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OCTAHEDRAL WEBS ON CLOSED MANIFOLDS

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1. Introduction, definitions and the statement of results. Webs on domains in $\boldsymbol{R}^n$ have been studied from the viewpoint of differential geometry and algebraic geometry. See Blaschke-Bol [1] and Chern-Griffiths [2]. In this paper we classify octahedral webs on closed manifolds under a certain assumption on the fundamental groups of underlying manifolds. Our method depends on the fact that octahedral webs are always analytic, and consists in the construction of developing maps and total holonomy homomorphisms.

Now let M be a C^r manifold, $r = 0, 1, \cdots, \infty, \omega$, with $\dim M \geqq 2$.

DEFINITION 1. An ordered family $\mathscr{W} = (\mathscr{F}_1, \cdots, \mathscr{F}_k)$ of codimension-one C^r foliations on M is called a C^r *web* on M if for each $x \in M$ the tangent spaces $T_x\mathscr{F}_1, \cdots, T_x\mathscr{F}_k$ of the foliations $\mathscr{F}_i$ at x are in general position in T_xM. Of course for the C^0 case an appropriate modification is needed.

DEFINITION 2. A C^r web $\mathscr{W} = (\mathscr{F}_1, \cdots, \mathscr{F}_k)$ on M is called *octahedral* if $k = \dim M + 1$ and for each $x \in M$ there is a C^r chart $\varphi: U \to \boldsymbol{R}^n$ such that $x \in U$ and $\mathscr{F}_i|U = \varphi^*\mathscr{G}_i$, where $\mathscr{G}_i$ is a family of parallel hyperplanes in $\boldsymbol{R}^n$.

DEFINITION 3. Let $\mathscr{W}^i = (\mathscr{F}_1^i, \cdots, \mathscr{F}_{k_i}^i)$ be a C^r web on C^r manifolds M_i for $i=1, 2$. Then $\mathscr{W}_1$ is C^r *isomorphic* to $\mathscr{W}_2$ if $k_1=k_2$ and there is a C^r diffeomorphism $f: M_1 \to M_2$ such that $\mathscr{W}_1 = f^*\mathscr{W}_2$, that is, $\mathscr{F}_j^1 = f^*\mathscr{F}_j^2$ for $j = 1, \cdots, k_1$.

We construct some examples.

EXAMPLE 1. Consider $\boldsymbol{R}^n$ as M. Let

$$\mathscr{G}_i^0 = \{x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n | x_i = c\}_{c \in \boldsymbol{R}}$$

for $i=1, \cdots, n$, and

$$\mathscr{G}_{n+1}^0 = \{x \in \boldsymbol{R}^n | x_1 + \cdots + x_n = c\}_{c \in \boldsymbol{R}} .$$

Then $\mathscr{W}^0 = (\mathscr{G}_1^0, \cdots, \mathscr{G}_{n+1}^0)$ is clearly an octahedral C^ω web. We call $\mathscr{W}^0$ the *standard web.*

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NOTATION. We denote by $\mathrm{OW}(n)$ the group consisting of diffeomorphisms $f_{r,a}: \boldsymbol{R}^n \to \boldsymbol{R}^n$ defined by

$$f_{r,a}(x) = rx + a \quad \text{for all} \quad x \in \boldsymbol{R}^n$$

where $r \in \boldsymbol{R} - \{0\}$ and $a \in \boldsymbol{R}^n$. Let $\mathrm{POW}(n) = \{f_{r,a} \mid r > 0,\ a \in \boldsymbol{R}^n\}$. Note that each element of $\mathrm{OW}(n)$ preserves $\mathscr{W}^0$ and that each element of $\mathrm{POW}(n)$ preserves the transverse orientations of all $\mathscr{G}_i^0$.

EXAMPLE 2. Let Γ be a lattice group of $\boldsymbol{R}^n$. We consider Γ as a subgroup of $\mathrm{POW}(n)$. Since each element of Γ preserves $\mathscr{W}^0$, we can consider the quotient web $\mathscr{W}^0/\Gamma = (\mathscr{G}_1^0/\Gamma, \cdots, \mathscr{G}_{n+1}^0/\Gamma)$ on the torus $\boldsymbol{R}^n/\Gamma$. We write $\mathscr{W}(\Gamma) = \mathscr{W}^0/\Gamma$. Note that each foliation in $\mathscr{W}(\Gamma)$ is without holonomy. Let $A(n)$ be the set of all conjugacy classes of lattice groups in $\mathrm{POW}(n)$. Since each element $h \in \mathrm{POW}(n)$ induces a C^ω isomorphism $h_*: \boldsymbol{R}^n/\Gamma \to \boldsymbol{R}^n/h\Gamma h^{-1}$ of webs $\mathscr{W}(\Gamma)$ and $\mathscr{W}(h\Gamma h^{-1})$, we can attach to each conjugacy class $\gamma = [\Gamma] \in A(n)$ a C^ω isomorphism class $W(\gamma)$ of the octahedral web $\mathscr{W}(\Gamma)$.

EXAMPLE 3. Let $n \geqq 3$. Let $\boldsymbol{R}_+ = \{r \in \boldsymbol{R} \mid r > 0\}$. For $r \in \boldsymbol{R}_+ - \{1\}$ let $m_r: \boldsymbol{R}^n - \{0\} \to \boldsymbol{R}^n - \{0\}$ be the map defined by $m_r(x) = rx$ for all $x \in \boldsymbol{R}^n - \{0\}$. Since m_r preserves the restricted web $\mathscr{W}^0 | \boldsymbol{R}^n - \{0\}$, we have the quotient web $\mathscr{W}(r) = (\mathscr{W}^0 | \boldsymbol{R}^n - \{0\})/m_r$ on the manifold $(\boldsymbol{R}^n - \{0\})/m_r$ diffeomorphic to $S^1 \times S^{n-1}$. Note that each foliation in $\mathscr{W}(r)$ consists of 2 Reeb components. Since $\mathscr{W}(r) = \mathscr{W}(1/r)$, it is sufficient to consider the isomorphism class $W(r)$ of the octahedral web $\mathscr{W}(r)$ for each $r \in B(n) = (0, 1)$.

