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THE VIRTUAL SINGULARITY THEOREM AND THE LOGARITHMIC BIGENUS THEOREM

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Introduction. When we study non-singular algebraic varieties V defined over $\boldsymbol{C}$ the field of complex numbers, it is very important to know the *logarithmic Kodaira dimension* $\bar{\kappa}(V)$ of them V. In order to compute $\bar{\kappa}(V)$ of a non-singular algebraic variety V, we have to find a complete non-singular algebraic variety $\bar{V}^*$ and a divisor $D^{\#}$ with normal crossings on $\bar{V}^*$, such that $V=\bar{V}^*-D^{\#}$. Then by definition, $\bar{\kappa}(V)=\kappa(K(\bar{V}^*)+D^{\#}, \bar{V}^*)$. Here $\kappa(X, \bar{V})$ denotes the *X-dimension* of $\bar{V}$ (see [1]).

Occasionally, V is given as a complement of a reduced divisor D on a complete non-singular algebraic variety $\bar{V}$. In practice, it is very laborious to transform D into $D^{\#}$ with normal crossings by a finite succession of blowing ups with non-singular centers. However, in general,

$$\bar{\kappa}(V) \leqq \kappa(K(\bar{V})+D,\ \bar{V})\ .$$

In many examples, we observe that the equality above holds actually. In such a case, we say that *the virtual singularity theorem holds for the pair* $(\bar{V}, D)$. For example, when D has only normal crossings, the virtual singularity theorem holds by definition. If $\kappa(\bar{V}) \geqq 0$, the virtual singularity theorem holds with any effective divisor D. In this case, however, the strong virtual singularity theorem will be proved in Theorem 1. Moreover, even if $\bar{V}$ is a non-singular non-rational ruled surface, we can prove the virtual singularity theorem for $(\bar{V}, D)$ in Theorem 2.

On the other hand, when $\bar{V}$ is a rational surface (which is always assumed to be non-singular), the virtual singularity theorem does not hold in general. But even in this case, if D has very bad singularities, we have the virtual singularity theorem (Theorem 4). This is a generalization of a theorem of Wakabayashi [10].

THEOREM (Wakabayashi). *Let C be an irreducible curve of degree d in $\boldsymbol{P}^2$.*

(1) *If C is not rational and $d \geqq 4$, or*

(2) *if C is a rational curve which has at least two singular*

points such that one of those points is not a cusp, or

(3) *if C is a rational curve with at least three cusps, then $\bar{\kappa}(\boldsymbol{P}^2 - C) = 2$, i.e., $\boldsymbol{P}^2 - C$ is an algebraic surface of hyperbolic type (or, as Mumford calls it, logarithmic general type).*

Furthermore, if C is a rational curve with at least two cusps, then $\bar{\kappa}(\boldsymbol{P}^2 - C) \geqq 0$.

REMARK. The above theorem is reformulated as (i) $\bar{\kappa}(\boldsymbol{P}^2 - C) \geqq \kappa^{\sharp}(C)$, and (ii) $\kappa^{\sharp}(C) = 1$ implies that $\bar{\kappa}(\boldsymbol{P}^2 - C) = 2$ or C is a rational curve with only one singular point. Here, $\kappa^{\sharp}(W)$ denotes the *singular Kodaira dimension* of W, which is defined to be $\bar{\kappa}(\mathrm{Reg}\ W)$.

The latter part of Wakabayashi's theorem is extended to the "Bigenus theorem" (Theorem 3).

Trigenus theorem and Kodaira dimension of graphs of the third kind will be discussed in a forthcoming paper.

Finally, we make the following

CONJECTURE. *Let $\bar{V}$ be a complete non-singular rational variety and W a subvariety of codimension 1 of $\bar{V}$.*

(1) *If $\kappa^{\sharp}(W) \geqq 0$, then $\bar{\kappa}(\bar{V} - W) \geqq 0$,*

(2) *If $\kappa^{\sharp}(W) = n - 1$, then $\bar{\kappa}(\bar{V} - W) \geqq n - 1$.*

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1. Let V be a non-singular algebraic variety and let $(\bar{V}, B)$ be a ∂-manifold whose interior is V, i.e., $\bar{V}$ is a non-singular complete algebraic variety and B is a divisor with normal crossings such that $V = \bar{V} - B$. Now let D be a reduced divisor on V and denote by $\bar{D}$ the closure of D in $\bar{V}$. We choose a proper birational morphism $\rho\colon \bar{V}^* \to \bar{V}$ such that $\rho^{-1}(B + \bar{D})$ has only normal crossings with $\bar{V}^*$ being non-singular. Define V^* to be $\rho^{-1}(V)$, and D^* to be the proper transform of D by $\mu = \rho | V^*$. If the equality:

$$\bar{\kappa}(V^* - D^*) = \kappa(K(\bar{V}) + B + \bar{D},\ \bar{V})$$

holds, we say that *the strong virtual singularity theorem holds for the pair (V, D).*

