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BOUNDED AND ALMOST PERIODIC SOLUTIONS OF CERTAIN NONLINEAR ELLIPTIC EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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1. Let us consider the nonlinear elliptic equation

(E) $$u_{xx} + u_{yy} = f(x, y, u) ,$$

in the infinite strip

(D) $$-\infty < x < \infty , \qquad 0 \leqq y \leqq 1 ,$$

with boundary value conditions

$(\mathrm{BVC})_0$ $$u(x, 0) = 0 , \quad u(x, 1) = 0 , \qquad x \in R .$$

The more general boundary value conditions

(BVC) $$u(x, 0) = u_0(x) , \quad u(x, 1) = u_1(x) , \qquad x \in R ,$$

with u_0, $u_1 \in C^{(2)}(R)$, can be reduced to $(\mathrm{BVC})_0$ by choosing $v = u - u_0 - y(u_1 - u_0)$ as a new unknown function.

We will assume that f is a continuous map from $D \times R$ into R, satisfying further conditions to be specified below. It is worth to be pointed out that the change of variable indicated above preserves the basic assumptions to be made on f.

Let us notice that existence results for (E), under conditions (BVC), have been recently obtained by Schmitt, Thompson and Walter [4]. Their approach is based on the "method of lines," the discretization being taken with respect to the variable x. To build up the solution, the authors solve a boundary value problem for a countable system of ordinary differential equations.

Our aim is to attach a system of n ordinary differential equations to (E), for each natural n, such that the solution $u(x, y)$ of (E), $(\mathrm{BVC})_0$ is approximated with any degree of accuracy by the solution of the system. This aim will be achieved under reasonable assumptions on f and the solution, provided we associate with (E), $(\mathrm{BVC})_0$ the system

(E_n) $$d^2u/dx^2 + (n + 1)^2 A_n u = f_n(x, u) , \qquad x \in R ,$$

where

$$(1) \qquad u = \mathrm{col}(u_1, u_2, \cdots, u_n)\,, \quad f = \mathrm{col}(f(x, y_1, u_1), \cdots, f(x, y_n, u_n))\,,$$

and $y_k = k/(n+1)$, $k = 1, 2, \cdots, n$. The matrix A_n is given by the $n \times n$ matrix

$$(2) \qquad A_n = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

It is agreed that $u_k(x)$, $k = 1, 2, \cdots, n$, represents the approximate value for $u(x, y_k)$. We obtain (E_n) if we write (E) for $y = y_k$, $k = 1, 2, \cdots, n$, and then substitute to the second derivative $u_{yy}(x, y_k)$, the second difference $(n+1)^2[u(x, y_{k+1}) - 2u(x, y_k) + u(x, y_{k-1})]$. Of course, the conditions $(\mathrm{BVC})_0$ generate $u_0(x) = u_{n+1}(x) = 0$.

It is of primary interest that (E_n) is a conservative system. Indeed, A_n is a symmetric matrix and we can write

$$(3) \qquad A_n u = \mathrm{grad}_u\{\langle A_n u, u\rangle/2\}\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in R^n. Furthermore, $f_n(x, u)$ can be represented as

$$(4) \qquad f_n(x, u) = \mathrm{grad}_u \phi_n(x, u)\,,$$

with

$$(5) \qquad \phi_n(x, u) = \sum_{m=1}^{n} \int_0^{u_m} f(x, y_m, \xi)\,d\xi\,.$$

Therefore, from (3), (4), and (5), we easily find for (E_n) the following form:

$$(6) \qquad d^2u/dx^2 = \mathrm{grad}_u\{-(n+1)^2\langle A_n u, u\rangle/2 + \phi_n(x, u)\}\,,$$

which shows that (E_n) is conservative.

The matrix A_n, defined by (2), has the eigenvalues [3]

$$(7) \qquad \lambda_k = -4\sin^2(k\pi/2(n+1))\,, \quad k = 1, 2, \cdots, n\,.$$

Therefore, $-A_n$ is a positive definite matrix, a feature that will play a significant role in the sequel. Let us point out that the eigenvalues of the matrix $(n+1)^2 A_n$ are

$$(8) \qquad \lambda'_k = -4(n+1)^2\sin^2(k\pi/2(n+1))\,, \qquad k = 1, 2, \cdots, n\,.$$

Obviously, λ'_1 is the greatest eigenvalue of $(n+1)^2 A_n$. From (8) one easily derives that $\lambda'_1 \to -\pi^2$ as $n \to \infty$.