EXAMPLE 4. Now we consider $\boldsymbol{R}^2 - \{0\}$. The map $q: \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}^2 - \{0\}$ defined by

$$q(t, \theta) = (t \cos 2\pi\theta,\ t \sin 2\pi\theta)$$

for all $t \in \boldsymbol{R}_+$ and $\theta \in \boldsymbol{R}$ is a universal covering map. Define the maps $f,\ g: \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}_+ \times \boldsymbol{R}$ by

$$f(t, \theta) = (rt, \theta)\ , \quad g(t, \theta) = (st, \theta + d)\ ,$$

where $r \in \boldsymbol{R}_+ - \{1\}$, $s \in \boldsymbol{R}_+$ and $d \in \boldsymbol{Z}_+ = \boldsymbol{Z} \cap \boldsymbol{R}_+$. Since f and g preserve the induced web $q^*(\mathscr{W}^0 | \boldsymbol{R}^2 - \{0\})$, we have the quotient web $\mathscr{W}(r, s, d) = q^*(\mathscr{W}^0 | \boldsymbol{R}^2 - \{0\})/\{f, g\}$ on the manifold $\boldsymbol{R}_+ \times \boldsymbol{R}/\{f, g\}$ diffeomorphic to $S^1 \times S^1$. Note that each foliation in $\mathscr{W}(r, s, d)$ consists of $2d$ Reeb components. Let $B(2) = \{(r, s, d) \mid 0 < r < s \leqq 1,\ d \in \boldsymbol{Z}_+\}$. Clearly it suffices to consider the C^ω isomorphism class $W(r, s, d)$ of the octahedral web $\mathscr{W}(r, s, d)$ for all $(r, s, d) \in B(2)$.

Now we can state the results.

THEOREM 1. *The list of C^ω isomorphism classes of octahedral webs on closed manifolds in Examples* 2, 3 *and* 4 *has no duplication, that is,* $W(\gamma_1) \neq W(\gamma_2)$ *if* $\gamma_1 \neq \gamma_2$ *where* $\gamma_1, \gamma_2 \in A(n) \cup B(n)$.

REMARK. By the argument in § 2 we see that every C^0 isomorphism of octahedral webs is a C^ω isomorphism.

THEOREM 2. *Let* $\mathscr{W}$ *be an octahedral* C^r *web on a closed connected* C^r 2*-manifold* M *such that some foliation in* $\mathscr{W}$ *is transversely orientable. Then* M *is* C^r *diffeomorphic to* $S^1 \times S^1$ *and* $\mathscr{W}$ *is* C^r *isomorphic to* $\mathscr{W}(\gamma)$ *for some* $\gamma \in A(2) \cup B(2)$.

THEOREM 3. *Let* $\mathscr{W}$ *be an octahedral* C^r *web on a closed connected* C^r *manifold* M *of dimension* $n \geqq 3$ *such that some foliation in* $\mathscr{W}$ *is transversely orientable. Suppose that* M *satisfies one of the following conditions.*

(1) $n = 3$ *and* $\pi_1(M)$ *has non-exponential growth.*

(2) $\pi_1(M)$ *is abelian.*

(3) $\pi_1(M)$ *is generated by at most* n *elements.*

Then $\mathscr{W}$ *is* C^r *isomorphic to* $\mathscr{W}(\gamma)$ *for some* $\gamma \in A(n) \cup B(n)$. *Furthermore,* M *is* C^r *diffeomorphic to* $S^1 \times \cdots \times S^1$ *in the case* $\gamma \in A(n)$ *and to* $S^1 \times S^{n-1}$ *in the case* $\gamma \in B(n)$.

REMARK. If a C^1 manifold M has an octahedral web $\mathscr{W}$ such that some foliation in $\mathscr{W}$ is transversely orientable, then M is parallelizable.

The following problem is open.

PROBLEM. *Does there exist an octahedral* C^r *web on a closed orientable* C^r *manifold whose fundamental group is non-abelian?*

In § 2 we study fundamental properties of octahedral webs. The theorems are partially proved in § 3 and § 4 and the proof is completed in § 5.

2. Fundamental properties of octahedral webs. Let M be a connected C^r manifold of dimension $n \geqq 2$ and $\mathscr{W} = (\mathscr{F}_1, \cdots, \mathscr{F}_{n+1})$ an octahedral C^r web on M.

First we show that we can choose special charts of M depending on $\mathscr{W}$.

DEFINITION 4. A C^r chart $(U, \varphi: U \to \boldsymbol{R}^n)$ of M is called *admissible* with respect to $\mathscr{W}$ if $\mathscr{W}|U = \varphi^* \mathscr{W}^0$ where $\mathscr{W}^0$ is the standard web on $\boldsymbol{R}^n$.

PROPOSITION 1. (1) *For each* $x \in M$ *there is an admissible* C^r *chart*

(U, φ) *with* $x \in U$.

(2) *If* (U, φ) *and* (V, ψ) *are admissible* C^r *charts such that* $U \cap V$ *is non-empty and connected, then there is* $f \in \mathrm{OW}(n)$ *such that* $\psi = f \circ \varphi$ *on* $U \cap V$.

(3) *Let* U *and* V *be open connected subsets of* $\boldsymbol{R}^n$. *If a* C^r *diffeomorphism* $f: U \to V$ *satisfies* $\mathscr{W}^0 | U = f^*(\mathscr{W}^0 | V)$, *then there is* $\bar{f} \in \mathrm{OW}(n)$ *with* $\bar{f} | U = f$.

(4) *If* $f \in \mathrm{OW}(n)$ *preserves a transverse orientation of some* $\mathscr{G}_i^0$, *then* $f \in \mathrm{POW}(n)$.

