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A MODIFICATION OF TEICHMÜLLER'S MODULE THEOREM AND ITS APPLICATION TO A DISTORTION PROBLEM IN n-SPACE

KAZUO IKOMA

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1. A distortion theorem of Hölder type for certain quasiconformal mappings of the unit disc onto itself was considered first by Lavrentieff, Ahlfors, and then the best estimate (Theorem B) was established by Mori [4] who used his module theorem as a tool. Afterwards, Lehto-Virtanen [3] showed a modification (Theorem A) of Teichmüller's module theorem which implies Mori's module theorem, and presented an alternative proof of Theorem B by applying Theorem A.

THEOREM A. *If a ring R separates z_1 and z_2 from 0 and ∞ in the complex plane, then*

$$\operatorname{mod} R \leqq \log \Phi_2(\{2(|z_1| + |z_2|)\}^{1/2}/|\sqrt{z_1} - \sqrt{z_2}|) ,$$

where $\log \Phi_2(a)$ *denotes the module of the plane Grötzsch ring and* $\sqrt{z_1}$, $\sqrt{z_2}$ *belong to the same branch of the square root, single valued in* R.

THEOREM B. *Let w be a K-quasiconformal mapping of the unit disc onto itself, normalized by $w(0) = 0$. Then, for every pair of points z_1, z_2 with $|z_1| \leqq 1$, $|z_2| \leqq 1$, we have*

$$|w(z_1) - w(z_2)| \leqq 16 |z_1 - z_2|^{1/K} ,$$

where 16 *cannot be replaced by any smaller number if the inequality is to hold for all K.*

2. Since for $n \geqq 3$ there is no 1-quasiconformal mapping in the n-dimensional case corresponding to analytic branches $w = \pm\sqrt{z}$ used in the proof of Theorem A, we used previously two branches $y = y_+(x)$ and $y_-(x)$: $y_1 = r\cos(\theta/2)$, $y_2 = r\sin(\theta/2)$, $y_j = x_j$ $(3 \leqq j \leqq n)$ for $-\pi \leqq \theta < \pi$ and $\pi \leqq \theta < 3\pi$, respectively, which are called foldings and are 2-quasiconformal. And we deduced an estimate for the module of a ring in n-space corresponding to Theorem A. This estimate means a modification of Teichmüller's module theorem in n-space. Then, it follows that the estimate obtained by using such a modification for certain K-quasiconformal mappings in n-space corresponding to Theorem B has the exponent $1/2K$ (see [2]).

The main purpose of this paper is to improve on the exponent in the latter estimate. That is to say, we establish, as Theorem 1, another modification of Teichmüller's module theorem under an additional condition that the unbounded component of the complement of the ring in n-space contains the ball $B^n(0, r_0)$ with certain radius r_0 centered at the origin, and, as its application, we obtain in Theorem 2 an estimate, where the exponent can be taken to be $1/K$, for certain K-quasiconformal mappings in n-space corresponding to Theorem B.

THEOREM 1. *Suppose that a ring R in n-space separates a pair of points α and β from the origin and the point at infinity, and that the unbounded component of the complement of R contains the ball $\{x \mid |x| \leqq r_0\}$ for certain positive number r_0. Then, we have*

$$\operatorname{mod} R \leqq \log \Phi_n(\{2\{|\alpha|^2(|\beta| + r_0)^2 + |\beta|^2(|\alpha| + r_0)^2\}\}^{1/2}/r_0|\alpha - \beta|) ,$$

where $\log \Phi_n(a)$ *denotes the module of the Grötzsch ring in n-space.*

THEOREM 2. *Let y be an n-dimensional K-quasiconformal mapping of the unit ball onto itself normalized by $y(0) = 0$. Then, for every pair of points α and β with $|\alpha| \leqq 1$, $|\beta| \leqq 1$, we have*

$$|y(\alpha) - y(\beta)| \leqq c|\alpha - \beta|^{1/K} ,$$

where $c = 2^{1+1/K}(1 + 1/\rho_0)\lambda_n$, and λ_n is such a bound that $\Phi_n(a) \leqq \lambda_n a$, $\rho_0 = 1/\Phi_n^{-1}[\{\Phi_n(4)\}^K]$, and Φ_n^{-1} is the inverse function of Φ_n.

