(\int_\theta^t |E(t, s)|^p ds\right)^{1/p} \leqq \gamma'(\theta)\ , \quad 1 \leqq p\ , \tag{3.2}$$

for some $\gamma'(\theta) > 0$. But (3.2) implies, in turn, both (1.1) and (1.2), i.e., $A \in L^p S \cap \mathrm{US}$ and since (Conti [5])

$$\mathrm{ES} = L^p S \cap \mathrm{US}\ , \quad 1 \leqq p\ , \tag{3.3}$$

we see that (3.1) and (3.2) are equivalent.

The class ES corresponds to equations (A) whose solutions are exponentially (or uniformly asymptotically) stable for $t \to +\infty$.

From (3.3) it follows

$$\mathrm{ES} \subset \bigcap_{1 \leqq p} L^p S\ . \tag{3.4}$$

We want to show that the inclusion is strict. In fact, taking $\delta_k = 2^{-k^2}$, $\gamma_k = 2^k$ in the preceding example we have $\sum_{k=1}^\infty \delta_k \gamma_k^p = \sum_{k=1}^\infty (2^{p-k})^k < +\infty$ for every $p \geqq 1$. Then, by virtue of (2.2), the corresponding A belongs to $L^p S$ for every $p \geqq 1$, i.e., $A \in \bigcap_{1 \leqq p} L^p S$.

On the other hand from (2.3) we have

$$\left(\int_{t_k}^{t_k+\delta_k} |E(t_k + \delta_k, s)^p| ds\right)^{1/p} \geqq 2^{-1} e^{-1/2} 2^{-k^2/p} 2^k\ ,$$

hence, as $p \to +\infty$,

$$\sup\{|E(t_k + \delta_k, s)| : t_k \leqq s \leqq t_k + \delta_k\} \geqq 2^{-1} e^{-1/2} 2^k\ .$$

Therefore (1.2) cannot hold, i.e., $A \notin \mathrm{US}$, and from (3.3) it follows that (3.4) is a strict inclusion.

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ISTITUTO MATEMATICO U. DINI
VIALE MORGAGNI 67/A
50134 FIRENZE, ITALY