PROOF. (1) Let $x_0 \in M$. By definition we have a C^r chart (U, φ) such that $x_0 \in U$ and $\mathscr{F}_i | U = \varphi^* \mathscr{G}_i$ for $i = 1, \cdots, n+1$, where $\mathscr{G}_i$ is a family of parallel hyperplanes in $\boldsymbol{R}^n$. We denote by $G_i(y)$ the leaf of $\mathscr{G}_i$ passing $y \in \boldsymbol{R}^n$. Let $y_0 = \varphi(x_0) \in \boldsymbol{R}^n$. Since $G_1(y_0), \cdots, G_n(y_0)$ are in general position, each intersection

$$L_i = G_1(y_0) \cap \cdots \cap G_{i-1}(y_0) \cap G_{i+1}(y_0) \cap \cdots \cap G_n(y_0)$$

is a line passing y_0. Choose a point $y_1 \in L_1 - \{y_0\}$ and, for $i = 2, \cdots, n$, let y_i be the intersecting point of L_i and $G_{n+1}(y_1)$. Since the vectors $y_1 - y_0, \cdots, y_n - y_0$ are linearly independent, there is a linear map $F: \boldsymbol{R}^n \to \boldsymbol{R}^n$ sending $y_i - y_0$ to $(0, \cdots, 0, \underset{\widehat{i}}{1}, 0, \cdots, 0)$ for $i = 1, \cdots, n$. We define a map $\varphi': U \to \boldsymbol{R}^n$ by

$$\varphi'(x) = F(\varphi(x) - y_0)$$

for all $x \in U$. It is easy to see that the chart (U, φ') is admissible.

(2) follows from (3), and (4) is clear.

(3) For each $x \in \boldsymbol{R}^n$ we denote by $F_i(x)$ the leaf of $\mathscr{G}_i^0$ passing x. Let $L_i(x) = F_1(x) \cap \cdots \cap F_{i-1}(x) \cap F_{i+1}(x) \cap \cdots \cap F_n(x)$ for $i = 1, \cdots, n$. Clearly $L_i(x)$ is a line parallel to the i-th axis of $\boldsymbol{R}^n$. Now let $a \in U$ and choose $b \in L_i(a) - \{a\}$ such that the line segment between a and b is contained in U. Let $j \neq i$ and consider the following equation

$$(*) \qquad L_i(L_j(a) \cap F_{n+1}(x)) \cap F_{n+1}(b) = L_j(x) \cap F_{n+1}(b)$$

for the unknown $x \in L_i(a)$. The equation $(*)$ has the unique solution $x = a + (b-a)/2$. Transforming the equation $(*)$ by f, we have the following equation

$$(**) \qquad L_i(L_j(f(a)) \cap F_{n+1}(y)) \cap F_{n+1}(f(b)) = L_j(y) \cap F_{n+1}(f(b))$$

for the unknown $y \in L_i(f(a))$. Since the equation $(**)$ has the unique solution $y = f(a) + (f(b) - f(a))/2$, we have $f(a + (b-a)/2) = f(a) + (f(a) - f(b))/2$. This implies that for all $a \in U$ and sufficiently small $\varepsilon > 0$ the map f sends points in $L_i(a) \cap U(a, \varepsilon)$ separated by intervals of the same length

to such points in $L_i(f(a))$, where $U(a, \varepsilon)$ is the open disk centered at a and of radius ε, and furthermore that f has the form $f(x) = rx + c$ for all $x \in L_i(a) \cap U(a, \varepsilon)$, where $r = \|f(b) - f(a)\|/\|b - a\|$ for all $b \in L_i(a) \cap U(a, \varepsilon) - \{a\}$ and $c = f(a) - ra$. In the above $\|\cdot\|$ is the usual norm of $\boldsymbol{R}^n$.

We show that $r = r(i, a)$ does not depend on i. Let $j \neq i$ and consider the identity

$$(***) \qquad f(L_j(a) \cap F_{n+1}(b)) = L_j(f(a)) \cap F_{n+1}(f(b)) ,$$

where $b \in L_i(a) \cap U(a, \varepsilon) - \{a\}$. Let $b' = L_j(a) \cap F_{n+1}(b)$. Then the triangle determined by a, b and b' is similar to that determined by $f(a)$, $f(b)$ and $f(b')$ because of the identity $(***)$. Therefore we see that $r(i, a) = \|f(b) - f(a)\|/\|b - a\| = \|f(b') - f(a)\|/\|b' - a\| = r(j, a)$.

For all $x \in U(a, \varepsilon) \subset U$ we have the identities $f(F_i(x) \cap L_i(a)) = F_i(f(x)) \cap L_i(f(a))$ for $i = 1, \cdots, n$, which implies that $f(x) = rx + c$. Therefore f is an analytic map in a neighborhood of each point $a \in U$. By the unicity theorem of analytic maps, it follows that $f(x) = rx + c$ for all $x \in U$. Let $\bar{f} = f_{r,c} \in \mathrm{OW}(n)$. Then $f = \bar{f}|U$.

REMARK. By Proposition 1 we can consider octahedral webs as $\mathrm{OW}(n)$-structures. An $\mathrm{OW}(n)$-structure is clearly a similarity structure.

The following corollary follows immediately from Proposition 1 and we omit the proof.

COROLLARY 1. (1) *Let $\mathscr{A}$ be the set of all admissible C^r charts of M with respect to $\mathscr{W}$. Then $\mathscr{A}$ determines an analytic structure $\bar{\mathscr{A}}$ of M and $\mathscr{W}$ is analytic with respect to $\bar{\mathscr{A}}$.*

(2) *Let $\mathscr{W}'$ be an octahedral C^r web on another C^r manifold M' and $f: M \to M'$ a C^r diffeomorphism with $\mathscr{W} = f^*\mathscr{W}'$. Then f is analytic with respect to the analytic structures of the manifolds determined by admissible charts.*

From now on we assume that all octahedral webs are of class C^ω and we omit the term "C^ω". By using the analyticity of octahedral webs, we can develop it over $\boldsymbol{R}^n$ as follows. Refer to Thurston [4].