REMARK 1. It should be noted that the exponent $1/K$ in Theorem 2 cannot be replaced by any larger number. Because, if we consider the following K-quasiconformal mapping $y = y_0(r, \theta_1, \cdots, \theta_{n-1})$:

$$\begin{aligned} y_1 &= r^{1/K} \cos \theta_1 , \\ y_j &= r^{1/K} \sin \theta_1 \cdots \sin \theta_{j-1} \cos \theta_j \qquad (j = 2, 3, \cdots, n-1) \\ y_n &= r^{1/K} \sin \theta_1 \cdots \sin \theta_{n-2} \sin \theta_{n-1} , \end{aligned}$$

then there exists such a point α that $|y_0(\alpha) - y_0(0)| > c|\alpha|^t$, $|\alpha| < 1$ for each constant c provided that t is larger than $1/K$.

3. As regards the definition of the module of a ring in the case of dimension $n \geqq 3$ and its fundamental properties, i.e., the superadditivity of the module, Grötzsch's and Teichmüller's module theorems, and the definition of a K-quasiconformal mapping and its fundamental properties, we refer the reader to Mostow [5] and Väisälä [6]. We here note only that a K-quasiconformal mapping in this paper is equivalent to a K^{n-1}-quasiconformal one in the sense of Väisälä.

4. Proof of Theorem 1. We may assume, without loss of generality, that the point α lies on the positive x_1-axis, since this can be obtained by a suitable rotation around the origin and the inequality to be established is invariant with respect to such rotations. Now, let $\alpha = (a, 0, \cdots, 0)$ $(a > 0)$ and $\beta = (b_1, \cdots, b_n)$.

First, let us map the ring R into the ball $B^n(0, r_0)$ by the inversion $\xi = f_1(x) = r_0^2 x/|x|^2$ with respect to the sphere $S^{n-1}(0, r_0)$. Denote by α', β' the images of α, β, under $\xi = f_1(x)$, respectively. Then $\alpha' = (r_0^2/a, 0, \cdots, 0)$ and $\beta' = (r_0 b_1/|\beta|^2, \cdots, r_0 b_n/|\beta|^2)$.

Next, let us map the ball $B^n(0, r_0)$ onto the half space $\{\eta \mid \eta_1 \geqq r_0\}$ by the inversion $\eta = f_2(\xi) = -r_0 e_1 + 4r_0^2(\xi + r_0 e_1)/|\xi + r_0 e_1|^2$ with respect to the sphere $S^{n-1}(-r_0 e_1, 2r_0)$, where $e_1 = (1, 0, \cdots, 0)$. Let α'', β'' be the images of α', β', respectively, by $\eta = f_2(\xi)$. Then we have $\alpha'' = (-r_0 + 4r_0/(r_0/a + 1), 0, \cdots, 0)$, $\beta'' = (-r_0 + 4(|\beta|^2 + r_0 b_1)r_0/|\beta + r_0 e_1|^2, 4r_0^2 b_j/|\beta + r_0 e_1|^2)$ $(2 \leqq j \leqq n)$.

Finally, let α''', β''' be the images of α'', β'' by the translation $\zeta = f_3(\eta) = \eta + r_0 e_1$. Then

$$(1)\quad \begin{cases} \alpha''' = (4r_0/(r_0/a + 1), 0, \cdots, 0)\ , \\ \beta''' = (4(|\beta|^2 + r_0 b_1)r_0/|\beta + r_0 e_1|^2, 4r_0^2 b_j/|\beta + r_0 e_1|^2) \qquad (2 \leqq j \leqq n)\ . \end{cases}$$

We set hereafter $4r_0/(r_0/a + 1)$ as α_1''' for simplicity sake. Then, the ring R is mapped onto the ring R' in the half space $\{\zeta \mid \zeta_1 \geqq 2r_0\}$ by the composite mapping of $\xi = f_1(x)$, $\eta = f_2(\xi)$, $\zeta = f_3(\eta)$. Let R'' be the ring symmetric to R' with respect to the hyperplane $\{\zeta \mid \zeta_1 = 0\}$. Then, $\mathrm{mod}\, R = \mathrm{mod}\, R' = \mathrm{mod}\, R''$.

