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ASYMPTOTIC EXPANSIONS IN SCALAR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

MITSUNOBU KURIHARA

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0. Introduction. We consider a scalar linear functional differential equation

$$\dot{x}(t) = F(t, x_t) . \tag{0.1}$$

Hereafter the following notations are used: ω is a nonnegative number. C denotes the space of all complex valued functions continuous on the interval $[-\omega, 0]$ with the norm $||\phi|| = \sup\{|\phi(\theta)|; -\omega \leqq \theta \leqq 0\}$ for any ϕ in C. If $x = x(t)$ is a complex valued function continuous in t on the interval $[\sigma - \omega, \sigma + \gamma]$ for some $\gamma \geqq 0$, the symbol x_t denotes the element in C with $x_t(\theta) = x(t + \theta)$ for $-\omega \leqq \theta \leqq 0$ and $\sigma \leqq t \leqq \sigma + \gamma$. Moreover, the following hypotheses are imposed on the equation (0.1). $F(t, \phi)$ is a complex valued functional which is continuous in $t \geqq 0$ and ϕ in C, linear in ϕ and has the asymptotic expansion of the form

$$F(t, \phi) \sim \sum_{n=0}^{\infty} L_n(\phi) t^{-n} \qquad \text{as} \quad t \to \infty , \tag{0.2}$$

where L_n $(n = 0, 1, \cdots)$ are complex valued bounded linear functionals on the space C of the form

$$L_n(\phi) = \int_{-\omega}^{0} \phi(\theta) d\eta_n(\theta) \qquad (n = 0, 1, \cdots) \tag{0.3}$$

for any ϕ in C and some functions $\eta_n(\theta)$ $(n = 0, 1, \cdots)$ of bounded variation on the interval $[-\omega, 0]$. The asymptotic expansion (0.2) means that for any nonnegative integer N there exist constants $\gamma_N \geqq 0$ and $\sigma_N \geqq 0$ satisfying the relation

$$\left| F(t, \phi) - \sum_{n=0}^{N} L_n(\phi) t^{-n} \right| \leqq \gamma_N t^{-(N+1)} ||\phi|| \text{ for any } t \geqq \sigma_N \text{ and any } \phi \text{ in } C .$$

The linear functional differential equation

$$\dot{u}(t) = L_0(u_t) \tag{0.4}$$

is called the homogeneous equation corresponding to (0.1). The equation in the variable λ

$$\Delta(\lambda) = \lambda - \int_{-\omega}^{0} e^{\lambda\theta} d\eta_0(\theta) = 0 \tag{0.5}$$

is called the characteristic equation of (0.4). The roots λ of (0.5) are called the characteristic values of (0.4).

In the present paper we prove the following theorems:

THEOREM 1. *If λ is a simple characteristic value of the equation* (0.4), *then the equation* (0.1) *has a formal solution $x = x(t)$ of the type*

$$e^{\lambda t} t^r \sum_{m=0}^{\infty} c_m t^{-m} , \tag{0.6}$$

where the coefficient c_0 may be chosen arbitrarily.

THEOREM 2. *Let λ be a simple characteristic value of the equation* (0.4). *Suppose that any other characteristic value with its real part equal to* $\operatorname{Re}\lambda$ *is simple and that the equation* (0.1) *has a formal solution of the type* (0.6). *Then there exists a constant $\sigma \geqq 0$ such that the equation* (0.1) *has a solution $x = x(t)$ for t on the interval $[\sigma - \omega, \infty)$ with the asymptotic expansion*

$$x(t) \sim e^{\lambda t} t^r \sum_{m=0}^{\infty} c_m t^{-m} \quad \text{as} \quad t \to \infty . \tag{0.7}$$

For a linear differential difference equation

$$\dot{x}(t) = a(t)x(t) + b(t)x(t - \omega) , \tag{0.8}$$

which is a special case of the equation (0.1), assume that the coefficients $a(t)$ and $b(t)$ have the asymptotic expansions

$$a(t) \sim \sum_{n=0}^{\infty} a_n t^{-n} \quad \text{and} \quad b(t) \sim \sum_{n=0}^{\infty} b_n t^{-n} \quad \text{as} \quad t \to \infty .$$