The above fact shows that the family of matrices $\{(n+1)^2 A_n;\ n = 1, 2, \cdots\}$ is uniformly stable (i.e., the set of their eigenvalues has a negative upper bound, say $-\pi^2 + \varepsilon$, with $\varepsilon > 0$ arbitrarily small).

The following basic problems arise in connection with the approximating procedure devised above:

I. Find conditions guaranteeing the existence and uniqueness of solution to the system (E_n).

Let us notice that the solution must be defined on the entire real axis, if we look for the approximation of a solution of (E), defined in D.

II. Prove that the solution of the system (E_n) is approximating the solution of (E), in a convenient norm, with any degree of accuracy.

We shall be concerned with the supremum norm, though alternate norms could provide somewhat better results.

III. Find estimates for the solutions of the approximating equations, independent of n, and then use them in proving the existence for the partial differential equation.

We are not going to consider in this paper Problem III. As mentioned above, under assumptions that are more restrictive than those in the sequel, the existence for (E), $(\mathrm{BVC})_0$ has been proved in [4].

2. Let us consider now Problem I. In other words, we must provide conditions that assure the existence and uniqueness of the solution of (E_n). Since the equivalent form of (E_n) is (6), we are going to consider the system

$$d^2u/dx^2 = \mathrm{grad}_u F(x, u)\ , \quad x \in R\ , \tag{9}$$

with $u \in R^n$ and F a map from $R \times R^n$ into R. Our aim is to find adequate conditions on F, such that (9) possesses a unique bounded solution on R.

First, let us remark that a special case of (9) is provided by the linear system

$$d^2u/dx^2 - Mu = f(x)\ , \qquad x \in R\ , \tag{10}$$

where $u \in R^n$, $M > 0$, and $f(x)$ is a continuous and bounded map from R into R^n. Then (10) has a unique bounded (on R) solution, given by

$$\bar{u}(x) = -\frac{1}{2\sqrt{M}}\left\{e^{\sqrt{M}x}\int_x^\infty e^{-\sqrt{M}t}f(t)dt + e^{-\sqrt{M}x}\int_{-\infty}^x e^{\sqrt{M}t}f(t)dt\right\}\ , \tag{11}$$

such that

$$\sup|\bar{u}(x)| \leqq M^{-1}\sup|f(x)|\ , \qquad x \in R\ . \tag{12}$$

It is now convenient to take the space $C(R, R^n)$, consisting of all continuous and bounded maps from R into R^n, as underlying space. The norm to be used is the supremum norm.

On each $\Sigma_r \subset C(R, R^n)$, with $\Sigma_r = \{v; v \in C(R, R^n), |v(t)| \leqq r, r > 0\}$, we shall define an operator T_r by means of the following equation

$$d^2u/dx^2 - Mu = \mathrm{grad}_u F(x, v) - Mv \ , \tag{13}$$

where u denotes the only solution in $C(R, R^n)$ of the above equation, and $v \in \Sigma_r$. The number $M > 0$ will be chosen below. We denote in this case, $u = T_r v$. Hence, $T_r \colon \Sigma_r \to C(R, R^n)$.

Let us point out that equation (13) has the form (10), and a unique solution in $C(R, R^n)$ is guaranteed as soon as its right hand side belongs to the same space. Obviously, this implies that $\mathrm{grad}_u F(x, v(x))$ to has be bounded for any $v \in C(R, R^n)$. Such a situation occurs if $\mathrm{grad}_u F(x, u)$ is bounded on any set of the form $R \times B_r$, with $B_r = \{u; u \in R^n, |u| \leqq r, r > 0\}$. In other words, for any $r \geqq 0$, there exists $A(r) > 0$, such that

$$|\mathrm{grad}_u F(x, u)| \leqq A(r) \ , \qquad (x, u) \in R \times B_r \ . \tag{14}$$

Another condition we must assume in regard to $F(x, u)$ is concerned with the Hessian matrix attached to $F(x, u)$:

$$H(x, u) = (\partial^2 F/\partial u_j \partial u_i) \ , \quad i, j = 1, 2, \cdots, n \ . \tag{15}$$

Namely, we assume the existence of a positive number $m > 0$, and a function $M(r) > 0$, such that

$$mI \leqq H(x, u) \leqq M(r)I \ , \qquad (x, u) \in R \times B_r \ , \tag{16}$$

where I stands for the unit matrix of order n.

