Tôhoku Math. Journ.
32 (1980), 225-233.

TWO POINT BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS IN HILBERT SPACES

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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(Received May 1, 1979, revised June 21, 1979)

1. Introduction. This paper is devoted to the study of existence and uniqueness of solutions for the Picard boundary value problem

$$x''(t) + kx'(t) + f(t, x(t), x'(t)) = 0\ , \qquad t \in [0, \pi]\ , \tag{1.1}$$

$$x(0) = x(\pi) = 0\ , \tag{1.2}$$

with $f: I \times H \times H \to H$, $k \in \boldsymbol{R}$, $I = [0, \pi]$ and H is a real Hilbert space. The results are motivated by and improve the ones given in [6] for the case $H = \boldsymbol{R}^n$, where references to the corresponding literature are also given. One can add a recent paper by Brown and Lin [2] for the scalar case.

We shall essentially consider two types of regularity assumptions for f. In Theorem 1 we suppose that f is completely continuous, which allows an existence proof based upon Leray-Schauder's theorem. The required a priori bounds for the possible solutions are obtained via L^2-estimates and an extension of a Nagumo-type condition of Lasota and Yorke [5] given in Lemma 1. A condition for the uniqueness of the solution given in Theorem 2 suggests then to replace the complete continuity of f by some monotonicity-type condition, and this is done in Theorem 3. To prove this result we use the approach introduced in [7] which consists in approximating (1.1)-(1.2) by suitable finite-dimensional differential equations which can be solved using Theorem 1 and then using the monotonicity to obtain an exact solution from those approximate ones.

Obvious modifications allow replacing (1.2) by other boundary conditions, homogeneous or not. Moreover, we shall refer to [6] with easy adaptations for the obtention of various interesting special cases of the main results given here.

2. p-Nagumo functions and an extension of a result of Lasota-Yorke. We first introduce the following definition.

DEFINITION. If $p \geqq 1$ is a real number, a *p-Nagumo function* will be a continuous function $h: \boldsymbol{R}_+ \to \boldsymbol{R}_+ \setminus \{0\}$ such that

$$\int_0^\infty \frac{s^{(2/p)-1}}{h(s)} ds = +\infty . \tag{2.1}$$

REMARKS 1. For $p = 1$, this is the usual definition for a Nagumo function (see e.g., [4]).

2. If $h(s) = s^q + C$, with $C > 0$, it is easy to check that h is a *p-Nagumo* function if and only if $pq \leqq 2$.

Condition (2.1) and the positiveness of h imply that the continuous function

$$\chi: (x, u) \mapsto \int_{u/\pi}^{x} \frac{s^{(2/p)-1}}{h(s)} ds$$

is increasing in x for each fixed u and such that $\lim_{x\to+\infty} \chi(x, u) = +\infty$. Therefore, the equation in x

$$\chi(x, u) = u$$

has a unique solution $x = g(u)$ for each $u \in \boldsymbol{R}_+$. This mapping g is thus defined on $\boldsymbol{R}_+$ by the relation

$$\int_{u/\pi}^{g(u)} \frac{s^{(2/p)-1}}{h(s)} ds = u . \tag{2.2}$$

Clearly one must have, on $\boldsymbol{R}_+$, $g(u) \geqq u/\pi$. Now, if $v > u$, one has

$$\int_{v/\pi}^{g(v)} \frac{s^{(2/p)-1}}{h(s)} ds = v > u = \int_{u/\pi}^{g(u)} \frac{s^{(2/p)-1}}{h(s)} ds ; \tag{2.3}$$

if $g(u) < v/\pi$, then $g(u) < v/\pi \leqq g(v)$, and if $g(u) \geqq v/\pi$ and $g(v) \leqq g(u)$, then $[v/\pi, g(v)]$ is strictly contained in $[u/\pi, g(u)]$ which is in contradiction to (2.3). Thus g is increasing on $\boldsymbol{R}_+$.