The characteristic equation of $\dot{x}(t) = a_0 x(t) + b_0 x(t - \omega)$ is

$$\Delta(\lambda) = \lambda - (a_0 + b_0 e^{-\lambda\omega}) = 0 \tag{0.9}$$

and the roots of (0.9) are the characteristic values. Bellman [1] as well as Bellman and Cooke [2] [3] studied the equation (0.8) and proved the existence of a formal solution of (0.8) of the type (0.6) for any simple characteristic value λ and for the constant $r = (a_1 + b_1 e^{-\lambda\omega})/(1 + b_1 e^{-\lambda\omega})$. Moreover, they proved the existence of an exact solution of (0.8) with the asymptotic expansion of the form (0.7) for any simple characteristic value λ under some other conditions. Our main theorems are generalizations of these results to the case of linear functional differential equations.

For a system of linear ordinary differential equations whose coefficients have the asymptotic expansions. Hukuhara [6] proved the existence of a solution with its asymptotic expansion equal to the formal solution. The method in our proof of Theorem 2 is based on that by Hukuhara [6].

In Section 1 we give a proof of Theorem 1 by the formal power series expansion of a solution. In order to prove Theorem 2, we state, in Section 2, some facts due to Hale [4] [5] concerning linear functional differential equations. We then convert the problem of solving our equation (0.1) to that of solving an integral equation in Section 3. In Section 4 we prove an existence theorem and a uniqueness theorem for the integral equation derived in the previous section. In Section 5 we complete the proof of Theorem 2.

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1. Proof of Theorem 1. Let λ be a simple characteristic value of (0.4). Thus we have (0.5) as well as

$$\Delta'(\lambda) = 1 - \int_{-\omega}^{0} \theta e^{\lambda\theta} d\eta_0(\theta) \neq 0 . \tag{1.1}$$

Substituting the series (0.6) into the equation (0.1) with the expansion (0.2), we obtain

$$e^{\lambda t}t^r\left\{\lambda c_0 + \sum_{m=0}^{\infty}[\lambda c_m + (r - m + 1)c_{m-1}]t^{-m}\right\}$$
$$= e^{\lambda t}t^r \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{r-m}{k}\left[\int_{-\omega}^{0} e^{\lambda\theta}\theta^k d\eta_n(\theta)\right]c_m t^{-(m+n+k)} ,$$

where

$$\binom{r-m}{k} = (r-m)(r-m-1)\cdots(r-m-k+1)/k! .$$

Comparing the coefficients of $e^{\lambda t}t^r$ and $e^{\lambda t}t^{r-1}$, respectively, we have $\Delta(\lambda)c_0 = 0$ and $\Delta(\lambda)c_1 + [\Delta'(\lambda)r - \Delta_1(\lambda)]c_0 = 0$, where

$$\Delta_1(\lambda) = \int_{-\omega}^{0} e^{\lambda\theta} d\eta_1(\theta) .$$

Then we choose c_0 arbitrarily and let $r = \Delta_1(\lambda)/\Delta'(\lambda)$, which is justified by (1.1). Furthermore, comparing the coefficient of $e^{\lambda t}t^{r-m}$, we have

$$\Delta(\lambda)c_m + \{[\Delta'(\lambda)r - \Delta_1(\lambda)] - (m-1)\Delta'(\lambda)\}c_{m-1} + H(c_0, \cdots, c_{m-2}) = 0 \tag{1.2}$$

for $m \geqq 2$, where $H(c_0, \cdots, c_{m-2})$ denotes the sum of the terms containing the coefficients $c_0, \cdots, c_{m-2}$ alone. It follows that the coefficients c_m,

$m \geqq 1$ can be determined recursively starting from an arbitrary c_0. Thus we are done.

