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POINTWISE INEQUALITIES AND CONTINUATION OF SOLUTIONS OF AN n^{th} ORDER EQUATION

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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1. Introduction. Burton [1] has discovered a necessary condition for the continuation of solutions of

$$(1) \qquad x^{(n)} + a(t)g(x) = 0$$

to be that

$$(2) \qquad \int^{\infty} G_{n-1}^{-1/n} = +\infty \quad \text{and} \quad \int^{-\infty} [(-1)^n G_{n-1}]^{-1/n} = -\infty\ ,$$

where

$$G_{k+1}(x) = \frac{1}{(k+1)!}\int_0^x (x-t)^k g(t)dt;\ xg(x) > 0\ , \qquad x \neq 0\ ;$$

and $a(t)$ is somewhere negative.

For $n = 2$ this is known to be a sufficient condition and Burton showed that with an extra assumption it also holds for $n = 3$. We here extend that result to all n and clarify the extra assumption that is to be made. The point of the proof is that one can estimate the relative growth of the derivatives of a function that is unbounded at a finite point. These inequalities are of interest in themselves. The simplest case is: if $x(0) = x'(0) = 0$; $x''(0) \geqq 0$, $x'''(t) \geqq 0$ then $2xx'' \geqq (x')^2$ for $t \geqq 0$; a result that is easily proved with the mean value theorem. This is what Burton used. Our generalization is

THEOREM 1. *If $f \in M_n$ and $1 < j \leqq n$, $k \geqq 1$, then*

$$(3) \qquad nf(t)f^{(j)}(t) \geqq (n-j+1)f'(t)f^{(j-1)}(t)$$

and

$$(4) \qquad \int_0^t f^k(s)f^{(j)}(s)ds \geqq \frac{n-j+1}{n(k+1)-j+1} f^k(t)f^{(j-1)}(t)\ .$$

Both are equalities if and only if $f(t) = C(t-t_0)_+^n$.

Here $x_+ = \max(0, x)$ and $M_n \equiv \{f \mid f \in C^{n-1}(0, T);\ f^{(j)}(0) = 0$ for $j \leqq$

$n-1$; $f^{(n-1)}$ *is convex and increasing on* $[0, T)\}$.

The inequalities in Theorem 1 are very special cases of some results which will be dealt with elsewhere. The proofs of Theorem 1 and relevant corollaries are in the next section. The differential equation is discussed in section 3. We acknowledge helpful discussions with Max Jodeit, Jr.

2. Inequalities. In order to provide a straightforward proof of Theorem 1, we note that if $f \in M_n$, then $f^{(n-1)}$ being convex, has a representation

$$f^{(n-1)}(x) = \int_0^x (x-t)d\mu(t) \tag{5}$$

where μ is a non-negative Borel measure. If $f \in C^{(n+1)}$ then $d\mu(t) = \delta_0 f^{(n)}(0) + f^{(n+1)}(t)dt$ where δ_0 is the unit mass at 0. By repeated integration one then has

$$n!f(x) = \int_0^x (x-t)^n d\mu(t) = \int_0^\infty (x-t)_+^n d\mu(t) . \tag{6}$$

PROOF OF THEOREM 1. The inequality (3) can be written equivalently by using (6). The result is

$$\begin{aligned} &\int_0^\infty \int_0^\infty (x-t)_+^n (x-s)_+^{n-j} d\mu(t)d\mu(s) \\ &\qquad \geqq \int_0^\infty \int_0^\infty (x-t)_+^{n-1}(x-s)_+^{n-j+1} d\mu(t)d\mu(s) . \end{aligned} \tag{7}$$

There is no question of convergence, and interchange of order of integration follows easily since the integrands are non-negative. It is convenient to symmetrize (7) by switching the roles of s and t and adding the result to (7). Thereby we must show that

$$\begin{aligned} &(x-t)_+^n(x-s)_+^{n-j} + (x-s)_+^n(x-t)_+^{n-j} \\ &\qquad \geqq (x-t)_+^{n-1}(x-s)_+^{n-j+1} + (x-s)_+^{n-1}(x-t)_+^{n-j+1} . \end{aligned} \tag{8}$$