From (16), one obtains $\langle \mathrm{grad}_u F(x, u) - \mathrm{grad}_u F(x, v), u - v \rangle \geqq m|u - v|^2$, which shows the strict monotonicity of the right hand side of (9).

Under assumptions (14) and (16), one can easily show that the operator T_r is a contraction on Σ_r, and that

$$T_r \Sigma_r \subset \Sigma_r \ , \quad \text{for} \quad mr \geqq A(0) \ . \tag{17}$$

The number M in (13) is precisely $M(r)$ occuring in (16).

Let $u = T_r v$ and $\bar{u} = T_r \bar{v}$ be two couples of corresponding elements, with $v, \bar{v} \in \Sigma_r$. Then (13) implies

$$\begin{aligned}(d^2/dx^2)(u - \bar{u}) - M(u - \bar{u}) &= \mathrm{grad}_u F(x, v(x)) - \mathrm{grad}_u F(x, \bar{v}(x)) \\ &\quad - M[v(x) - \bar{v}(x)] \ .\end{aligned} \tag{18}$$

Therefore, $u - \bar{u}$ satisfies an equation of the form (10) and is bounded on R. The formula (12) implies

$$\sup|u(x)-\bar{u}(x)| \leqq M^{-1}\sup|\mathbf{grad}_u F(x, v(x)) - \mathbf{grad}_u F(x, \bar{v}(x)) \\ -M[v(x)-\bar{v}(x)]| , \tag{19}$$

the supremum being taken for $x \in R$. But

$$\mathbf{grad}_u F(x, v(x)) - \mathbf{grad}_u F(x, \bar{v}(x)) = H(x, w(x))[v(x)-\bar{v}(x)] , \tag{20}$$

where $H(x, w(x))$ is the Hessian matrix, taken at an intermediate point $w(x)$, between $v(x)$ and $\bar{v}(x)$. Hence, (19) yields

$$\sup|u(x)-\bar{u}(x)| \leqq M^{-1}\sup|[MI - H(x, w)][v(x)-\bar{v}(x)]| . \tag{21}$$

The matrix $MI - H(x, w)$ is a symmetric one, and we can write

$$|[MI-H][v-\bar{v}]| \leqq |MI-H||v-\bar{v}| , \tag{22}$$

where $|MI-H|$ denotes the operator norm of $MI-H$. It is well known that

$$|MI-H| = \sup_{|\xi|=1} \langle [MI-H]\xi, \xi\rangle . \tag{23}$$

By (16) and (23), one obtains

$$|MI-H| \leqq M - m , \tag{24}$$

which combined with (21) leads to

$$\sup|u(x)-\bar{u}(x)| \leqq ((M-m)/M)\sup|v(x)-\bar{v}(x)| , \qquad x \in R . \tag{25}$$

Inequality (25) proves that T_r is a contraction map on Σ_r.

We must prove now that (17) holds true. Indeed, the right hand side of (13) can be written as

$$\mathbf{grad}_u F(x, v(x)) - Mv(x) = -[MI - H(x, v^*(x))]v(x) + \mathbf{grad}_u F(x, u)|_{u=0} . \tag{26}$$

Applying again (12) to (13), and manipulating in a similar way to the one shown above, one obtains

$$\sup|u(x)| \leqq (M(r)-m)r/M(r) + A(0)/M(r) , \qquad x \in R , \tag{27}$$

which shows that $u \in \Sigma_r$, provided $mr \geqq A(0)$.

Let us summarize the above discussion regarding equation (9) into the following

THEOREM 1. *Assume $F: R \times R^n \to R$ is continuous, and of class $C^{(2)}$ in u, such that* (14) *and* (16) *hold true. Then equation* (9) *has a unique solution, bounded on R.*

REMARK. The uniqueness in $C(R, R^n)$ follows from the fact that the solution is unique in each Σ_r, $mr \geqq A(0)$.

COROLLARY. *Assume further that* $\mathbf{grad}_u F(x, u)$ *is* (*Bohr*) *almost* *periodic in* x, *uniformly with respect to* $u \in B_r$, *for each* $r > 0$. *Then the unique bounded solution of* (9) *is* (*Bohr*) *almost periodic too.*

Indeed, $\bar{u}(x)$ given by (11) is almost periodic as soon as $f(x)$ is, on behalf of (12). Instead of $C(R, R^n)$, one can use as underlying space for (9) the space $AP(R, R^n)$ of (Bohr) almost periodic functions with values in R^n, [2], [5].