2. Linear functional differential equations. We state some facts, due to Hale [4] [5], on linear functional differential equations, which we need for the proof of Theorem 2. If $\sigma \geqq 0$ is a given real number and ϕ is a given function defined on the interval $[0-\omega, 0]$, a solution of the equation (0.1) with initial value ϕ at σ is defined to be any continuous extension of $\phi(\theta-\sigma)$ on $[\sigma-\omega, \sigma]$ to the right of σ which satisfies the equation (0.1). It is well known that for any given ϕ in C there exists a unique solution with initial value ϕ at σ defined for $t \geqq \sigma$ and the solution is continuous and linear in ϕ under the hypotheses stated in Section 0. If $u(\phi)$ is the solution of the equation (0.4) with initial value ϕ at zero, we define the family of linear operators $U(t)$, $t \geqq 0$ by $U(t)\phi = u_t(\phi)$. Let X_0 be the function on $[-\omega, 0]$ defined by $X_0(\theta)=0$ for $-\omega \leqq \theta < 0$ and $X_0(0)=1$. Then the solution $x=x(t)$ of the equation (0.1) with initial value ϕ at σ has the integral representation

$$x_t(\theta) = U(t-\sigma)\phi(\theta) + \int_\sigma^t U(t-\tau)X_0(\theta)F(\tau, x_\tau)d\tau$$

for $-\omega \leqq \theta \leqq 0$ or, in a more compact form,

$$x_t = U(t-\sigma)\phi + \int_\sigma^t U(t-\tau)X_0F(\tau, x_\tau)d\tau \ . \tag{2.1}$$

For any characteristic value λ with multiplicity $m(\lambda)$, there are exactly $m(\lambda)$ linearly independent solution of the equation (0.4) of the form $p_j(\lambda, t)e^{\lambda t}$ for $j=1, \cdots, m(\lambda)$ and $-\infty < t < \infty$, where $p_j(\lambda, t)$ are polynomials in t. We define the functions $\phi_j(\lambda)$ in C by the relation $\phi_j(\lambda)(\theta) = p_j(\lambda, \theta)e^{\lambda\theta}$ for $j=1, \cdots, m(\lambda)$ and $-\omega \leqq \theta \leqq 0$. Let $\Phi_\lambda = (\phi_1(\lambda), \cdots, \phi_{m(\lambda)}(\lambda))$. Then there exists a square matrix B_λ of order $m(\lambda)$ whose characteristic values are λ alone such that

$$\Phi_\lambda(\theta) = \Phi_\lambda(0)\exp[B_\lambda\theta] \qquad \text{for} \quad -\omega \leqq \theta \leqq 0 \ . \tag{2.2}$$

Furthermore, if $\phi = \Phi_\lambda a$ for some constant vector a and if u is a solution of the equation (0.4) with initial value ϕ at zero, then $u_t = \Phi_\lambda \exp[B_\lambda t]a$.

The equation adjoint to (0.4) is defined to be

$$\dot{v}(\tau) = -\int_{-\omega}^0 v(\tau-\theta)d\eta_0(\theta) \ . \tag{2.3}$$

C^* denotes the space of complex valued continuous functions defined on the interval $[0, \omega]$. For any ψ in C^* and ϕ in C we define

$$(2.4)\qquad (\psi, \phi) = \psi(0)\phi(0) - \int_{-\omega}^{0}\int_{0}^{\theta}\psi(\xi - \theta)\phi(\xi)d\xi d\eta_0(\theta)\ .$$

The characteristic equation for the adjoint equation (2.3) is also defined by (0.5). For any characteristic value λ with multiplicity $m(\lambda)$, there exist also exactly $m(\lambda)$ linearly independent solutions of the equation (2.3) of the form $q_j(\lambda, \tau)e^{-\lambda\tau}$ for $j = 1, \cdots, m(\lambda)$ and $-\infty<\tau<\infty$. We define functions $\psi_j(\lambda)$ in C^* by $\psi_j(\lambda)(\theta)=q_j(\lambda, \theta)e^{-\lambda\theta}$ for $j=1, \cdots, m(\lambda)$ and $0 \leqq \theta \leqq \omega$. If $\Psi_\lambda = \mathrm{col}(\psi_1(\lambda), \cdots, \psi_{m(\lambda)}(\lambda))$, then the matrix $(\Psi_\lambda, \Phi_\lambda) = ((\psi_j(\lambda), \phi_k(\lambda))\ ; j, k = 1, \cdots, m(\lambda))$ is nonsingular and hence, without any loss of generality, can be assumed to be the identity.