We can now prove the following lemma, essentially due, for $p = 1$ and $H = \boldsymbol{R}^n$, to Lasota and Yorke [5].

LEMMA 1. *Let $p \geqq 1$ be a real, h a p-Nagumo function and g the function given by* (2.2). *If $x \in C^2([0, \pi], H)$, with H a real Hilbert space, is such that, for all $t \in [0, \pi]$, one has*

$$|(x'(t), x''(t))| \leqq p^{-1} h(|x'(t)|^p) |x'(t)|^p , \tag{2.4}$$

with $(,)$ the inner product in H and $|\cdot|$ the corresponding norm, then, for all $t \in [0, \pi]$, one has

$$|x'(t)|^p \leqq g\left(\int_0^\pi |x'(u)|^p du\right) .$$

PROOF. By (2.4) one has, for every $u, v \in [0, \pi]$,

$$\int_0^\pi |x'(t)|^p dt \geqq \left|\int_u^v \frac{p(x'(t), x''(t))}{h(|x'(t)|^p)} dt\right| = \left|\int_{|x'(u)|^p}^{|x'(v)|^p} \frac{s^{(2/p)-1}}{h(s)} ds\right| . \tag{2.5}$$

The last equality follows from the change of variable defined by the absolutely continuous transformation $s = |x'(t)|^p$ (see e.g., [3]), and the fact that, almost everywhere, $(d/dt)|x'(t)|^p = p|x'(t)|^{p-2}(x'(t), x''(t))$. Now, by the mean value theorem, there exists $u \in [0, \pi]$ such that

$$\int_0^\pi |x'(t)|^p dt = \pi |x'(u)|^p ,$$

and by the continuity of x', there exists $v \in [0, \pi]$ such that $|x'(v)| = \max_{t \in [0,\pi]} |x'(t)|$. With such a choice for u and v, it follows from (2.5) and the definition of g that, by letting

$$w = \int_0^\pi |x'(t)|^p dt$$

$$\int_{w/\pi}^{g(w)} \frac{s^{(2/p)-1}}{h(s)} ds = w \geqq \int_{w/\pi}^{|x'(v)|^p} \frac{s^{(2/p)-1}}{h(s)} ds ,$$

and hence

$$\left(\max_{t \in [0,\pi]} |x'(t)|^p\right) = |x'(v)|^p \leqq g(w) = g\left(\int_0^\pi |x'(t)|^p dt\right) ,$$

which completes the proof.

COROLLARY 1. *The conclusion of Lemma* 1 *holds if* (2.4) *is replaced by*

$$|x''(t)| \leqq p^{-1} h(|x'(t)|^p) |x'(t)|^{p-1} , \qquad t \in [0, \pi] .$$

3. Existence for the case of a completely continuous f and a uniqueness condition. Keeping the terminology and notations of Sections 1 and 2, we shall now prove an existence theorem for (1.1)-(1.2) when f is completely continuous, i.e., continuous and such that it takes bounded subsets into relatively compact subsets.

THEOREM 1. *Assume that the following conditions hold.*

(1) *f is completely continuous on $I \times H \times H$.*

(2) *There exist nonnegative numbers a, b, c with $a + b < 1$, such that*

$$(x, f(t, x, y)) \leqq a|x|^2 + b|x||y| + c|x|$$

for all $(t, x, y) \in I \times H \times H$.

(3) *There exists a continuous function $h: \boldsymbol{R}_+ \to \boldsymbol{R}_+ \setminus \{0\}$ such that*

$h_k = h(\cdot) + 2|k|$ is a 2-Nagumo function and such that

$$|2(y, f(t, x, y))| \leqq h(|y|^2)|y|^2 \tag{3.1}$$

for all $t \in I$, $y \in H$ and $x \in H$ such that $|x| \leqq \pi(1-a-b)^{-1}c$.