PROPOSITION 2. *Let $p: \tilde{M} \to M$ be the universal covering map of M. For all $x \in \boldsymbol{R}^n$ and $y \in \tilde{M}$ there is a map $D: \tilde{M} \to \boldsymbol{R}^n$ such that $D(y) = x$ and, for each sufficiently small open set U and each section $s: U \to \tilde{M}$ of p, the map $D \circ s: U \to \boldsymbol{R}^n$ is an admissible chart.*

PROOF. Let $y \in \tilde{M}$ and $x \in \boldsymbol{R}^n$. Choose a neighborhood U of y in $\tilde{M}$ such that $p|U$ is a diffeomorphism onto $p(U)$. By Proposition 1 we can take an admissible chart (V, φ) with $p(y) \in V \subset p(U)$. Let $U_1 = (p|U)^{-1}(V)$

and $f = f_{1, x-\varphi\circ p(y)} \in \mathrm{POW}(n)$. Then the composed map $D_1 = f\circ\varphi\circ p\colon U_1 \to \boldsymbol{R}^n$ satisfies $D_1(y) = x$ and, for the section $s = (p|U_1)^{-1}\colon V \to U_1$, the map $D_1\circ s = f\circ\varphi$ is an admissible chart. It is easy to extend D_1 to a map $D\colon \tilde{M} \to \boldsymbol{R}^n$ with the desired property by analytic continuation.

DEFINITION 5. The map $D\colon \tilde{M} \to \boldsymbol{R}^n$ in Proposition 2 is called a *developing map* of the octahedral web $\mathscr{W}$.

The covering transformation group $G(\tilde{M}|M)$ of the universal covering map $p\colon \tilde{M} \to M$ and a developing map of $\mathscr{W}$ are related as follows. The proof is routine and we omit it.

PROPOSITION 3. *Let $D\colon \tilde{M} \to \boldsymbol{R}^n$ be a developing map of $\mathscr{W}$. Then there is a unique homomorphism $H\colon G(\tilde{M}|M) \to \mathrm{OW}(n)$ such that $D\circ\alpha = H(\alpha)\circ D$ for all $\alpha \in G(\tilde{M}|M)$.*

DEFINITION 6. The homomorphism $H\colon G(\tilde{M}|M) \to \mathrm{OW}(n)$ in Proposition 3 is called the *total holonomy homomorphism* associated with D, and the image $\mathrm{Im}\, H$ of H the *total holonomy group* with respect to D.

PROPOSITION 4. *The set of all total holonomy groups of $\mathscr{W}$ forms a conjugacy class of subgroups of* $\mathrm{POW}(n)$ *if some foliation in $\mathscr{W}$ is transversely orientable, and of* $\mathrm{OW}(n)$ *otherwise.*

The proof is easy and we omit it.

3. The case where a total holonomy group of $\mathscr{W}$ consists of parallel translations of $\boldsymbol{R}^n$. In § 3 and § 4 we consider an octahedral web $\mathscr{W}$ on a closed connected manifold M such that some foliation in $\mathscr{W}$ is transversely orientable. Let $p\colon \tilde{M} \to M$ be the universal covering map, $D\colon \tilde{M} \to \boldsymbol{R}^n$ a developing map of $\mathscr{W}$ and $H\colon G(\tilde{M}|M) \to \mathrm{POW}(n)$ the total holonomy homomorphism associated with D.

In this section we suppose that $\mathrm{Im}\, H$ consists of parallel translations of $\boldsymbol{R}^n$. First we show that D is a covering map. As before we denote by $U(x, r)$ the open disk in $\boldsymbol{R}^n$ centered at x and of radius r. Let K be a compact subset of $\tilde{M}$ such that $p(K) = M$. Since D is locally a diffeomorphism, we can take $\varepsilon > 0$ so that for all $y \in K$ there is a neighborhood U_y of y in $\tilde{M}$ such that $D|U_y$ is a diffeomorphism onto $U(D(y), \varepsilon)$.

LEMMA 1. *The map $D\colon \tilde{M} \to \boldsymbol{R}^n$ is a covering map.*

PROOF. We show that D is surjective. Since D is locally a diffeomorphism, it follows that $\mathrm{Im}\, D$ is open in $\boldsymbol{R}^n$. Let $x \in \mathrm{Cl}(\mathrm{Im}\, D)$. Then there is $y \in \tilde{M}$ with $D(y) \in U(x, \varepsilon)$. Furthermore, there is $\alpha \in G(\tilde{M}|M)$ with $\alpha(y) \in K$. Transforming the diffeomorphism $D|U_{\alpha(y)}\colon U_{\alpha(y)} \to U(D\alpha(y), \varepsilon)$

by α^{-1}, we have a diffeomorphism $D|\alpha^{-1}U_{\alpha(y)}\colon \alpha^{-1}U_{\alpha(y)} \to H(\alpha)^{-1}U(D\alpha(y), \varepsilon)$. Since $H(\alpha)$ is a parallel translation, it follows that $H(\alpha)^{-1}U(D\alpha(y), \varepsilon) = U(D(y), \varepsilon)$. Thus we see that $x \in U(D(y), \varepsilon) \subset \operatorname{Im} D$ and $\operatorname{Im} D$ is closed in $\boldsymbol{R}^n$, too. By the connectivity of $\boldsymbol{R}^n$ we conclude that $\operatorname{Im} D = \boldsymbol{R}^n$.

We show that for each $x \in \boldsymbol{R}^n$ the neighborhood $U(x, \varepsilon/2)$ is evenly covered by D. Let $y \in D^{-1}U(x, \varepsilon/2)$. By the above argument we have a neighborhood U_y of y in $\tilde{M}$ such that $D|U_y$ is a diffeomorphism onto $U(D(y), \varepsilon)$. Since $U(x, \varepsilon/2) \subset U(D(y), \varepsilon)$, it follows that $D|(D|U_y)^{-1}U(x, \varepsilon/2)$ is a diffeomorphism onto $U(x, \varepsilon/2)$. Hence $U(x, \varepsilon/2)$ is evenly covered by D, and $D\colon \tilde{M} \to \boldsymbol{R}^n$ is a covering map.