Suppose $\Lambda = \{\lambda_1, \cdots, \lambda_k\}$ is a finite set of characteristic values of (0.4). Let $\{\Phi_{\lambda_1}, \cdots, \Phi_{\lambda_k}\}$ and $\{\Psi_{\lambda_1}, \cdots, \Psi_{\lambda_k}\}$ be the corresponding sets of functions in C and those in C^*, respectively, defined above. If we let $\Phi_\Lambda = (\Phi_{\lambda_1}, \cdots, \Phi_{\lambda_k})$ and $\Psi_\Lambda = \mathrm{col}(\Psi_{\lambda_1}, \cdots, \Psi_{\lambda_k})$, then the matrix $(\Psi_\Lambda, \Phi_\Lambda)$ is nonsingular and may be assumed to be the identity. Thus the matrix $B = \mathrm{diag}(B_{\lambda_1}, \cdots, B_{\lambda_k})$, where $B_{\lambda_1}, \cdots, B_{\lambda_k}$ are as defined in (2.2), is such that $\Phi_\Lambda(\theta) = \Phi_\Lambda(0)\exp[B\theta]$ for $-\omega\leqq\theta\leqq 0$. If $\phi = \Phi_\Lambda a$ for some constant vector a and if $u(\phi)$ is the solution of the equation (0.4) with the initial value ϕ at zero, then we have $u_t(\phi) = \Phi_\Lambda \exp[Bt]a$ for $-\infty < t < \infty$.

The above facts allow us to conclude that any ϕ in C has a unique decomposition of the form $\phi = \phi^P + \phi^Q$ with ϕ^P in P and with ϕ^Q in Q, where $P = P(\Lambda) = \{\phi$ in $C; \phi = \Phi_\Lambda b$ for a constant vector $b\}$ and $Q = Q(\Lambda) = \{\phi$ in C; $(\Psi_\Lambda, \phi) = 0\}$. In fact, $\phi^P = \Phi_\Lambda(\Psi_\Lambda, \phi)$. If we make this decomposition on the integral equation (2.1), we have the equivalent equation

$$(2.5)\qquad x_t = U(t-\sigma)\phi^P + \int_\sigma^t U(t-\tau)X_0^P F(\tau, x_\tau)d\tau + U(t-\sigma)\phi^Q + \int_\sigma^t U(t-\tau)X_0^Q F(\tau, x_\tau)d\tau\ ,$$

where $X_0^P = \Phi_\Lambda(\Psi_\Lambda, X_0) = \Phi_\Lambda\Psi_\Lambda(0)$ and $X_0^Q = X_0 - X_0^P$.

3. Conversion to integral equations. It is well known that for any formal power series of the form $\sum_{m=0}^\infty c_m t^{-m}$, there exists an analytic function $q(t)$ with the asymptotic expansion $q(t) \sim \sum_{m=0}^\infty c_m t^{-m}$ as $t \to \infty$. A proof of the fact is given, for example, in Wasow [7].

Suppose there exists a formal solution of the type (0.6) of the equation (0.1). Then we have an analytic function $h(t)$ in t on an interval $[\sigma_0, \infty)$ for some $\sigma_0 > \omega$, which has the asymptotic expansion

$$(3.1)\qquad h(t) \sim e^{\lambda t}t^r \sum_{m=0}^{\infty} c_m t^{-m} \qquad \text{as} \quad t \to \infty\ .$$

Changing the variable in the equation (0.1) by

$$x(t) = y(t) + h(t)\ , \tag{3.2}$$

we obtain

$$\dot{y}(t) = L_0(y_t) + G(t, y_t) + g(t)\ , \tag{3.3}$$

where

$$G(t, \phi) = F(t, \phi) - L_0(\phi) \quad \text{and} \quad g(t) = -\dot{h}(t) + F(t, h_t) \tag{3.4}$$

for any $t \geqq 0$ and ϕ in C.