Then problem (1.1)-(1.2) *has at least one solution, and all its solutions are such that*

$$\begin{aligned} \left(\int_0^\pi |x'(t)|^2 dt\right)^{1/2} &\leqq \pi^{1/2}(1-a-b)^{-1}c , \\ \max_{t\in[0,\pi]} |x(t)| &\leqq \pi(1-a-b)^{-1}c , \\ \max_{t\in[0,\pi]} |x'(t)| &\leqq [g_k(\pi(1-a-b)^2c^2)]^{1/2} , \end{aligned} \tag{3.2}$$

where g_k is the function associated to h_k by formula (2.2).

PROOF. We shall apply the Leray-Schauder theorem in its simplest form (see e.g., [6]). If G is the scalar Green function associated to the problem

$$-x''(t) - kx'(t) = b(t) , \qquad x(0) = x(\pi) = 0 ,$$

then, for each $\lambda \in]0, 1[$, the boundary value problem

$$x''(t) + kx'(t) + \lambda f(t, x(t), x'(t)) = 0 , \qquad t \in [0, \pi] , \tag{3.3}$$

$$x(0) = x(\pi) = 0 , \tag{3.4}$$

is equivalent to the fixed point problem

$$x(t) = \lambda\int_0^\pi G(t, s)f(s, x(s), x'(s))ds = \lambda T(x)(t) , \tag{3.5}$$

in the space $C^1(I, H)$ of functions from I into H of class C^1, with the norm

$$|x|_1 = \max_{t\in I}|x(t)| + \max_{t\in I}|x'(t)| = |x|_0 + |x'|_0 .$$

Now assumption (1) and a classical argument show that the mapping $T: C^1(I, H) \to C^1(I, H)$ is completely continuous. The result will then follow from Leray-Schauder's theorem if we can show that all possible solutions of (3.5), or equivalently of (3.3)-(3.4) are a priori bounded independently of λ and of the solution. To show this, let us denote by $\langle\, ,\rangle$ the inner product

$$\langle x, y\rangle = \int_0^\pi (x(t), y(t))dt$$

in the Hilbert space $\mathscr{H} = L^2(0, \pi; H)$ and by $\|\cdot\|$ the corresponding norm. If x is a possible solution of (3.3)-(3.4), then, for all $t \in I$,

$(x''(t), x(t)) + k(x'(t), x(t)) + \lambda(f(t, x(t), x'(t)), x(t)) = 0$, and hence, using assumption (2), $(x''(t), x(t)) + k(x'(t), x(t)) + \lambda a|x(t)|^2 + \lambda b|x(t)||x'(t)| + \lambda c|x(t)| \geqq 0$, $t \in I$. Integrating over I we obtain, after integration by parts and use of the boundary conditions and of Schwarz inequality,

$$\|x'\|^2 \leqq a\|x\|^2 + b\|x\|\,\|x'\| + \pi^{1/2}c\|x\| \,. \tag{3.6}$$

Now, using the following inequalities of Poincaré and Sobolev type, whose proof given e.g., in [6] for $H = \boldsymbol{R}^n$ trivially extend to the general case,

$$\|x\| \leqq \|x'\| \,, \qquad |x|_0 \leqq \pi^{1/2}\|x'\| \,,$$

we deduce from (3.6) that

$$\|x'\| \leqq \pi^{1/2}(1 - a - b)^{-1}c \tag{3.7}$$

and

$$|x|_0 \leqq \pi(1 - a - b)^{-1}c \,. \tag{3.8}$$

It follows now from (3.3) that, for all $t \in I$, $(x''(t), x'(t)) + k|x'(t)|^2 + \lambda(f(t, x(t), x'(t)), x'(t)) = 0$, and hence, using (3.7), (3.8) and assumption (3),

$$|x'|_0^2 \leqq g_k(\|x'\|^2) \leqq g_k(\pi(1 - a - b)^2c^2) \,, \tag{3.9}$$

as g_k is increasing. The proof is now complete.

Let us now consider the problem of the uniqueness of the solution.