Of course, if $F(x, u)$ is periodic in x, one can ask whether the solution constructed above has the same property. The answer is positive, because $\bar{u}(x)$ in (11) is periodic, whenever $f(x)$ is. The periodic case has been investigated in [1], using another approach. The hypotheses in [1] are somewhat less restrictive than above.

From Theorem 1 we can obtain immediately the answer to Problem I formulated above. We must secure conditions (14) and (16) for the function occurring in system (E_n) or (6), namely

$$F(x, u) = -(n + 1)^2 \langle A_n u, u \rangle / 2 + \phi_n(x, u) . \tag{28}$$

If one takes into account the properties of the matrix A_n listed above and formula (5), one finds out that $F(x, u)$ given by (28) satisfies both (14) and (16), provided $f(x, y, u)$ is such that $f_u(x, y, u)$ exists on $D \times R$, and for a positive μ,

$$-\pi^2 < -\mu \leqq f_u(x, y, u) \leqq C(|u|) , \qquad (x, y, u) \in D \times R , \tag{29}$$

with $C(|u|)$ bounded on each compact interval.

Condition (29) implies the boundedness of $f(x, y, u)$ on each set $D \times [-r, r]$, $r > 0$, provided

$$f(x, y, 0) \text{ is bounded in } D . \tag{30}$$

To check condition (16), it is worth to remark that the Hessian matrix corresponding to $F(x, u)$, given by (28), is

$$H(x, u) = -(n + 1)^2 A_n + \mathrm{diag}(f_u(x, y_1, u_1), \cdots, f_u(x, y_n, u_n)) . \tag{31}$$

Consequently, the answer to Problem I is positive, provided (29) and (30) hold true for $f(x, y, u)$.

3. We can now consider Problem II formulated above, i.e., to investigate the convergence of the approximating proposed scheme. Let us find first the system verified by the error function

$$\varepsilon(x) = \mathrm{col}(u_1(x) - u(x, y_1), \cdots, u_n(x) - u(x, y_n)) , \tag{32}$$

where $u_k(x)$ are defined by (E_n), and $u(x, y)$ is a solution of (E) in (D), verifying $(\mathrm{BVC})_0$. It will be subject to further assumptions we shall formulate below.

Let us remark that (E) implies

(33) $(d^2/dx^2)u(x, y_k) + u_{yy}(x, y_k) = f(x, y_k, u(x, y_k))\,,\quad x \in R\,,\quad k = 1, 2, \cdots, n\,.$

By subtracting the equations of (33) from those of the system (E_n), one obtains

(34) $d^2\varepsilon(x)/dx^2 = -(n+1)^2 A_n \varepsilon(x) + f_n(x, u(x)) - f_n(x, u(x, \cdot)) + r_n(x)\,,$

with

(35) $r_n(x) = \mathrm{col}(u_{yy}(x, y_k) - (n+1)^2[u(x, y_{k+1}) - 2u(x, y_k) + u(x, y_{k-1})])_{k=1}^{n}\,.$

Before proceeding further, let us obtain a "linearized" version of (34). This can be achieved if we take into account

(36) $$f_n(x, u(x)) - f_n(x, u(x, \cdot)) = G_n(x)[u(x) - u(x, \cdot)]\,,$$

where G_n is given by

(37) $$G_n(x) = \mathrm{diag}(g_1(x), g_2(x), \cdots, g_n(x))\,,$$

with

(38) $$g_k(x) = \int_0^1 f_u(x, y_k, u(x, y_k) + t\varepsilon_k(x))dt\,,\qquad k = 1, 2, \cdots, n\,.$$

Hence, (34) becomes

(39) $$d^2\varepsilon(x)/dx^2 = -(n+1)^2 A_n \varepsilon(x) + G_n(x)\varepsilon(x) + r_n(x)\,,$$

with $G_n(x)$ given by (37).

It is worth to point out that each $g_k(x)$, $k = 1, 2, \cdots, n$, satisfies by (29) and (38)

(40) $$g_k(x) \geqq -\mu\,,\qquad x \in R\,.$$

The linearized system (39) is such that its matrix of coefficients

(41) $$-(n+1)^2 A_n + G_n(x)$$

is symmetric and has only positive eigenvalues. More precisely, the smallest eigenvalue of (41) is at least $(\pi^2 - \mu)/2 > 0$, provided n is chosen sufficiently large.