Let us convert the problem of solving the equation (3.3) to that of solving an integral equation by making use of the facts stated in Section 2. Choose any number $\sigma \geqq \sigma_0$. If we let $y(t) = 0$ for $t \leqq \sigma$, we have by (2.1) the integral representation of a solution of the equation (3.3)

$$y_t = \int_\sigma^t U(t-\tau)X_0[G(\tau, y_\tau) + g(\tau)]d\tau\ . \tag{3.5}$$

Let λ be a simple characteristic value of (0.4) and let $\mathrm{Re}\,\lambda = \mu$. Put $\Lambda = \{\nu;\ \Delta(\nu) = 0,\ \mathrm{Re}\,\nu \geqq \mu\}$, which is known to be finite, and denote by $P = P(\Lambda)$ and $Q = Q(\Lambda)$ the spaces in C corresponding to Λ. Therefore we obtain the unique decomposition of C by the subspaces P and Q. Hence we have

$$\begin{aligned} y_t = &\int_\sigma^t U(t-\tau)X_0^P[G(\tau, y_\tau) + g(\tau)]d\tau \\ &+ \int_\sigma^t U(t-\tau)X_0^Q[G(\tau, y_\tau) + g(\tau)]d\tau\ . \end{aligned} \tag{3.6}$$

Suppose that any other characteristic value with its real part equal to μ is simple. It can be shown that there exist constants $K \geqq 0$ and $\varepsilon > 0$ such that

$$\| U(t)X_0^P \| \leqq Ke^{\mu t} \qquad \text{for} \quad t \leqq 0 \tag{3.7}$$

and

$$\| U(t)X_0^Q \| \leqq \mathrm{K}e^{(\mu-\varepsilon)t} \qquad \text{for} \quad t \geqq 0\ . \tag{3.8}$$

If the integral

$$-\int_\sigma^\infty U(t-\tau)X_0^P[G(\tau, y_\tau) + g(\tau)]d\tau \tag{3.9}$$

is convergent, it is a solution of the equation (0.4). Adding the integral (3.9) and a continuous function $f_t(\theta) = f(t+\theta)$ for $t \geqq \sigma$ and $-\omega \leqq \theta \leqq 0$ to the right-hand side of the equation (3.6), we have the integral equation

$$(3.10)\qquad y_t = f_t - \int_t^\infty U(t-\tau)X_0^P[G(\tau, y_\tau) + g(\tau)]d\tau + \int_\sigma^t U(t-\tau)X_0^Q[G(\tau, y_\tau) + g(\tau)]d\tau \,.$$

A solution $y = y(t)$ of the integral equation (3.10) is also a solution of the functional differential equation (3.3) if $f_t = 0$ for $t \geqq \sigma$ and if the integral (3.9) is convergent. Hence the function $x = x(t)$ in (3.2) is a solution of our fuctional differential equation (0.1).

4. **Existence and uniqueness theorem.** It follows from the hypothesis (0.2) and the relation (3.4) that $G(t, \phi) \sim \sum_{n=1}^{\infty} L_n(\phi)t^{-n}$ as $t \to \infty$ for any ϕ in C. Then there exist constants $\sigma_1 \geqq \sigma_0 > \omega$ and $A \geqq 0$ such that

$$(4.1)\qquad |G(t, \phi)| \leqq At^{-1}\|\phi\| \quad \text{for} \quad t \geqq \sigma_1 \text{ and } \phi \text{ in } C\,.$$

Moreover, for any nonnegative integer N there exist constants B_N and σ_N satisfying $|g(t)| \leqq B_N e^{\mu t}t^{\rho - N}$ for $t \geqq \sigma_N$. Here $g(t)$ is the function defined in (3.4) and

$$(4.2)\qquad \operatorname{Re}\lambda = \mu \quad \text{and} \quad \operatorname{Re} r = \rho\,.$$

Here is a theorem concerning the existence of solutions of the integral equation (3.10).