THEOREM 2. *(1.1)-(1.2) has at most one solution if*

(A) *f is continuous on $I \times H \times H$.*

(B) *There exist nonnegative numbers a, b with $a + b < 1$, such that, for all $t \in I$ and x, y, u, v in H, one has*

$$(x - u, f(t, x, y) - f(t, u, v)) \leqq a|x - u|^2 + b|x - u|\,|y - v| \,. \tag{3.10}$$

PROOF. Define $L\colon \operatorname{dom} L \subset \mathscr{H} \to \mathscr{H}$ and $N\colon \operatorname{dom} N \subset \mathscr{H} \to \mathscr{H}$ by

$$\operatorname{dom} L = \{x \in \mathscr{H}\colon x \text{ and } x' \text{ are absolutely continuous, } x'' \in \mathscr{H} \text{ and } x(0) = x(\pi) = 0\} \,, \quad Lx = -x'' - kx' \,, \tag{3.11}$$

$$\operatorname{dom} N = C^1(I, H) \,, \qquad Nx = -f(\cdot, x(\cdot), x'(\cdot)) \,, \tag{3.12}$$

so that problem (1.1)-(1.2) is equivalent to the equation

$$Lx + Nx = 0$$

in $\operatorname{dom} L$. We shall show that, with respect to the inner product of $L^2(0, \pi; H)$, $L + N$ is strongly monotone (see e.g., [8] for the corresponding definition). Using again the Schwarz and Poincaré inequalities, we obtain, for every x and u in $\operatorname{dom} L$, by (3.10),

$$\begin{aligned}
&\langle (L+N)x-(L+N)u, x-u\rangle \\
&= \int_0^\pi |x'(t)-u'(t)|^2 dt - \int_0^\pi (f(t, x(t), x'(t)) - f(t, u(t), u'(t)), x(t)-u(t))dt \\
&\geqq \int_0^\pi |x'(t)-u'(t)|^2 dt - a\int_0^\pi |x(t)-u(t)|^2 - b\int_0^\pi |x(t)-u(t)||x'(t)-u'(t)|dt \\
&\geqq (1-a-b)\|x'-u'\|^2 \geqq (1-a-b)\|x-u\|^2 .
\end{aligned}$$

Thus, $L+N$ is strongly monotone on $\operatorname{dom} L$, and together with the continuity of any solution, this immediately implies the uniqueness.

COROLLARY 2. *Assume that condition* (B) *of Theorem* 2 *and conditions* (1) *and* (3) *of Theorem* 1 *with* $c = \max_{t\in I}|f(t, 0, 0)|$ *hold. Then problem* (1.1)-(1.2) *has a unique solution.*

PROOF. By (3.10) with $u = v = 0$, we obtain

$$(x, f(t, x, y)) \leqq a|x|^2 + b|x||y| + |x||f(t, 0, 0)| \leqq a|x|^2 + b|x||y| + c|x| ,$$

so that conditions (1) to (3) of Theorem 1 as well as conditions (A) and (B) of Theorem 2 are satisfied.

4. Existence and uniqueness for the case of a continuous f.

We shall now show that the uniqueness condition (B) of Theorem 2 together with a Nagumo-type condition of type (3) of Theorem 1 implies the existence and the uniqueness of a solution for (1.1)-(1.2) under a mere continuity assumption for f.

THEOREM 3. *Assume that conditions* (A) *and* (B) *of Theorem* 2, *condition* (3) *of Theorem* 1 *with* $c = \max_{t\in I}|f(t, 0, 0)|$, *and the following assumption hold.*

(C) *The set*

$$\{f(t, x, y) \in H : t\in I, |x| \leqq \pi(1-a-b)^{-1}c\ ,\ |y| \leqq [g_k(\pi(1-a-b)^{-2}c^2)]^{1/2}\}$$

is bounded in H.

Then, problem (1.1)-(1.2) *has a unique solution.*

PROOF. We denoted for brevity by $\mathscr{H}$ the Hilbert space $L^2(0, \pi; H)$; of course, $\operatorname{dom} L$ and $\operatorname{dom} N$ as defined in (3.11) and (3.12) are vector subspaces of $\mathscr{H}$ and problem (1.1)-(1.2) is equivalent to the equation

$$Lx + Nx = 0$$

in $\operatorname{dom} L \subset \mathscr{H}$. The proof proceeds in three steps.