Letting $u = (x-t)_+$, $v = (x-s)_+$, and cancelling common factors, (8) becomes

$$u^j + v^j \geqq u^{j-1}v + uv^{j-1} \tag{9}$$

or

$$(u^{j-1} - v^{j-1})(u-v) \geqq 0 , \tag{10}$$

thus (3) is proved. To prove (4), simply multiply (3) by f^{k-1} and integrate from 0 to t and simplify the right hand side by parts. If equality holds in (4), then a differentiation yields equality in (3). Now note that in

(10), the inequality is strict unless $u = v$. That is inequality holds in (7) unless the measure $d\mu(s)d\mu(t)$ lives on the diagonal. So equality in (3) implies that μ is a unit mass.

The application to the differential equation requires a more complicated inequality. In order to introduce G_{n-1} one multiplies (1) by $x'(t)$ and integrates, repeating this the appropriate number of times. With $y = x'$, applying this procedure to the first term of (1) leads to a consideration of

$$(11) \quad I_n(y) = \int_0^t y(u_1) \int_0^{u_1} y(u_2) \cdots \int_0^{u_{n-2}} y(u_{n-1}) \int_0^{u_{n-1}} y(u_n) y^{(n)}(u_n) du_n \cdots du_1 \, ,$$

where $n - 1$ has been replaced by n.

THEOREM 2. *If $f \in M_n$, then*

$$(12) \qquad (f(t))^{n+1}(n+1)^{-n} \leqq I_n(f) \leqq (f(t))^{n+1} \, .$$

Equality holds on the left if and only if $f(t) = (t - t_0)_+^n$.

PROOF. Consider the special case of (4) when $j = n - k + 1$ and $1 \leqq k \leqq n$.

$$(13) \qquad \int_0^t f^k(s) f^{(n-k+1)}(s) ds \geqq \frac{1}{n+1} f^k(t) f^{(n-k)}(t) \, .$$

Then

$$\int_0^{u_{n-1}} f(u_n) f^{(n)}(u_n) du_n \geqq \frac{1}{n+1} f(u_{n-1}) f^{(n-1)}(u_{n-1}) \quad \text{and}$$

$$\int_0^{u_{n-2}} f(u_{n-1}) \int_0^{u_{n-1}} f(u_n) f^{(n)}(u_n) du_n du_{n-1}$$

$$\geqq \int_0^{u_{n-2}} \frac{1}{n+1} f^2(u_{n-1}) f^{(n-1)}(u_{n-1}) du_{n-1} \geqq \frac{1}{(n+1)^2} f^2(u_{n-2}) f^{(n-2)}(u_{n-2}) \, .$$

An iteration gives the left hand inequality. Equality holds if each step using (13) has equality, so f is $(t - t_0)_+^n$. The right hand inequality follows by replacing each $f(u_i)$ by $f(t)$ and then integrating.

Theorem 2 has limited use because of the severe zero requirements on f. A version without these requirements follows easily.

COROLLARY 1. *If $f \in C^{(n+1)}[0, T)$, $T < \infty$; and $f^{(j)}(t) \geqq 0$ for $j \leqq n + 1$ then*

$$(f(t))^{n+1} \leqq (n+1)^n I_n(f) + R_n(f(t))$$

where R_n is a polynomial of degree $\leqq n$ whose the coefficients depend on f.

PROOF. Letting $p = \sum_{j=0}^{n-1} f^{(j)}(0)t^j/j!$ and $h(t) = f(t) - p(t)$, then $h \in M_n$ and we may apply the theorem. Then $I_n(h)$ becomes a sum of terms, one of which is $I_n(f)$. In each other term, replace each factor $f(u_i)$ by $f(t)$ and each factor $p(u_i)$ by $\|p\| = \sup_{0 \leq t \leq T} |p(t)|$.

3. The differential equation. The standard assumptions on the differential equation (1) are

(i) $g(x)$ is continuous, $xg(x) > 0$ for $x \neq 0$, g is unbounded on R; and

(ii) $a(t)$ is continuous and has isolated zeros, and is somewhere negative on $[0, \infty)$.