THEOREM 3. *Suppose that there exist constants* $N > \rho + 1$, $\sigma \geqq \sigma_1$ *and* $\alpha \geqq 0$ *satisfying the relations*

$$(4.3)\qquad 2AK/(N - \rho - 1) < 1/2\,,$$

$$(4.4)\qquad \varepsilon\sigma > N - \rho - 1\,,$$

$$(4.5)\qquad 2AK/(\varepsilon\sigma - N + \rho + 1) < 1/2\,,$$

$$(4.6)\qquad |g(t)| \leqq B_N e^{\mu t}t^{\rho+1-N} \quad \text{for} \quad t \geqq \sigma\,,$$

$$(4.7)\qquad \|f_t\| \leqq \alpha e^{\mu t}t^{\rho+1-N} \quad \text{for} \quad t \geqq \sigma$$

and

$$(4.8)\qquad (2A\alpha + B_N)K[1/(N-\rho-1) + 1/(\varepsilon\sigma - N + \rho + 1)] \leqq \alpha\,.$$

Then the equation (3.10) *has a solution* $y = y(t)$ *continuous in* t *on the interval* $[\sigma - \omega, \infty)$ *satisfying the relation*

$$(4.9)\qquad \|y_t - f_t\| \leqq \alpha e^{\mu t}t^{\rho+1-N} \quad \text{for} \quad t \geqq \sigma\,.$$

PROOF. Denote by S the class of continuous functions $y = y(t)$ in t on the interval $[\sigma - \omega, \infty)$ which satisfy the relation (4.9). On S we define an operator T by $w(t) = (Ty)(t)$ for $t \geqq \sigma - \omega$, where

$$w_t = f_t - \int_t^\infty U(t-\tau)X_0^P[G(\tau, y_\tau) + g(\tau)]d\tau + \int_\sigma^t U(t-\tau)X_0^Q[G(\tau, y_\tau) + g(\tau)]d\tau . \tag{4.10}$$

$w = Ty$ is well-defined for any y in S and is continuous on the interval $[\sigma - \omega, \infty)$. For any member $y = y(t)$ in S we obtain

$$\|y_t\| \leqq 2\alpha e^{\mu t} t^{\rho+1-N} \qquad \text{for} \quad t \geqq \sigma \tag{4.11}$$

by (4.7) and (4.9). Thus by (4.10) we have

$$\|w_t - f_t\| \leqq (2A\alpha + B_N)K\int_t^\infty \tau^{\rho-N}d\tau + (2A\alpha + B_N)Ke^{(\mu-\varepsilon)t}\int_\sigma^t e^{\varepsilon\tau}\tau^{\rho-N}d\tau$$

using (3.7), (3.8), (4.1) and (4.6). On the other hand, we have the inequality $e^{\varepsilon t}t^{\rho-N} \leqq (d/dt)(e^{\varepsilon t}t^{\rho+1-N})/(\varepsilon\sigma - N + \rho + 1)$ for $t \geqq \sigma$ by (4.4). Then we obtain

$$\|w_t - f_t\| \leqq (2A\alpha + B_N)K[1/(N-\rho-1) + 1/(\varepsilon\sigma - N + \rho + 1)]e^{\mu t}t^{\rho+1-N} \qquad \text{for} \quad t \geqq \sigma . \tag{4.12}$$

Thus from (4.8) it follows that T is a mapping from S to S.

Moreover, we see that the mapping $T: S \to S$ is continuous with respect to the topology of uniform convergence on any compact subinterval of the interval $[\sigma - \omega, \infty)$ and that the class S is closed with respect to the same topology. It can be also proved that the family $T(S)$ is uniformly bounded and equicontinuous on any compact subinterval of the interval $[\sigma - \omega, \infty)$. It is clear that the class S is convex. Therefore we conclude that there exists a member $y = y(t)$ in S which is invariant under our mapping T by applying the following lemma proved by Hukuhara [6]. The function $y = y(t)$ is the desired solution of the integral equation (3.10). This proves Theorem 3.

LEMMA. *Let S be a convex family of continuous functions in t on an interval I. Suppose that a transformation T from S to S is continuous with respect to the topology of uniform convergence on any compact subinterval of I and that S is closed with respect to the same topology. Moreover, suppose that the family $T(S)$ is uniformly bounded and equi-continuous on any compact subinterval of I. Then there exists at least one function which is invariant under the transformation T, that is, a function $x(t)$ in S such that $T\{x(t)\} = x(t)$.*

We have the following uniqueness theorem.