Since $\boldsymbol{R}^n$ is simply connected, the map $D\colon \tilde{M} \to \boldsymbol{R}^n$ is a diffeomorphism. Therefore $H\colon G(\tilde{M}|M) \to \mathrm{POW}(n)$ is a monomorphism and M is homeomorphic to the quotient space $\boldsymbol{R}^n/\operatorname{Im} H$. Thus $\Gamma = \operatorname{Im} H$ is a lattice group of $\boldsymbol{R}^n$ and $\mathscr{W}$ is isomorphic to $\mathscr{W}(\Gamma)$.

Now we investigate the isomorphism class of $\mathscr{W}(\Gamma)$ for a lattice group Γ of $\boldsymbol{R}^n$.

PROPOSITION 5. *Let Γ_1 and Γ_2 be lattice groups of $\boldsymbol{R}^n$. Then $\mathscr{W}(\Gamma_1)$ is isomorphic to $\mathscr{W}(\Gamma_2)$ if and only if Γ_1 is conjugate to Γ_2 in* $\mathrm{POW}(n)$.

PROOF. The "if" part is already shown in Example 2. Suppose that there is a homeomorphism $f\colon \boldsymbol{R}^n/\Gamma_1 \to \boldsymbol{R}^n/\Gamma_2$ with $\mathscr{W}(\Gamma_1) = f^*\mathscr{W}(\Gamma_2)$. By the theory of covering maps, we can find a diffeomorphism $\tilde{f}\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ making the diagram

$$\begin{array}{ccc} \boldsymbol{R}^n & \xrightarrow{\tilde{f}} & \boldsymbol{R}^n \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \boldsymbol{R}^n/\Gamma_1 & \xrightarrow[f]{} & \boldsymbol{R}^n/\Gamma_2 \end{array}$$

commutative, where p_1 and p_2 are the projections. Since $\tilde{f}^*\mathscr{W}^0 = \tilde{f}^* \circ p_2^*\mathscr{W}(\Gamma_2) = p_1^* \circ f^*\mathscr{W}(\Gamma_2) = p_1^*\mathscr{W}(\Gamma_1) = \mathscr{W}^0$, it follows that $\tilde{f} \in \mathrm{OW}(n)$. Composing it with $f_{-1,0} \in \mathrm{OW}(n)$ if necessary, we may assume that $\tilde{f} \in \mathrm{POW}(n)$. It is easy to see that $\Gamma_2 = \tilde{f}\Gamma_1\tilde{f}^{-1}$, which completes the proof of Proposition 5.

4. The case where a total holonomy group of $\mathscr{W}$ contains a contraction. Let $p\colon \tilde{M} \to M$, $D\colon \tilde{M} \to \boldsymbol{R}^n$ and $H\colon G(\tilde{M}|M) \to \mathrm{POW}(n)$ be as in § 3. In this section we suppose that $\operatorname{Im} H$ is generated by elements $f, f_1, \cdots, f_k$ such that for all $x \in \boldsymbol{R}^n$

$f(x) = rx$ where $0 < r < 1$,

$f_i(x) = r_i x + a_i$ where $r_i > 0$, $a_i \in \boldsymbol{R}^n$ for $i = 1, \cdots, k$.

Let S be the subspace spanned by $a_1, \cdots, a_k$ in $\boldsymbol{R}^n$. Furthermore, suppose that $\dim S < n$.

First we investigate the restriction $D|\tilde{M} - D^{-1}(S)$: $\tilde{M} - D^{-1}(S) \to \boldsymbol{R}^n - S$. Note that every element of $\operatorname{Im} H$ maps the space S to itself, a line perpendicular to S to another, and a sphere with the center in S to another.

Let K be a compact subset of $\tilde{M}$ with $p(K) = M$. Let

$$L = \operatorname{Max}\{d(D(y), S) \mid y \in K\}$$

where d is the usual metric in $\boldsymbol{R}^n$. Clearly $L > 0$. We can take a number $\varepsilon > 0$ with $0 < \varepsilon < L$ so that for each $y \in K$ there is a neighborhood U_y of y in $\tilde{M}$ such that $D|U_y$ is a diffeomorphism onto $U(D(y), \varepsilon)$.

LEMMA 2. (1) *For each $y \in \tilde{M} - D^{-1}(S)$ there is a neighborhood V_y of y in $\tilde{M} - D^{-1}(S)$ such that $D|V_y$ is a diffeomorphism onto $U(D(y), \varepsilon_y)$, where $\varepsilon_y = \varepsilon \cdot d(D(y), S)/L$.*

(2) *If $x \in D(\tilde{M}) - S$ then $U(x, \varepsilon_x) \subset D(\tilde{M})$ where $\varepsilon_x = \varepsilon \cdot d(x, S)/L$.*

PROOF. It is sufficient to prove (1). For each $y \in \tilde{M} - D^{-1}(S)$ there is $\alpha \in G(\tilde{M}|M)$ with $\alpha(y) \in K$. By the remark above concerning elements of $\operatorname{Im} H$, we see that $H(\alpha)$ maps spheres of radius s to those of radius $s \cdot d(D\alpha(y), S)/d(D(y), S)$. Since $D|U_{\alpha(y)}$ is a diffeomorphism onto $U(D\alpha(y), \varepsilon)$, the map $D|\alpha^{-1}U_{\alpha(y)}$ is a diffeomorphism onto $U(D(y), \varepsilon \cdot d(D(y), S)/d(D\alpha(y), S))$. Since $L \geqq d(D\alpha(y), S)$, the image of $D|\alpha^{-1}U_{\alpha(y)}$ contains $U(D(y), \varepsilon_y)$. Then $V_y = (D|\alpha^{-1}U_{\alpha(y)})^{-1}U(D(y), \varepsilon_y)$ will do.