Under these conditions the equation (1) has local solutions. If $a(t) \geqq 0$, then all solutions extend to $[0, \infty)$. However, if $a(t)$ is somewhere negative this is no longer the case, since for example, $y^{(n)} - n!\, y^{n+1} = 0$ has solutions $y = (1 - t)^{-1}$. The growth of $g(y)$ is important. Burton has showed that if all solutions of (1) can be continued to $[0, \infty)$ under (i) and (ii) then (2) must hold. For example, $g(y) = |y|^\alpha \operatorname{sgn}(y)$, $0 < \alpha < 1$ satisfies this. It is now our purpose to investigate the converse of this statement.

It is known that for $n = 2$ this is correct. In this case the condition (2) is a Nagumo condition. We sketch a proof in order to motivate the rest of our work. Suppose $a(t) \leqq 0$ on $[0, T]$ and

$$x'' + a(t)g(x) = 0 \ . \tag{14}$$

Then for t near T say $(T - \varepsilon, T)$, $x'' \geqq 0$ and so $x'(t) \to \infty$ as $t \to T^-$. Thus on $(T - \varepsilon, T)$, $x'x'' = -a(t)g(x(t))x'(t) \leqq Mg(x(t))x'(t)$ and

$$\frac{x'(t)^2}{2} - \frac{x'(T-\varepsilon)^2}{2} = \int_{T-\varepsilon}^t x'x'' \leqq MG_1(x(t)) - MG_1(x(T-\varepsilon)) \ . \tag{15}$$

Thus (15) gives for $t \in (T - \varepsilon, T)$ that $x'(t)^2 \leqq \bar{M}G_1(x(t))$ and

$$\int_{T-\varepsilon}^t G_1(x(s))^{-1/2}x'(s)ds \leqq \bar{M}\varepsilon \ .$$

Thus

$$\int_{x(T-\varepsilon)}^{x(t)} G_1(u)^{-1/2}du \leqq \bar{M}\varepsilon \ .$$

If (2) holds then $x(t)$ must remain finite. So (2) suffices to have all solutions extend. If one tries the above for the third order, one is hampered by the presence of the term $-\int_0^t (x'')^2$ on the left. This is the reason for the inequalities.

We are now prepared to set the stage for our general argument. Suppose that $n > 2$ and (1) has a solution $x(t)$ which cannot be extended to $[0, \infty)$. Then $\limsup_{t\to T^-} |x(t)| = +\infty$, say $\limsup_{t\to T^-} x(t) = +\infty$. If $a(t) \geqq 0$ near T, this cannot happen since then $x^{(n)} \leqq 0$ there. To apply the inequalities one needs to have functions which have constant sign derivatives. Now a solution may fail to continue because it oscillates. If this occurs we can construct another non-continuable solution with the required sign conditions. Let $z(t)$ be a solution of (1) such that $z(T-\varepsilon) = x(T-\varepsilon) + 1$, $z^{(i)}(T-\varepsilon) > x^{(i)}(T-\varepsilon)$ $i = 1, \cdots, n-1$ and $z^{(i)}(T-\varepsilon) \geqq 0$. If $w = z - x$, then $w^{(n)}(t) = -a(t)[g(z) - g(x)]$ so $ww^{(n)} > 0$ on $[T-\varepsilon, T)$ provided g is increasing and $w \neq 0$. We assume that g is increasing. Then if w is zero somewhere, let ξ_0 be the least zero on $[T-\varepsilon, T)$. By repeated application of the mean value theorem (noting that $w^{(i)}(T-\varepsilon) > 0$), one arrives at $T - \varepsilon < \xi_{n-1} < \cdots < \xi_1 < \xi_0 < T$ such that $w^{(i)}(\xi_i) = 0$. This implies that $w^{(n)} = 0$ somewhere on $[T-\varepsilon, \xi_{n-1})$ and hence w has a zero there too. This condition shows that $w > 0$ on $[T-\xi, T)$ so $z > x$ there and $\lim_{t\to T} z(t) = +\infty$. It follows that $z^{(i)}(t) \to \infty$ as $t \to T^-$ for $i = 1, \cdots, n$.

REMARK. The above argument shows that if g is bounded on R then the continuation problem is trivial. This is the reason for the unbounded requirement in (i).