THEOREM 4. *Suppose that there exists a solution $y = y(t)$ of the* *equation* (3.10), *continuous in* t *on the interval* $[\sigma-\omega, \infty)$, *which satisfies the relation*

$$\|y_t\| \leqq \beta e^{\mu t} t^{\rho+1-N} \quad \textit{for} \quad t \geqq \sigma \tag{4.13}$$

and for some constant $\beta \geqq 0$, *where* $N > \rho+1$ *and* $\sigma \geqq \max\{\sigma_1, 1\}$ *satisfy*

$$\varepsilon\sigma - N + \rho + 1 > 0 \tag{4.14}$$

and

$$AK[1/(N-\rho-1) + 1/(\varepsilon\sigma - N + \rho + 1)] \leqq 1 . \tag{4.15}$$

Then the solution $y = y(t)$ *is unique.*

PROOF. Let $y = y(t)$ and $y' = y'(t)$ be continuous solutions in t, on the interval $[\sigma-\omega, \infty)$, of the equation (3.10) which satisfy, respectively, (4.13) and

$$\|y'_t\| \leqq \beta' e^{\mu t} t^{\rho+1-N'} \quad \text{for} \quad t \geqq \sigma \tag{4.16}$$

and for some constants $\beta \geqq 0$ and $\beta' \geqq 0$, where $N > \rho + 1$, $N' > \rho + 1$ and $\sigma \geqq \max\{\sigma_1, 1\}$ satisfy (4.14), (4.15),

$$\varepsilon\sigma - N' + \rho + 1 > 0 \tag{4.17}$$

and

$$AK[1/(N'-\rho-1) + 1/(\varepsilon\sigma - N' + \rho + 1)] < 1 . \tag{4.18}$$

The function $z = y - y'$ is a solution of the integral equation

$$z_t = -\int_t^\infty U(t-\tau)X_0^P G(\tau, z_\tau)d\tau + \int_\sigma^t U(t-\tau)X_0^Q G(\tau, z_\tau)d\tau , \tag{4.19}$$

since the functional $G(t, \phi)$ is linear in ϕ. On the other hand, the solution $z = z(t)$ satisfies

$$\|z_t\| \leqq \|y_t\| + \|y'_t\| \leqq \beta'' e^{\mu t} t^{\rho+1-N''} \quad \text{for} \quad t \geqq \sigma , \tag{4.20}$$

where $\beta'' = \beta + \beta'$ and $N'' = \min\{N, N'\}$ by (4.13) and (4.16). Using the relations (3.7), (3.8), (4.1), (4.14), (4.15), (4.17) and (4.18) for the equation (4.19), we have

$$\|z_t\| \leqq AK[1/(N''-\rho-1) + 1/(\varepsilon\sigma - N'' + \rho + 1)]\beta'' e^{\mu t} t^{\rho+1-N''} \quad \text{for} \quad t \geqq \sigma \tag{4.21}$$

by the same argument as in the proof of Theorem 3. Repeating the same argument, we have, for any positive integer m,

$$\|z_t\| \leqq \{AK[1/(N''-\rho-1) + 1/(\varepsilon\sigma - N'' + \rho + 1)]\}^m \beta'' e^{\mu t} t^{\rho+1-N''} \quad \text{for} \quad t \geqq \sigma .$$

This implies that $z(t)=0$ for $t \geqq \sigma-\omega$ by (4.15) or (4.18). This proves Theorem 4.

5. **Proof of Theorem 2.** Now we are in a position to prove Theorem 2. Under the hypotheses stated in Section 0 for the equation (0.1) and the assumptions in Theorem 2 for a characteristic value λ and a formal solution of the type (0.6) of the equation (0.1), we consider the integral equation (3.10) with $f_t=0$ for $t \geqq \sigma$, where $G(t, \phi)$ and $g(t)$ are as defined in (3.4) and (3.1). Note the relations (3.7), (3.8), (4.1) and (4.2). First we choose a nonnegative integer $N>\rho+1$ satisfying the relation (4.3), and next choose a constant $\sigma \geqq \max\{\sigma_1, 1\}$ satisfying the relations (4.4), (4.5) and (4.6). Finally we choose a constant $\alpha \geqq 0$ satisfying the relation (4.8). The assumption (4.7) is automatically satisfied. Then it follows from Theorem 3 that there exists a solution $y=y(t)$, continuous in t on the interval $[\sigma-\omega, \infty)$, of the equation (3.10) with $f_t=0$ for $t \geqq \sigma$ satisfying the relation (4.9). Thus we have

$$\|y_t\| \leqq \alpha e^{\mu t} t^{\rho+1-N} \quad \text{for} \quad t \geqq \sigma . \tag{5.1}$$

Since the integral (3.9) is clearly convergent for the solution $y=y(t)$, it is also a solution of the functional differential equation (3.3), for which the function $x=x(t)$ defined in (3.2) is a solution of our equation (0.1) on the interval $[\sigma-\omega, \infty)$.