First step. For each finite-dimensional vector subspace F of H, let us denote by $P_F: H \to H$ the orthogonal projector onto F. Define the corresponding orthogonal projector $\mathscr{P}_{\mathscr{F}}$ on $\mathscr{H}$ by

$$(\mathscr{P}_{\mathscr{F}}u)(t) = P_F(u(t)) , \qquad t \in I ,$$

and let us write $\mathscr{F} = \operatorname{Im} \mathscr{P}_{\mathscr{F}}$. It is immediately checked that, in $\operatorname{dom} L \cap \mathscr{F}$, the equation

$$\mathscr{P}_{\mathscr{F}}(Lx_{\mathscr{F}} + Nx_{\mathscr{F}}) = 0 , \qquad x_{\mathscr{F}} \in \operatorname{dom} L \cap \mathscr{F} ,$$

is equivalent to the boundary value problem

$$x''_{\mathscr{F}}(t) + kx'_{\mathscr{F}}(t) + P_F f(t, x_{\mathscr{F}}(t), x'_{\mathscr{F}}(t)) = 0 , \qquad t \in I , \tag{4.1}$$

$$x_{\mathscr{F}}(0) = x_{\mathscr{F}}(\pi) = 0 . \tag{4.2}$$

By condition (B) and the fact that P_F is an orthogonal projector, one has, for all $x, y, u, v \in F$ and all $t \in I$,

$$\begin{aligned}(x - u, P_F f(t, x, y) - P_F f(t, u, v)) &= (x - u, f(t, x, y) - f(t, u, v)) \\ &\leqq a|x - u|^2 + b|x - u||y - v| ,\end{aligned}$$

and condition (3) of Theorem 1 is satisfied for $P_F f$. Finally, $P_F f$ continuous on $I \times F \times F$ will take bounded subsets into bounded subsets, hence relatively compact subsets, of the finite-dimensional space F. Thus, all the conditions of Corollary 2 are satisfied and, for each finite-dimensional vector subspace F of H, there will exist a (unique) solution $x_{\mathscr{F}} \in \operatorname{dom} L \cap \mathscr{F}$ of (4.1)-(4.2) verifying (3.2) and therefore, by (4.1) and assumption (C), such, that P_F having norm one,

$$|x''_{\mathscr{F}}|_0 = \max_{t \in I} |x''_{\mathscr{F}}(t)| \leqq K \tag{4.3}$$

where K depends only upon a, b, c, k and h.

Second step. Let us denote by Λ the collection of all the vector subspaces of $\mathscr{H}$ formed by the set of functions in $L^2(0, \pi; H)$ whose range is contained in a given finite-dimensional vector subspace of H. For every $\mathscr{F}_0 \in \Lambda$, let us write

$$V_{\mathscr{F}_0} = \{x_{\mathscr{F}} : x_{\mathscr{F}} \text{ is the unique solution of (4.1)-(4.2) obtained in the first step, } \mathscr{F} \in \Lambda \text{ and } \mathscr{F} \supset \mathscr{F}_0\} ,$$