CONTINUATION LEMMA. *If* (2) *holds and* $a(t) \leqq 0$ *on* $[a, b]$ *then every solution of* (1) *that satisfies* $xx^{(n+1)} \geqq 0$ *can be continued across the interval, i.e.,* $x^{(j)}(b)$ $0 \leqq j \leqq n$ *is defined.*

PROOF. Since on $[a, b]$, $xx^{(n)} \geqq 0$ and $xx^{(n+1)} \geqq 0$, a non-continuable solution must satisfy $x^{(j)} \to +\infty$ as $t \to T^-$, for some $T \in (a, b]$, $j = 0, \cdots, n$ and therefore for an interval of the form $(T - \delta, T)$, $x^{(j)}(t) \geqq 0$ $j = 0, \cdots, n+1$. We now estimate $I_{n-1}(x')$ in two ways. First according to the corollary $(x'(t))^n \leqq n^{n-1} I_{n-1}(x') + R_{n-1}(x')$. On the other hand letting $y = x'$ and $M = \sup_{[a,b]} |a(t)|$ we have from (1) $yy^{(n-1)} \leqq Mg(x(t))x'(t)$ so that

$$\int_a^{u_{n-1}} y(u_n) y^{(n-1)}(u_n) du_n \leqq MG_1(x(u_{n-1})) + C_1 .$$

Then

$$\begin{aligned}\int_a^{u_{n-2}} y(u_{n-1}) &\int_a^{u_{n-1}} y(u_n) y^{(n-1)}(u_n) du_n du_{n-1} \\ &\leqq M \int_a^{u_{n-2}} G_1(x(u_{n-1})) x'(u_{n-1}) du_{n-1} + C_1 \int_a^{u_{n-2}} x' \\ &\leqq MG_2(x(u_{n-2})) + C_2 y(u_{n-2}) + C_3 .\end{aligned}$$

This process is repeated, terminating in

$$n^{-(n-1)}y(t)^n \leqq I_{n-1}(y) + n^{-(n-1)}R_{n-1}(y) \leqq MG_{n-1}(x(t)) + \bar{R}_{n-1}(y)$$

where $\bar{R}_{n-1}$ is a polynomial in y of degree $\leqq n-1$. Since $y(t) \to \infty$, $\bar{R}_{n-1}(y)/y(t)^n \to 0$ as $t \to T^-$ so we get an inequality of the form $y(t)^n \leqq \bar{M}G_{n-1}(x(t))$ for t sufficiently near T. Thus

$$\int_a^t G_{n-1}(x(u))^{-1/n}x'(u)du \leqq \bar{M}(t-a) \leqq \bar{M}(b-a)\ , \quad \text{and}$$

$$\int_{x(a)}^{x(t)} G_{n-1}(u)^{-1/n}du \leqq \bar{M}(b-a)\ .$$

Since $x(t) \to +\infty$ as $t \to T^-$ this violates (2).

The Continuation lemma gives a criterion for all solutions to be extendable to $[0, \infty)$. However the extra requirement that $xx^{(n+1)} \geqq 0$ is troublesome. It is needed to get Corollary 1. On the other hand, the condition that

$$x^{(n+1)} = -a(t)g'(x(t))x'(t) - a'(t)g(x(t)) \geqq 0$$

when $x(t) \geqq 0$ is not easy to verify. Consider for example, the condition when $a(t) = 0$. The resolution of this problem lies elsewhere. The key is to compare with other equations. Suppose for example that $a(t)$ were a negative constant. Then $x^{(n+1)} = c^2g'(x(t))x'(t) \geqq 0$ when $g' \geqq 0$ and $x' \geqq 0$. Thus in this case the proof of the Continuation lemma works without the assumption $xx^{(n+1)} \geqq 0$.

COMPARISON LEMMA 1. *Let $x_i(t)$ be solutions to*

$$(16)_i \qquad x_i^{(n)}(t) + a_i(t)g(x_i(t)) = 0 \qquad i = 1, 2$$

on $[c, d)$ where $a_2(t) < a_1(t) \leqq 0$, and $0 < x_1^{(j)}(c) = x_2^{(j)}(c) = b_j$ $j = 0, \cdots, n-1$ with $g(b_0) > 0$. If g is increasing then $x_1(t) < x_2(t)$ on $[c, d)$.