This feature of the system (39) can be exploited to get an estimate for $|\varepsilon(x)|$ in terms of $|r_n(x)|$. Indeed, if we take the scalar product of both sides in (39) by $\varepsilon(x)$, and use

(42) $$\langle d^2\varepsilon/dx^2, \varepsilon\rangle = (d^2/dx^2)|\varepsilon(x)|^2/2 - |\varepsilon'(x)|^2$$

and

$$(43)\qquad -(n+1)^2\langle A_n\varepsilon, \varepsilon\rangle + \langle G_n\varepsilon, \varepsilon\rangle \geqq (\pi^2-\mu)|\varepsilon|^2/2\ ,$$

then we obtain, for sufficiently large n,

$$(44)\qquad (d^2/dx^2)|\varepsilon(x)|^2/2 \geqq (\pi^2-\mu)|\varepsilon(x)|^2/2 - |r_n(x)||\varepsilon(x)|\ , \qquad x\in R\ .$$

We claim that (44) implies

$$(45)\qquad |\varepsilon(x)| \leqq 2(\pi^2-\mu)^{-1}\sup|r_n(x)|\ , \qquad x\in R\ .$$

Of course, we make the assumption that $u(x, y)$ is such a solution of (E), for which the right hand side in (45) is finite (actually, we need $|r_n(x)|\to 0$ as $n\to\infty$, uniformly in x, $x\in R$).

Let us process $r_n(x)$, in order to get a better idea about the conditions we have to impose on $u(x, y)$, such that we shall get a positive answer to Problem II. Let us assume that the solution $u(x, y)$ of (E), $(\mathrm{BVC})_0$ is such that

$$(46)\qquad |u_{yy}(x, z) - u_{yy}(x, w)| \leqq \eta(|z-w|)\ , \qquad (x, y), (x, w)\in D\ ,$$

where $\eta(\delta)$ is the modulus of continuity for u_{yy}, satisfying

$$(47)\qquad \lim\sqrt{n}\,\eta(1/(n+1)) = 0 \quad \text{as} \quad n\to\infty\ .$$

In particular, for $\eta(\delta) = K\delta^\alpha$, $1/2 < \alpha \leqq 1$, one obtains Hölder continuity, and (47) is obviously verified.

It is worth to point out that any solution of (E), $(\mathrm{BVC})_0$, under assumptions (29), (30) and (46), is bounded in D. Indeed, from $u(x, 0) = 0$, $x\in R$, we derive $u_{xx}(x, 0) = 0$, $x\in R$. Hence, taking $y = 0$ in (E), we find $u_{yy}(x, 0) = f(x, 0, 0)$, which together with (46) prove the boundedness of u_{yy} in D. Furthermore, for each $x\in R$, from $u(x, 0) = 0 = u(x, 1)$, one derives the existence of $y_x\in(0, 1)$ such that $u_y(x, y_x) = 0$. Consequently, $u_y(x, y) = u_y(x, y) - u_y(x, y_x) = (y - y_x)u_{yy}(x, \tilde{y}_x)$, which shows that $u_y(x, y)$ is also bounded in D. Finally, $u(x, y) = u(x, y) - u(x, 0) = yu_y(x, \bar{y})$, from which we get the boundedness of $u(x, y)$ in D.

We can show that, under assumptions (46), (47), the answer to Problem II stated in Section 1 is positive. Since

$$(48)\quad |u_{yy}(x, y_k) - (n+1)^2[u(x, y_{k+1}) - 2u(x, y_k) + u(x, y_{k-1})]| \leqq \eta(1/(n+1))\ ,$$
$$k = 1, 2, \cdots, n\ , \qquad x\in R\ ,$$

one obtains

$$(49)\qquad |r_n(x)| < \sqrt{n}\,\eta(1/(n+1))\ , \qquad x\in R\ .$$

Combining (45) and (49), there results

$$(50)\qquad |\varepsilon(x)| \leqq 2(\pi^2-\mu)^{-1}\sqrt{n}\,\eta(1\ (n+1)), \qquad x\in R\ ,$$

which together with (47) provide the desired answer to Problem II.

We must now prove that (44) implies (45), under the only assumption that $\varepsilon(x)$ is bounded on R. This will follow from the next lemma.