Now, denote by C_0', C_0'' the bounded components of complements of R', R'', respectively, and let R_0 be the ring with C_0', C_0'' as its complementary components. Then, R' and R'' are disjoint rings each of which separates the boundary components of R_0, and hence we have $\mathrm{mod}\, R' + \mathrm{mod}\, R'' \leqq \mathrm{mod}\, R_0$ by the superadditivity of the module of a ring, so that

$$(2)\qquad \mathrm{mod}\, R \leqq (1/2)\, \mathrm{mod}\, R_0\ .$$

Put $\alpha_+''' = \alpha'''$ and $\beta_+''' = \beta'''$, and let α_-''', β_-''' be the points symmetric to α_+'', β_+'', respectively, with respect to the hyperplane $\{\zeta \mid \zeta_1 = 0\}$. Then $\alpha_-''' = (-4r_0/(r_0/a + 1), 0, \cdots, 0)$, $\beta_-''' = (-4(|\beta|^2 + r_0 b_1)r_0/|\beta + r_0 e_1|^2, 4r_0^2 b_j/|\beta + r_0 e_1|^2)$ $(2 \leqq j \leqq n)$, which belong to the unbounded component C_1' of the complement of R'.

Here, we consider the auxiliary Möbius transformation

$$\begin{cases} y_1 = \left\{\sum_{j=1}^{n} \zeta_j^2 - \alpha_1'''^2\right\} \Big/ \left\{(\zeta_1 + \alpha_1'''^2)^2 + \sum_{j=2}^{n} \zeta_j^2\right\} \\ y_j = 2\alpha_1'''\zeta_j \Big/ \left\{(\zeta_1 + \alpha_1''')^2 + \sum_{j=2}^{n} \zeta_j^2\right\} \qquad (2 \leqq j \leqq n)\ , \end{cases}$$

which maps R_0 onto a ring $\tilde{R}_0$ and carries α_+''', α_-''' into the origin and the point at infinity, respectively. Denote by $\tilde{\beta}_+$, $\tilde{\beta}_-$ the images of β_+''', β_-''', respectively, under the above Möbius transformation. Then the ring $\tilde{R}_0$ separates the origin and $\tilde{\beta}_+$ from the point at infinity and $\tilde{\beta}_-$. Thus, we can apply Teichmüller's module theorem in n-space to the ring $\tilde{R}_0$ to obtain the estimate

$$(3) \qquad \mathrm{mod}\, R_0 = \mathrm{mod}\, \tilde{R}_0 \leqq 2 \log \Phi_n(\{(|\tilde{\beta}_+| + |\tilde{\beta}_-|)/|\tilde{\beta}_+|\}^{1/2}) .$$

Since $\tilde{\beta}_+ = (\{\sum_{j=1}^n \beta_j'''^2 - \alpha_1'''^2\}/\{(\alpha_1''' + \beta_1''')^2 + \sum_{j=2}^n \beta_j'''^2\}, 2\alpha_1'''\beta_j'''/\{(\alpha_1''' + \beta_1''')^2 + \sum_{j=2}^n \beta_j'''^2\})$ and $\tilde{\beta}_- = (\{\sum_{j=1}^n \beta_j'''^2 - \alpha_1'''^2\}/\{(\alpha_1''' - \beta_1''')^2 + \sum_{j=2}^n \beta_j'''^2\}, 2\alpha_1'''\beta_j'''/\{(\alpha_1''' - \beta_1''')^2 + \sum_{j=2}^n \beta_j'''^2\})$ for $j = 2, 3, \cdots, n$, we have, after some elementary computations,

$$(4) \qquad (|\tilde{\beta}_+| + |\tilde{\beta}_-|)/|\tilde{\beta}_+| = 2(|\alpha'''|^2 + |\beta'''|^2)/|\alpha''' - \beta'''|^2 .$$

Substituting (1) into the right hand side and continuing elementary computations, we obtain

$$\begin{aligned} |\alpha'''|^2 + |\beta'''|^2 &= (4r_0)^2\{|\alpha|^2|\beta + r_0e_1|^2 + |\beta|^2(|\alpha| + r_0)^2\}/(|\alpha| + r_0)^2|\beta + r_0e_1|^2 \\ &\leqq (4r_0)^2\{|\alpha|^2(|\beta| + r_0)^2 + |\beta|^2(|\alpha| + r_0)^2\}/(|\alpha| + r_0)^2|\beta + r_0e_1|^2 \end{aligned}$$

and $|\alpha''' - \beta'''|^2 = (4r_0^2)^2|\alpha - \beta|^2/(|\alpha| + r_0)^2|\beta + r_0e_1|^2$. Consequently, we have from (3) and (4) into which these two relations are substituted,

$$\mathrm{mod}\, R_0 \leqq 2 \log \Phi_n(\{2\{|\alpha|^2(|\beta| + r_0)^2 + |\beta|^2(|\alpha| + r_0)^2\}\}^{1/2}/r_0|\alpha - \beta|) .$$

Combining it with the preceding (2), the statement of the theorem follows immediately.