LEMMA 3. *In the case $\dim S \leqq n - 2$ the map $D|\tilde{M} - D^{-1}(S)$ is a covering map onto $\boldsymbol{R}^n - S$. In the case $\dim S = n - 1$ the restriction of D to a connected component of $\tilde{M} - D^{-1}(S)$ is a covering map onto a connected component of $\boldsymbol{R}^n - S$.*

PROOF. We treat only the case $\dim S \leqq n - 2$, since the other case can be proved in a similar manner. First we show that the image of $D|\tilde{M} - D^{-1}(S)$ is $\boldsymbol{R}^n - S$. Since D is locally a diffeomorphism, the set $D(\tilde{M} - D^{-1}(S))$ is an open subset of $\boldsymbol{R}^n - S$. Let x be a point in the closure of $D(\tilde{M} - D^{-1}(S))$ with respect to the topology of $\boldsymbol{R}^n - S$. Choose $x' \in D(\tilde{M} - D^{-1}(S))$ with $||x' - x|| < \operatorname{Min}\{\varepsilon_x/2, d(x, S)/2\}$. Then it follows that $d(x', S) \geqq d(x, S) - ||x' - x|| > d(x, S)/2$ and $\varepsilon_{x'} = \varepsilon \cdot d(x', S)/L > \varepsilon \cdot d(x, S)/2L = \varepsilon_x/2$. By Lemma 2 (2) we see that $x \in U(x', \varepsilon_x/2) \subset U(x', \varepsilon_{x'}) \subset D(\tilde{M})$. Therefore $x \in D(\tilde{M} - D^{-1}(S))$, hence $D(\tilde{M} - D^{-1}(S))$ is closed in $\boldsymbol{R}^n - S$, too.

Since $\boldsymbol{R}^n - S$ is connected, it follows that $D(\tilde{M} - D^{-1}(S)) = \boldsymbol{R}^n - S$.

Secondly we show that each point $x \in \boldsymbol{R}^n - S$ has an evenly covered neighborhood. Let $\delta_x = \text{Min}\{\varepsilon_x/4, d(x, S)/2\}$. We look at the neighborhood $U(x, \delta_x)$ of x. Let $y \in D^{-1}U(x, \delta_x)$. By Lemma 2 (1) the map $D|V_y$ is a diffeomorphism onto $U(D(y), \varepsilon_y)$. Since $d(x, D(y)) < \delta_x \leqq \varepsilon_y/2$, it follows that $U(D(y), \varepsilon_y) \supset U(x, \delta_x)$. Furthermore, we see that the connected component of $D^{-1}U(x, \delta_x)$ containing y is $(D|V_y)^{-1}U(x, \delta_x)$. Therefore $U(x, \delta_x)$ is evenly covered by $D|\tilde{M} - D^{-1}(S)$. Hence $D|\tilde{M} - D^{-1}(S)$ is a covering map.

LEMMA 4. (I) *Suppose that* $D^{-1}(S) \neq \varnothing$. *Then the map* $D: \tilde{M} \to \boldsymbol{R}^n$ *is a diffeomorphism.*

(II) *Suppose that* $D^{-1}(S) = \varnothing$. *In the case* $\dim S = n - 1$ *the map* D *is a diffeomorphism onto a connected component of* $\boldsymbol{R}^n - S$. *In the case* $\dim S = n - 2$ *the map* $D: \tilde{M} \to \boldsymbol{R}^n - S$ *is a universal covering map. In the case* $\dim S \leqq n - 3$ *the map* $D: \tilde{M} \to \boldsymbol{R}^n - S$ *is a diffeomorphism.*

The proof of Lemma 4 is easy and we omit it. Now we examine each case of Lemma 4.

LEMMA 5. *The case* (I) *of Lemma* 4 *does not occur.*

PROOF. Suppose that it does. Then $H: G(\tilde{M}|M) \to \text{POW}(n)$ is a monomorphism and M is homeomorphic to the quotient space $\boldsymbol{R}^n/\text{Im}\, H$. Take a point $x \in \boldsymbol{R}^n - S$. Then the orbit $(\text{Im}\, H)\cdot x$ contains $x_n = r^n x$ for $n = 1, 2, \cdots$. Since the limit of the sequence $x_1, x_2, \cdots$ is $0 \notin (\text{Im}\, H)\cdot x$, the space $\boldsymbol{R}^n/\text{Im}\, H$ is not Hausdorff, a contradiction.

LEMMA 6. *The case* (II) *of Lemma* 4 *does not occur, unless* $\dim S = 0$.

PROOF. Suppose that it does when $\dim S > 0$. It follows that $a_j \neq 0$ for some j. Consider the case $\dim S \leqq n-3$. Then the projection p': $\boldsymbol{R}^n - S \to \boldsymbol{R}^n - S/\text{Im}\, H$ is a covering map. Take a point $x \in \boldsymbol{R}^n - S$. Then the orbit $(\text{Im}\, H)\cdot x$ contains $x_n = f^n f_j f^{-n}(x) = r_j x + r^n a_j$ for $n = 1, 2, \cdots$. Since the sequence $x_1, x_2, \cdots$ converges in $\boldsymbol{R}^n - S$, the projection p' cannot be a covering map, a contradiction.

For the case $\dim S = n - 1$ an argument similar to the one above is valid and we have a contradiction.