LEMMA. *Let* $y = y(x)$ *be a* $C^{(2)}$*-map from* R *into* $[0, \infty)$, *such that*

$$d^2y/dx^2 \geqq f(y)\ , \qquad x \in R\ , \tag{51}$$

with $f(y)$ *continuous on* $[0, \infty)$, *and satisfying*

$$f(y) > 0 \quad \textit{for} \quad y > M > 0\ . \tag{52}$$

If it is known that $y(x)$ *is bounded on* R, *then necessarily*

$$y(x) \leqq M\ , \qquad x \in R\ . \tag{53}$$

The proof of the lemma will be given in the Appendix.

Going back to (44), let us denote $|\varepsilon(x)|^2 = y(x)$. Then (44) leads to an inequality of the form (51), with $f(y) = (\pi^2 - \mu)y - 2\sup|r_n(x)|\sqrt{y}$. The lemma applies and one obtains (45). Consequently, the following result has been established:

THEOREM 2. *Consider equation* (E), *under conditions* $(\mathrm{BVC})_0$. *Let* $u(x, y)$ *be solution of the problem, such that* (46), (47) *hold true. Then the* (*unique*) *bounded solution of* (E_n) *is uniformly* (*on* R) *approximating* $u(x, y)$, *provided* $f(x, y, u)$ *in* (E) *satisfies conditions* (29) *and* (30).

4. By the above result, we shall discuss some properties of the solutions of equation (E), in connection with the properties of solutions to the approximating system (E_n). Our aim is to transfer qualitative properties from the ordinary differential system that approximates (E), $(\mathrm{BVC})_0$, to the partial differential equation under investigation.

A first result that can be easily proved regards the uniqueness of the solution for (E), $(\mathrm{BVC})_0$, under assumptions (29), (30), on $f(x, y, u)$, and assumptions (46), (47) on the solution $u(x, y)$. Indeed, it has been shown that (E_n) has a unique bounded solution on R, under assumptions (29) and (30) on $f(x, y, u)$. On the other hand, any solution $u(x, y)$ of (E), $(\mathrm{BVC})_0$, satisfying (46) and (47), is bounded in D and can be uniformly approximated by the solution of (E_n). More precisely, if $y_{k,n} = k/(n+1)$, with $n \geqq 1$ and $0 < k \leqq n$, then $u(x, y_{k,n}) = \lim u^{pk}_{p(n+1)}(x)$ as $p \to \infty$, uniformly with respect to $x \in R$, where $u^{pk}_{p(n+1)}$ denotes the pk-th coordinate of the unique bounded solution of the system $(\mathrm{E}_{n'})$, for $n' = p(n+1) - 1$. Hence, $u(x, y_{k,n})$ is uniquely determined for each $y = y_{k,n}$. From the continuity of u, we find easily that $u(x, y)$ is uniquely determined from the approximating procedure, and this shows the uniqueness of $u(x, y)$.

Let us consider now the problem of almost periodicity for the solution $u(x, y)$ of (E), $(\mathrm{BVC})_0$, under assumption of almost periodicity for $f(x, y, u)$ in x. In the case of Poisson's equation, i.e., $f(x, y, u) \equiv f(x, y)$, this problem has been discussed by Zaidman [6], under different hypotheses.

First, let us remark that the almost periodicity of $f(x, y, u)$ in x, uniformly with respect to $y \in [0, 1]$ and $u \in [-A, A]$, for each $A > 0$, implies the almost periodicity in x of $\mathrm{grad}_u F(x, u)$, with F given by (28). Of course, the almost periodicity is uniform with respect to u (now $u \in R^n$), for u in any compact set [2], [5].

Applying the Corollary to Theorem 1, one obtains the almost periodicity of the unique bounded solution of (E_n), for each n. In particular, this implies that $u(x, y_{k,n})$ is almost periodic in x, for each $y_{k,n} = k/(n+1)$, $n \geqq 1$, $0 < k \leqq n$.

The almost periodicity of $u(x, y)$ in x, uniformly with respect to $y \in [0, 1]$, will be the consequence of the boundedness of $u_y(x, y)$ in D (see Section 3), and of the almost periodicity of $u(x, y_{k,n})$. Indeed, the following inequality provides the support of the above statement:

$$|u(x+\xi, y) - u(x, y)| \leqq |u(x+\xi, y) - u(x+\xi, y_{k,n})| + |u(x+\xi, y_{k,n}) - u(x, y_{k,n})| + |u(x, y_{k,n}) - u(x, y)| .$$

This inequality is valid for each $(x, y) \in D$ and $\xi \in R$. For each $y \in [0, 1]$, $y_{k,n}$ should be the closest one (for fixed n) with respect to y, in the given subdivision of $[0, 1]$: $|y - y_{k,n}| \leqq 1/(n+1)$.