5. We need the following two lemmas together with Theorem 1 for the proof of Theorem 2.

LEMMA 1. *For $n \geqq 2$, $\Phi_n(a) \leqq \lambda_n a$ and $4 \leqq \lambda_n$.*

The upper bound for λ_n is omitted since it is not used immediately (see [1]).

REMARK 2. It is well known that $\Phi_n(a)$ is increasing and is continuous for $a > 1$ (see, for instance, Mostow [5, Sections 6 and 7]).

LEMMA 2. (A space analogue of Schwarz's lemma). *Let $y = f(x)$ be a K-quasiconformal mapping of $|x| < 1$ onto $|y| < 1$ in n-space normalized by $f(0) = 0$. Then, for $0 < |x| < 1$, we have*

$$\Phi_n(1/|f(x)|) \leqq \{\Phi_n(1/|x|)\}^K .$$

PROOF. Let R_x be the ring obtained by deleting from the unit ball $|x| < 1$ the segment connecting the point x to the origin, and let R_y be the ring, in y-space, similar to R_x. Then,

$$\begin{cases} \operatorname{mod} R_x = \log \Phi_n(1/|x|) , \\ \operatorname{mod} R_y = \log \Phi_n(1/|y|) . \end{cases} \tag{5}$$

It is well known that the inverse mapping $f^{-1}(y)$ of $f(x)$ is also K-quasiconformal. Denote by $f^{-1}(R_y)$ the image of R_y under $f^{-1}(y)$. Then we have $\operatorname{mod} f^{-1}(R_y) \leqq \operatorname{mod} R_x$ by Grötzsch's module theorem in n-space which is deduced by means of the spherical symmetrization (see Mostow [5, Sect. 8]). This together with the characterization of the K-quasiconformality $(1/K) \operatorname{mod} R_y \leqq \operatorname{mod} f^{-1}(R_y)$ yields

$$(1/K) \operatorname{mod} R_y \leqq \operatorname{mod} R_x .$$

Taking (5) into account, we have the desired relation.

REMARK 3. It follows from Lemma 2 that on $|x| = r_0$, $0 < r_0 < 1$, we have $|f(x)| \geqq 1/\Phi_n^{-1}[\{\Phi_n(1/r_0)\}^K]$. Since $1/\Phi_n^{-1}[\{\Phi_n(1/|x|)\}^K]$ is an increasing and continuous function in $|x|$, Lemma 2 and Remark 2 imply that the image of the ball $|x| \leqq r_0$ under $y = f(x)$ contains the ball $\{y \mid |y| \leqq 1/\Phi_n^{-1}[\{\Phi_n(1/r_0)\}^K]\}$.

6. Proof of Theorem 2. Since $|y(\alpha) - y(\beta)| \leqq |y(\alpha)| + |y(\beta)| \leqq 2$, it follows that for $|\alpha - \beta| \geqq 1/\lambda_n$,

$$|y(\alpha) - y(\beta)| \leqq 2 < c \leqq c\lambda_n(1/\lambda_n)^{1/K} \leqq c\lambda_n|\alpha - \beta|^{1/K} .$$

The theorem is trivial for $|\alpha - \beta| = 0$, and so it suffices to prove it for $0 < |\alpha - \beta| < 1/\lambda_n$. For that purpose, we consider the following two cases: (i) $|\alpha + \beta| \leqq 1$ and (ii) $|\alpha + \beta| > 1$. Note that $|\alpha - \beta|/2 < 1/2\lambda_n \leqq 1/8$ by Lemma 1.