PROOF. Let $z(t) = x_1(t) - x_2(t)$. Then $z^{(j)}(c) = 0$ $j = 0, \cdots, n-1$ and $z^{(n)}(t) = a_2(t)[g(x_2(t)) - g(x_1(t))] + [a_2(t) - a_1(t)]g(x_1(t))$. Now $z^{(n)}(c) = [a_2(c) - a_1(c)]g(b_0) < 0$. Therefore $z(t) < 0$ on some maximal interval $[c, e) \subset [c, d)$. On the interval (c, e) $x_1 < x_2$ and so $z^{(n)}(t) \leqq 0$ on this interval. Thus $z^{(j)}(t) \leqq 0$ on this interval $j = 0, \cdots, n-1$ and it is not possible that $z(e) = 0$. Thus e must be d.

COROLLARY 2. *If g is increasing then the Continuation lemma holds without the assumption that $xx^{(n+1)} \geqq 0$.*

PROOF. If (1) has a solution with $x(t) \to +\infty$ as $t \to T^-$ then compare (1) with

$$y^{(n)} + \left[\min_{T-\delta \leqq t \leqq T} a(t)\right] g(y) = 0 \,. \tag{17}$$

By the Comparison lemma 1, (17) has a solution that is unbounded, say $y(t) \to \infty$ as $t \to T_1^-$ for $T_1 \leqq T$. But for t near T_1, $y'(t) \geqq 0$ so $y^{(n)}$ is increasing so that the Continuation lemma applies to get a contradiction.

COMPARISON LEMMA 2. *Let* $x_i(t)$ *be solutions of* $x_2(a) = x_1(a) + \varepsilon$, $\varepsilon > 0$, *and*

$$(18)_i \quad x_i^{(n)}(t) + a(t) g_i(x(t)) = 0\,; \quad x_i^{(j)}(a) = b_j > 0 \quad i = 1, 2 \quad j = 1, \cdots, n-1$$

on $[a, b)$ *where* $a(t) \leqq 0$, *a satisfies* (ii) *and* g_i *satisfy* (i). *If* $g_1(x) \leqq g_2(x)$ $x \geqq 0$ *with* g_2 *increasing, then* $x_1(t) < x_2(t)$ *on* $[a, b)$.

PROOF. Again let $z(t) = x_1(t) - x_2(t)$ so that $z(a) < 0$, $z^j(a) = 0$ $1 \leqq j \leqq n-1$ and $z^{(n)}(t) = a(t)[g_2(x_2(t)) - g_2(x_1(t))] + a(t)[g_2(x_1(t)) - g_1(x_1(t))]$. Then $z^{(n)}(t) \leqq 0$ on any interval where $x_2 \geqq x_1$. Let $[a, c)$ be a maximal interval in $[a, b)$ where $z \leqq 0$. If $c < b$, then $z(c) = 0$ and $z^{(n)} \leqq 0$ on $[a, c)$. But then $z^{(n-1)} \leqq 0$ on (a, c), which in turn implies $z^{(n-1)} \leqq 0$ on (a, c), $\cdots$, $z' \leqq 0$ on (a, c). Thus $z(c) \leqq z(a) < 0$. Thus $c = b$.

Now let $\bar{g}$ be defined by $\bar{g}(x) = \sup_{0 \leqq t \leqq x} g(t)$, for $x \geqq 0$; and $\bar{g}(x) = \inf_{x \leqq t \leqq 0} g(t)$ for $x \leqq 0$. We have the main theorem.

THEOREM 3. *Let* (i) *and* (ii) *be satisfied and* $n > 2$. *Then*

(a) *If* $\bar{g}$ *satisfies* (2), *then all solutions of* (1) *continue to* $[0, \infty)$.

(b) *If* g *is increasing, then* (2) *is necessary and sufficient for all solutions of* (1) *to continue to* $[0, \infty)$.

PROOF. The necessity of (b) is Burton's result while the sufficiency is a use of Corollary 2. Part (a) follows from Comparison lemma 2 and Corollary 2.

REMARK. An examination of the proofs show that the hypotheses on g need only hold for $|x|$ sufficiently large, but this is a minor generalization.

REFERENCES

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