Summing up the discussion above, we can state the following result.

THEOREM 3. *Let us assume conditions of Theorem 2 are satisfied. Moreover, assume $f(x, y, u)$ is almost periodic in x, uniformly with respect to $(y, u) \in [0, 1]\times[-A, A]$, for each $A > 0$. Then the unique solution of the problem is almost periodic in x, uniformly with respect to $y \in [0, 1]$.*

REMARK. The almost periodicity is not the only property that can be transferred from the ordinary differential equations, that approximate (E), to the solution of (E). Some other kinds of behavior can be investigated, using the scheme described above. For instance, Theorem 1 has a correspondent in the case when we look for solutions with finite limit at $\pm\infty$. One can easily check that formula (11) furnishes a solution of this kind, provided $f(x)$ in equation (0) enjoys the property mentioned above. The extension to the nonlinear case follows the same lines as in Section 2. The conclusion for the solution $u(x, y)$ of (E), $(\mathrm{BVC})_0$ is the existence of the limits $\lim_{x\to\infty} u(x, y) = u_+(y)$, and $\lim_{x\to-\infty} u(x, y) = u_-(y)$.

5. Let us consider now equation (E) in the half-strip

$$(D_+) \qquad 0 \leqq x < \infty\,, \qquad 0 \leqq y \leqq 1$$

with boundary value conditions

$$(54) \qquad u(x, 0) = u(x, 1) = 0\,, \qquad 0 \leqq x < \infty$$

$$(54') \qquad u(0, y) = g(y)\,, \qquad 0 \leqq y \leqq 1\,,$$

or (54) and

$$(54'') \qquad u_x(0, y) - h(y)u(0, y) = k(y)\,, \qquad 0 \leqq y \leqq 1\,,$$

where g, h and k are continuous maps from $[0, 1]$ into R. A basic assumption on h is

$$(55) \qquad h(y) \geqq 0\,, \qquad 0 \leqq y \leqq 1\,.$$

In particular, (54″) could take the form $u_x(0, y) = k(y)$, $0 \leqq y \leqq 1$. In other words, both Dirichlet type and Neumann type boundary value conditions are involved.

Of course, the system (E_n) is again a candidate for the approximation procedure of solutions in D_+. This time we must look for solutions defined on R_+, satisfying the initial condition

$$(56) \qquad u(0) = g_n = \mathrm{col}(g(y_1), g(y_2), \cdots, g(y_n))\,,$$

in case of boundary value conditions (54), (54′), and

$$(57) \qquad u'(0) - H_n u(0) = k_n = \mathrm{col}(k(y_1), k(y_2), \cdots, k(y_n))\,,$$

in case of boundary value conditions (54), (54″). By H_n we denoted the matrix

$$(58) \qquad H_n = \mathrm{diag}(h(y_1), h(y_2), \cdots, h(y_n))\,.$$

As seen in the case of equation (E) with boundary value conditions $(\mathrm{BVC})_0$, the data on the half-lines $0 \leqq x < \infty$, $y = 0$ and $0 \leqq x < \infty$, $y = 1$, lead to $u_0(x) = 0 = u_{n+1}(x)$. This means that the approximating system has form (E_n).

The same basic problems, stated in Section 1, arise now in connection with (E) in D_+: existence and uniqueness for the approximating system; validity of the procedure.

We are not going to provide all the details involved in the proofs of results stated below. They are very much alike to those encountered when equation (E) has been discussed in D.

THEOREM 4. *Consider the system* (9), *and assume that* $F: R_+ \times R^n \to R$ *is continuous, of class* $C^{(2)}$ *in* u, *and satisfies* (14) *in* $R_+ \times R^n$, *and* (16) *in* $R_+ \times B_r$. *Then* (9) *has a unique solution bounded on* R_+, *satisfying either one of the initial conditions*

$$u(0) = u_0 \in R^n \; , \tag{59}$$

$$u'(0) - Hu(0) = u_1 \in R^n \; , \tag{60}$$

where $H = \mathrm{diag}(h_1, h_2, \cdots, h_n)$, $h_i \geqq 0$, $i = 1, 2, \cdots, n$.

From Theorem 4 one derives the needed result for (E_n), with initial conditions (56) or (57).