(i) *The case* $|\alpha + \beta| \leqq 1$. Consider the spherical ring $A = \{x \mid |\alpha - \beta|/2 < |x - (\alpha + \beta)/2| < 1/2\}$. Then A is contained in the unit ball, hence so is the image $y(A)$ of A under $y(x)$. Therefore, $y(A)$ is contained in the ball $\{y \mid |y - y(\alpha)| < 2\}$. Hence one of the complementary components of $y(A)$ contains both $y(\alpha)$ and $y(\beta)$, and the other contains the outside of a ball $\{y \mid |y - y(\alpha)| \geqq 2\}$. Thus, by the monotonicity of the module of a ring and Grötzsch's module theorem in n-space, we have

$$\operatorname{mod} y(A) \leqq \log \Phi_n(2/|y(\alpha) - y(\beta)|) .$$

Taking into account the module condition of the K-quasiconformality $(1/K) \operatorname{mod} A = (1/K) \log (1/|\alpha - \beta|) \leqq \operatorname{mod} y(A)$, we have $1/|\alpha - \beta|^{1/K} \leqq \Phi_n(2/|y(\alpha) - y(\beta)|)$. By means of Lemma 1, we have $1/|\alpha - \beta|^{1/K} \leqq 2\lambda_n/|y(\alpha) - y(\beta)|$, from which it follows that

$$|y(\alpha) - y(\beta)| \leqq 2\lambda_n|\alpha - \beta|^{1/K} < c\lambda_n|\alpha - \beta|^{1/K} .$$

(ii) *The case* $|\alpha+\beta|>1$. Consider then the ring $B=\{x\,|\,|\alpha-\beta|/2<|x-(\alpha+\beta)/2|<1/4\}$. It is either completely contained in $|x|<1$ or not. In the latter case, consider the following mapping $y^*(x)$, instead of $y(x)$, defined by

$$y^*(x)=\begin{cases} y(x)\,, & |x|\leqq 1\,,\\ y(x/|x|^2)/|y(x/|x|^2)|^2\,, & |x|>1\,,\end{cases}$$

representing an extension of y also outside of the unit ball as a K-quasiconformal mapping.

The ring B separates a pair of points α, β from the origin and the point at infinity. On the other hand, the unbounded component $C_1(B)$ of the complement of B contains the ball $|x|\leqq 1/4$. Hence Lemma 2 and Remark 3 yield that the unbounded component of the complement $C_1(y^*(B))$ of the ring $y^*(B)$ contains the ball $|y|\leqq\rho_0$, where $\rho_0=1/\Phi_n^{-1}[\{\Phi_n(4)\}^K]$. Thus, the ring $y^*(B)$ separates a pair of points $y(\alpha)$, $y(\beta)$ from the origin and the point at infinity, and $C_1(y^*(B))$ contains the ball $|y|\leqq\rho_0$. Consequently, we have by Theorem 1, $\operatorname{mod} y^*(B)\leqq\log\Phi_n(\{2\{|y(\alpha)|^2(|y(\beta)|+\rho_0)^2+|y(\beta)|^2(|y(\alpha)|+\rho_0)^2\}\}^{1/2}/\rho_0|y(\alpha)-y(\beta)|)=\log\Phi_n(\{2\{|y(\alpha)|^2(|y(\beta)|/\rho_0+1)^2+|y(\beta)|^2(|y(\alpha)|/\rho_0+1)^2\}\}^{1/2}/|y(\alpha)-y(\beta)|)$. Hence, we have

$$\operatorname{mod} y^*(B)\leqq\log\Phi_n(2(1+1/\rho_0)/|y(\alpha)-y(\beta)|)\,.$$

Combining it with the module condition of the K-quasiconformality $(1/K)\operatorname{mod} B\leqq\operatorname{mod} y^*(B)$, we obtain $(1/2|\alpha-\beta|)^{1/K}\leqq\Phi_n(2(1+1/\rho_0)/|y(\alpha)-y(\beta)|$. By virtue of Lemma 1, we have $(1/2|\alpha-\beta|)^{1/K}\leqq 2(1+1/\rho_0)\lambda_n/|y(\alpha)-y(\beta)|$, from which it follows that

$$|y(\alpha)-y(\beta)|\leqq 2(1+1/\rho_0)\lambda_n 2^{1/K}|\alpha-\beta|^{1/K}=c|\alpha-\beta|^{1/K}$$

as desired.

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DEPARTMENT OF MATHEMATICS
YAMAGATA UNIVERSITY
YAMAGATA, 990 JAPAN