and let us denote by $W_{\mathscr{F}_0}$ the weak closure of $V_{\mathscr{F}_0}$ in $\mathscr{H}$. Since $V_{\mathscr{F}_0}$ is bounded in $\mathscr{H}$, $W_{\mathscr{F}_0}$ is weakly compact, and it is immediate to check that the family $\{W_{\mathscr{F}_0} : \mathscr{F}_0 \in \Lambda\}$ has the finite intersection property. Therefore, there exists $x_0 \in \bigcap_{\mathscr{F}_0 \in \Lambda} W_{\mathscr{F}_0}$. Let $\mathscr{F}_0$ by any element of Λ; since $x_0 \in W_{\mathscr{F}_0}$, it follows from a lemma of Kaplansky (see e.g., [1], p. 81) that one can find a sequence $(\mathscr{F}_n)$ in Λ such that $\mathscr{F}_n \supset \mathscr{F}_0$ for every $n \in N^*$ and such that $x_{\mathscr{F}_n} \rightharpoonup x_0$, where $\rightharpoonup$ denotes the weak convergence in $\mathscr{H}$. By (4.3) and (3.2) which imply that the sequence $(x''_{\mathscr{F}_n} + kx'_{\mathscr{F}_n})$ is bounded in $\mathscr{H}$, we can assume, going if necessary to a subsequence, that $-x''_{\mathscr{F}_n} - kx'_{\mathscr{F}_n} \rightharpoonup v$, if $n \to \infty$, for some $v \in \mathscr{H}$. But, the graph of L is convex and closed, and hence weakly closed, so that $x_0 \in \operatorname{dom} L$ and $v = Lx_0$.

Third step. Let $\mathscr{F}_0 \in \Lambda$ and let $u \in \mathscr{F}_0 \cap \operatorname{dom} L$. As $L + N$ is monotone on $\operatorname{dom} L$, we have, for all $\mathscr{F} \in \Lambda$, $0 \leqq \langle (L + N)u - (L + N)x_{\mathscr{F}}, u - x_{\mathscr{F}} \rangle$, and therefore, if $\mathscr{F} \supset \mathscr{F}_0$,

$$\begin{aligned} 0 &\leqq \langle (L + N)u - (L + N)x_{\mathscr{F}}, \mathscr{P}_{\mathscr{F}}(u - x_{\mathscr{F}}) \rangle \\ &= \langle \mathscr{P}_{\mathscr{F}}(L + N)u - \mathscr{P}_{\mathscr{F}}(L + N)x_{\mathscr{F}}, u - x_{\mathscr{F}} \rangle = \langle (L + N)u, u - x_{\mathscr{F}} \rangle \,. \end{aligned}$$

Consequently, one has

$$0 \leqq \langle (L + N)u, u - x_0 \rangle \,, \tag{4.4}$$

for every $u \in \operatorname{dom} L \cap \mathscr{F}_0$ and every $\mathscr{F}_0 \in \Lambda$. Let us show now that (4.4) holds for every $u \in \operatorname{dom} L$. If $u \in \operatorname{dom} L$, it has the Fourier series

$$u(t) = \sum_{m=1}^{\infty} a_m \sin mt \,, \qquad t \in I$$

which converges uniformly on I to u, the series

$$\sum_{m=1}^{\infty} m a_m \cos mt$$

converging uniformly on I to u'. Thus, if, for each $n \in N^*$, we write

$$u_n(t) = \sum_{m=1}^{n} a_m \sin mt \,, \qquad u'_n(t) = \sum_{m=1}^{n} m a_m \cos mt \,,$$

u_n and u'_n belong, for each n, to some $\mathscr{F}_{0,n} \cap \operatorname{dom} L$, so that

$$0 \leqq \langle (L + N)u_n, u_n - x_0 \rangle \,, \tag{4.5}$$

and, by the continuity of f, Nu_n converges uniformly on I to Nu. On the other hand, Lu_n converges strongly in $\mathscr{H}$ to Lu, and it follows then from (4.5) that

$$0 \leqq \langle (L + N)u, u - x_0 \rangle \tag{4.6}$$

for every $u \in \operatorname{dom} L$. We now use Minty's trick (see e.g., [8]) be taking $u = x_0 + \tau v$, with $\tau > 0$ and $v \in \operatorname{dom} L$ in (4.6). This gives $0 \leqq \langle (L + N)(x_0 + \tau v), v \rangle$, and hence, if $\tau \to 0+$, $0 \leqq \langle (L + N)x_0, v \rangle$, which implies $(L + N)x_0 = 0$, as $\operatorname{dom} L$ is dense in $\mathscr{H}$, and completes the proof.

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