THEOREM 5. *Consider equation* (E) *in* D_+, *under conditions* (54), (54′) *or* (54), (54″). *Let* $u(x, y)$ *be a solution of the problem satisfying* (46) *in* D_+, *with* $\eta(\delta)$ *subject to* (47). *Then* $u(x, y)$ *is unique and can be uniformly approximated on* R_+ *by the bounded* (*on* R_+) *solution of* (E_n), *satisfying the initial condition* (56) *or* (57), *provided* $f(x, y, u)$ *verifies* (29) *in* $D_+ \times R$, *and* (30) *in* D_+.

In proving Theorem 5, the assertion in the remark following the proof of the lemma in Appendix has to be used, instead of the lemma itself.

It is interesting to point out that the method of lines can be used in different situations than those discussed above.

For instance, the equation

$$(r\partial/\partial r)(r\partial u/\partial r) + \partial^2 u/\partial\theta^2 = r^2 f(r, \theta, u) \; , \tag{E′}$$

in the sector

$$0 < r < \infty \; , \qquad 0 \leqq \theta \leqq T \; , \qquad T < 2\pi \; , \tag{D'}$$

with boundary value conditions

$$u(r, 0) = u_0(r) \; , \qquad u(r, T) = u_1(r) \; , \qquad 0 < r < \infty \; , \tag{61}$$

can be reduced to the case investigated above. Indeed, if one discretizes (E′) with respect to θ, and take $\tau = lnr$ as a new independent variable, one obtains the same system (E_n), with slightly different right hand side. The discussion can be also conducted when, instead of D', one takes the set $0 < r_0 \leqq r < \infty$, $0 \leqq \theta \leqq T$.

Appendix. PROOF OF THE LEMMA. Since $y(x)$ is bounded on R, only two possibilities can occur. First, when $\sup y(x)$ is attained at a certain point $\tilde{x} \in R$. In such case $y''(\tilde{x}) \leqq 0$, and (52) shows that we must have $y(\tilde{x}) \leqq M$. Hence, (53) holds true in such a case. Second, there is no point $\tilde{x} \in R$ such that $y(\tilde{x}) = \sup y(x)$, $x \in R$. In this case, at least one of the following situations must take place: either we can find a sequence $\{x_m\}$, $x_m \to \infty$, such that $y(x_m) \to Y = \sup y(x)$, $x \in R$, or a sequence $\{x'_m\}$, $x'_m \to -\infty$, such that $y(x'_m) \to Y$ as $m \to \infty$. Since changing x by $-x$ does not affect (51), we examine only the case when $Y = \lim y(x_m)$, $x_m \to +\infty$. Again, two distinct situations have to be discussed separately: first, when $\lim y(x) = Y$ as $x \to +\infty$ and secondly, when there exists another sequence $\{\xi_m\}$, $\xi_m \to +\infty$, such that $\lim y(\xi_m) = Y_0 < Y$, as $m \to \infty$. If the first situation occurs, and $y(x) < Y$ for any $x \in R$, then one can find a sequence $\{x_m\}$, $x_m \to +\infty$, such that $y''(x_m) \leqq 0$, $m = 1, 2, 3 \cdots$. Indeed, if we assume $y''(x) > 0$ for $x \geqq X$, then $x \to y(x)$ is a convex map. From the boundedness of $y(x)$ we easily obtain $y(x) \downarrow Y$, in contradiction with the fact that $y(x) < Y$ for all x. Hence $0 \geqq f(y(x_m))$, $m = 1, 2, 3, \cdots$, which implies $0 \geqq f(Y)$. On behalf of (52), we again obtain (53). If the second situation occurs, then from $y(x_m) \to Y$ and $y(\xi_m) \to Y_0 < Y$, as $m \to \infty$, one finds easily that a new sequence $\{\bar{x}_m\}$, $\bar{x}_m \to +\infty$, must exist, with the property $y(\bar{x}_m) \to Y$, as $m \to \infty$, and such that $y(\bar{x}_m)$ is a local maximum for $y(x)$. At such a point we shall have $y''(\bar{x}_m) \leqq 0$, and therefore $0 \geqq f(y(\bar{x}_m))$, $m = 1, 2, 3, \cdots$. Hence $0 \geqq f(Y)$ and (53) holds true in this case too. The lemma is thereby proven.

REMARK. If in the statement of the lemma we replace R by $R_+ = [0, \infty)$, then estimate (53) keeps validity, provided we require an adequate condition to be satisfied at $x = 0$: either $y(0) = 0$, or $y'(0) - hy(0) = 0$, $h \geqq 0$. This version of the lemma is applicable in the case of equation (E) in D_+.

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