To investigate the properties of the solution $x=x(t)$, we choose any nonnegative integer $N'>\rho+1$ satisfying $2AK/(N'-\rho-1)<1/2$. There exist constants $\sigma' \geqq \sigma$ and $B_{N'} \geqq 0$ satisfying the relations $\varepsilon\sigma'-N'+\rho+1>0$, $2AK/(\varepsilon\sigma'-N'+\rho+1)<1/2$,

$$|g(t)| \leqq B_{N'} e^{\mu t} t^{\rho-N'} \quad \text{for} \quad t \geqq \sigma' \tag{5.2}$$

and

$$e^{-\varepsilon t} \leqq t^{\rho+1-N'} \quad \text{for} \quad t \geqq \sigma' . \tag{5.3}$$

We consider another integral equation of the form

$$\begin{aligned} z_t = f_t &- \int_t^\infty U(t-\tau) X_0^P [G(\tau, z_\tau)+g(\tau)] d\tau \\ &+ \int_{\sigma'}^t U(t-\tau) X_0^Q [G(\tau, z_\tau)+g(\tau)] d\tau , \end{aligned} \tag{5.4}$$

where

$$f_t = \int_\sigma^{\sigma'} U(t-\tau) X_0^Q [G(\tau, y_\tau)+g(\tau)] d\tau \quad \text{for} \quad t \geqq \sigma' . \tag{5.5}$$

For the function (5.5) we have $\|f_t\| \leqq \beta e^{(\mu-\varepsilon)t} \leqq \beta e^{\mu t} t^{\rho+1-N'}$ for $t \geqq \sigma'$ by

(3.8), (4.1), (4.6), (5.1) and (5.3). It is clear that $y = y(t)$ is a solution, continuous in t on the interval $[\sigma - \omega, \infty)$, of the equation (5.4) and satisfies (5.1) for $t \geqq \sigma'$. On the other hand, since we can choose a constant $\alpha' \geqq \beta$ so that $(2A\alpha' + B_{N'})K[1/(N' - \rho - 1) + 1/(\varepsilon\sigma' - N' + \rho + 1)] \leqq \alpha'$, it follows that the conditions (4.3)-(4.5) and (4.8) of Theorem 3 are fulfilled for the constants N', σ' and α'. Then there exists a solution $z = z(t)$, continuous in t on the interval $[\sigma' - \omega, \infty)$, satisfying $\|z_t - f_t\| \leqq \alpha' e^{\mu t} t^{\rho+1-N'}$ for $t \geqq \sigma'$, which implies the relation $\|z_t\| \leqq \beta' e^{\mu t} t^{\rho+1-N'}$ for $t \geqq \sigma'$ and for some $\beta' \geqq 0$ by Theorem 3. Moreover, we have $y(t) = z(t)$ for $t \geqq \sigma' - \omega$ by Theorem 4.

Hence the solution $y = y(t)$ of the equation (3.10) with $f_t = 0$ for $t \geqq \sigma$ satisfies the asymptotic property $y(t) = O(e^{\mu t} t^{\rho+1-N'})$ as $t \to \infty$ for any nonnegative integer $N' \geqq N$. Then $e^{-\lambda t} t^{-r} y(t) \sim 0$ as $t \to \infty$. Thus it follows that the solution $x = x(t)$ of the equation (0.1), obtained in (3.2), has the same asymptotic expansion as that of the function $h(t)$. This implies the relation (0.7). This completes the proof of Theorem 2.

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Department of Computer Science
Faculty of Engineering
Yamanashi University
Kofu, 400 Japan