Consider the case $\dim S = n - 2$. We define a universal covering map $q: S \times \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}^n - S$, which can be regarded as a generalization of "q" in Example 4, as follows. Choose vectors $u, v \in \boldsymbol{R}^n$ perpendicular to S with $||u|| = ||v||$ and $u \cdot v = 0$. Let

$$q(x, t, \theta) = x + t \cdot \cos 2\pi\theta \cdot u + t \cdot \sin 2\pi\theta \cdot v$$

for all $(x, t, \theta) \in S \times \boldsymbol{R}_+ \times \boldsymbol{R}$. Since $D: \tilde{M} \to \boldsymbol{R}^n - S$ is also a universal covering map, there is a homeomorphism $h: \tilde{M} \to S \times \boldsymbol{R}_+ \times \boldsymbol{R}$ with $D = q \circ h$. It follows that $p \circ h^{-1}: S \times \boldsymbol{R}_+ \times \boldsymbol{R} \to M$ is a covering map. It is easy to see that the covering transformation group $G(S \times \boldsymbol{R}_+ \times \boldsymbol{R} | M)$ contains maps $\tilde{f}, \tilde{f}_1, \cdots, \tilde{f}_k$ having the form

$$f(x, t, \theta) = (rx, rt, \theta + \nu) ,$$
$$f_i(x, t, \theta) = (r_i x + a_i, r_i t, \theta + \nu_i) ,$$

where $\nu, \nu_1, \cdots, \nu_k$ are integers. For $t > 0$ the orbit $G(S \times \boldsymbol{R}_+ \times \boldsymbol{R} | M) \cdot (0, t, 0)$ contains $x_n = \tilde{f}^n \tilde{f}_j \tilde{f}^{-n}(0, t, 0) = (r^n a_j, r_j t, \nu_j)$ for $n = 1, 2, \cdots$. Since the sequence $x_1, x_2, \cdots$ converges, the map $p \circ h^{-1}$ cannot be a covering map. This contradiction completes the proof of Lemma 6.

By Lemmas 5 and 6 we see that $D^{-1}(S) = \varnothing$ and $\dim S = 0$. Now we can determine $\mathscr{W}$.

LEMMA 7. *In the case* $n = \dim M \geqq 3$ *the web* $\mathscr{W}$ *is isomorphic to* $\mathscr{W}(r)$ *for some* $r \in B(n) = (0, 1)$ *and the manifold* M *is diffeomorphic to* $S^1 \times S^{n-1}$.

PROOF. In this case $D: \tilde{M} \to \boldsymbol{R}^n - \{0\}$ is a diffeomorphism, the map $p \circ D^{-1}: \boldsymbol{R}^n - \{0\} \to M$ is a covering map and $(p \circ D^{-1})^* \mathscr{W} = \mathscr{W}^0 | \boldsymbol{R}^n - \{0\}$. Every diffeomorphism $g: \boldsymbol{R}^n - \{0\} \to \boldsymbol{R}^n - \{0\}$ preserving $\mathscr{W}^0 | \boldsymbol{R}^n - \{0\}$ has the form $g(x) = sx$ for all $x \in \boldsymbol{R}^n - \{0\}$, where $s \in \boldsymbol{R}_+$ if g preserves a transverse orientation of some foliation in $\mathscr{W}^0 | \boldsymbol{R}^n - \{0\}$. Therefore the covering transformation group $G(\boldsymbol{R}^n - \{0\} | M)$ is a cyclic group generated by such a map g. Then $\mathscr{W}$ is isomorphic to $\mathscr{W}(s)$, and we may assume that $0 < s < 1$ as in Example 3. Clearly M is diffeomorphic to $S^1 \times S^{n-1}$.

LEMMA 8. *In the case* $n = \dim M = 2$ *the web* $\mathscr{W}$ *is isomorphic to* $\mathscr{W}(s_1, s_2, d)$ *for some* $(s_1, s_2, d) \in B(2)$ *and the manifold* M *is diffeomorphic to* $S^1 \times S^1$.

PROOF. Clearly M is diffeomorphic to $S^1 \times S^1$. As in the case $\dim S = n - 2$ in the proof of Lemma 6, we consider the covering map $p \circ h^{-1}: \boldsymbol{R}_+ \times \boldsymbol{R} \to M$. Note that $(p \circ h^{-1})^* \mathscr{W} = q^*(\mathscr{W}^0 | \boldsymbol{R}^2 - \{0\})$, where $q: \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}^2 - \{0\}$ is the universal covering map defined in Example 4. Every diffeomorphism $h: \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}_+ \times \boldsymbol{R}$ preserving $q^*(\mathscr{W}^0 | \boldsymbol{R}^2 - \{0\})$ has the form $h(t, \theta) = (st, \theta + \mu)$ for all $(t, \theta) \in \boldsymbol{R}_+ \times \boldsymbol{R}$ where $s \in \boldsymbol{R}_+$ and $\mu \in \boldsymbol{Z}$. The group $G(\boldsymbol{R}_+ \times \boldsymbol{R} | M)$ is generated by two elements, say h_1 and h_2 having the form $h_i(t, \theta) = (s_i t, \theta + \mu_i)$. Let $d > 0$ be the greatest common divisor of μ_1 and μ_2. Changing the generators h_i, we may assume that $\mu_1 = 0$ and $\mu_2 = d$. Then it follows that $s_1 \neq 1$. Therefore $\mathscr{W}$ is isomorphic to

$\mathscr{W}(s_1, s_2, d)$. Furthermore, we may assume that $(s_1, s_2, d) \in B(2)$ as in Example 4.

5. Completion of the proof of the theorems.

PROOF OF THEOREM 1. It remains to prove that if $W(\gamma_1) = W(\gamma_2)$ for $\gamma_1, \gamma_2 \in B(n)$ then $\gamma_1 = \gamma_2$.

First consider the case $n \geqq 3$. Let $r, s \in B(n) = (0, 1)$. Suppose that there is a homeomorphism $f: \boldsymbol{R}^n - \{0\}/m_r \to \boldsymbol{R}^n - \{0\}/m_s$ with $\mathscr{W}(r) = f^*\mathscr{W}(s)$. By the theory of covering maps we can find a homeomorphism $\tilde{f}: \boldsymbol{R}^n - \{0\} \to \boldsymbol{R}^n - \{0\}$ with $p_s \circ \tilde{f} = f \circ p_r$, where p_r and p_s are the projections to $\boldsymbol{R}^n - \{0\}/m_r$ and $\boldsymbol{R}^n - \{0\}/m_s$, respectively. Since $\tilde{f}^*(\mathscr{W}^0 | \boldsymbol{R}^n - \{0\}) = \tilde{f}^* p_s^* \mathscr{W}(s) = p_r^* f^* \mathscr{W}(s) = p_r^* \mathscr{W}(r) = \mathscr{W}^0 | \boldsymbol{R}^n - \{0\}$, the map f has the form $f(x) = tx$ for all $x \in \boldsymbol{R}^n - \{0\}$, where $t \neq 0$. Furthemore, it follows that $m_s = f m_r f^{-1} = m_r$. Therefore $r = s$.

Now consider the case $n = 2$. Let $(r_1, s_1, d_1), (r_2, s_2, d_2) \in B(2)$. Suppose that there is a homeomorphism $h: \boldsymbol{R}_+ \times \boldsymbol{R}/\{f_1, g_1\} \to \boldsymbol{R}_+ \times \boldsymbol{R}/\{f_2, g_2\}$ with $\mathscr{W}(r_1, s_1, d_1) = h^*\mathscr{W}(r_2, s_2, d_2)$ where f_i and g_i are the maps defined by means of r_i, s_i and d_i as in Example 4. As above we can find a homeomorphism $\tilde{h}: \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}_+ \times \boldsymbol{R}$ with $p_2 \circ \tilde{h} = h \circ p_1$, where each p_i is the projection to $\boldsymbol{R}_+ \times \boldsymbol{R}/\{f_i, g_i\}$. Since $\tilde{h}^* q^*(\mathscr{W}^0 | \boldsymbol{R}^2 - \{0\}) = \tilde{h}^* p_2^* \mathscr{W}(r_2, s_2, d_2) = p_1^* h^* \mathscr{W}(r_2, s_2, d_2) = p_1^* \mathscr{W}(r_1, s_1, d_1) = q^*(\mathscr{W}^0 | \boldsymbol{R}^2 - \{0\})$, where $q: \boldsymbol{R}_+ \times \boldsymbol{R} \to \boldsymbol{R}^2 - \{0\}$ is the covering map in Example 4, the map $\tilde{h}$ has the form $\tilde{h}(t, \theta) = (rt, \theta + \nu)$ for all $(t, \theta) \in \boldsymbol{R}_+ \times \boldsymbol{R}$, where $r \in \boldsymbol{R} - \{0\}$ and $\nu \in \boldsymbol{Z}$. By an easy computation we see that $\tilde{h} f_2 \tilde{h}^{-1} = f_2$ and $\tilde{h} g_2 \tilde{h}^{-1} = g_2$. Hence the group generated by f_1 and g_1 is equal to the one generated by f_2 and g_2. Considering the forms of f_i and g_i, we conclude that $f_1 = f_2$ and $g_1 = g_2$. Therefore $(r_1, s_1, d_1) = (r_2, s_2, d_2)$ and we are done.

PROOF OF THEOREM 2. Since M has a codimension-one C^r foliation, the Euler number of M is zero. By Proposition 1 (4) it follows that M is orientable. Therefore M is C^r diffeomorphic to $S^1 \times S^1$. Now consider a developing map $D: \tilde{M} \to \boldsymbol{R}^n$ and the total holonomy homomorphism $H: G(\tilde{M} | M) \to \mathrm{POW}(n)$ associated with D. Since $G(\tilde{M} | M)$ is isomorphic to $\boldsymbol{Z} \oplus \boldsymbol{Z}$, the total holonomy group $\mathrm{Im}\, H$ is abelian. When $\mathrm{Im}\, H$ consists of parallel translations, the web $\mathscr{W}$ is C^r isomorphic to $\mathscr{W}(\Gamma)$ for some lattice group Γ by the argument in § 3. When $\mathrm{Im}\, H$ contains $f = f_{r,a}$ for some $r \in \boldsymbol{R}_+ - \{1\}$ and $a \in \boldsymbol{R}^n$, we may suppose that $a = 0$ by changing D, if necessary. Let $g \in \mathrm{Im}\, H$. Since $f \circ g = g \circ f$, it follows that $g(0) = 0$. Therefore $g = f_{s,0}$ for some $s \in \boldsymbol{R}_+$. Take elements $f_1, f_2 \in \mathrm{Im}\, H$ so that f, f_1 and f_2 generate $\mathrm{Im}\, H$. Then the subspace S defined by f_1, f_2 as in § 4 is $\{0\}$. It follows that $\mathscr{W}$ is C^r isomorphic to $\mathscr{W}(r, s, d)$

for some $(r, s, d) \in B(2)$ by the argument in § 4.

PROOF OF THEOREM 3. Note that $\pi_1(M)$ is isomorphic to $G(\tilde{M}|M)$. Consider the case (1) of Theorem 3. We need the following theorem of Inaba [3].

THEOREM (Inaba). *Let M be a closed orientable connected analytic 3-manifold such that $\pi_1(M)$ has non-exponential growth. If M has a transversely orientable analytic foliation of codimension one, then M is diffeomorphic to $S^1 \times S^2$ or to a torus bundle over S^1.*

By the theorem of Inaba and Corollary 1, the case (1) can be reduced to the case (3), since the fundamental group of a torus bundle over S^1 is generated by 3 elements.

Now consider the case (2). As in the proof of Theorem 2, when a total holonomy group of $\mathscr{W}$ consists of parallel translations, we see that $\mathscr{W}$ is C^r isomorphic to $\mathscr{W}(\Gamma)$ for some $\Gamma \in A(n)$. In the other case we can apply the argument in the case $\dim S = 0$ of § 4. We see that $\mathscr{W}$ is C^r isomorphic to $\mathscr{W}(r)$ for some $r \in B(n)$.

Finally consider the case (3). Suppose that a total holonomy group of $\mathscr{W}$ contains $f_{r,a}$ for some $r \in \boldsymbol{R}_+ - \{1\}$ and $a \in \boldsymbol{R}^n$. Then we can apply the argument in § 4, because $\dim S \leqq n-1$ in such a case. As above we can apply the argument in § 3 in the other case. Thus the proof of Theorem 3 is completed.

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