THEOREM 1. *Suppose that $\bar{\kappa}(V) \geqq 0$. Then the strong virtual singularity theorem holds for the pair (V, D).*

This was proved in [2]. But for the convenience of the reader, we give a sketch of the proof here. We use the above notation. By hypothesis, $\bar{\kappa}(V) = \kappa(K(\bar{V}) + B,\ \bar{V}) = \kappa(K(\bar{V}^*) + \rho^{-1}(B),\ \bar{V}^*) \geqq 0$. Hence, denoting by $D^{\#}$ the closure of D^* in $\bar{V}^*$ we have,

$$\begin{aligned}\bar{\kappa}(V^* - D^*) &= \kappa(K(\bar{V}^*) + \rho^{-1}(B) + D^{\#}, \bar{V}^*) \\ &= \kappa(\rho^*(K(\bar{V}) + B) + \bar{R}_\mu + D^{\#}, \bar{V}^*) ,\end{aligned}$$

where $\bar{R}_\mu$ is the logarithmic ramification divisor, by the logarithmic canonical bundle formula [1, p. 180]. This is equal to

$$\kappa(\rho^*(K(\bar{V}) + B) + N\bar{R}_\mu + D^{\#}, \bar{V}^*) \quad \text{for any} \quad N \geqq 1 .$$

Choose N so large that $N\bar{R}_\mu + D^{\#} \geqq (\mu^*D)^{\#}$, where $(\mu^*D)^{\#}$ denotes the closure of the divisor μ^*D in $\bar{V}^*$. Then,

$$\begin{aligned}&\kappa(\rho^*(K(\bar{V}) + B) + N\bar{R}_\mu + D^{\#}, \bar{V}^*) \\ &\quad \geqq \kappa(\rho^*(K(\bar{V}) + B) + (\mu^*D)^{\#}, \bar{V}^*) \\ &\quad \geqq \kappa(\rho^*(K(\bar{V}) + B + \bar{D}), \bar{V}^*) = \kappa(K(\bar{V}) + B + \bar{D}, \bar{V}) .\end{aligned}$$

However, in general,

$$\kappa(K(\bar{V}) + B + \bar{D}, \bar{V}) \geqq \bar{\kappa}(V^* - D^*) .$$

Thus, we establish

$$\kappa(V^* - D^*) = \kappa(K(\bar{V}) + B + \bar{D}, V) . \qquad \text{q.e.d.}$$

The following lemmas play the key role in our theory.

LEMMA 1. *Let $(\bar{V}, B)$ be a ∂-manifold whose interior is V and let D be a reduced divisor on $V = \bar{V} - B$. Suppose there exist a complete nonsingular algebraic variety $\bar{V}^1$ and a proper birational morphism $f: \bar{V}^1 \to \bar{V}$ such that*

(1) $f^{-1}(B + \bar{D})$ has only normal crossings,

(2) for $g = f|f^{-1}(V)$ and $D^1 = g^{-1}(D)$, there is a decomposition $D^1 = D^* + E$ with effective divisors D^* and E such that

(i) $\bar{\kappa}(f^{-1}(V) - D^*) \geqq 0$,

(ii) $f^*(B + \bar{D}) \leqq f^*(B) + D^{\#} + NE^{\#} + (R_g)^{\#}$ for some $N > 0$, where $D^{\#}$ and $E^{\#}$ are the closures of D^*, and E in $\bar{V}^1$, respectively and R_g is the ramification divisor of $g: f^{-1}(V) \to V$.

Then $\bar{\kappa}(f^{-1}(V) - D^*) = \bar{\kappa}(V - D) = \kappa(K(\bar{V}) + B + \bar{D}, \bar{V})$.

The following lemma is a bit more general than Lemma 1.

LEMMA 2. *Let B be a reduced divisor on $\bar{V}$, and D a reduced divisor on $V = \bar{V} - B$. Suppose there exists a complete non-singular algebraic variety $\bar{V}^1$ and a proper birational morphism $f: \bar{V}^1 \to \bar{V}$ such that*

(1) $f^{-1}(B + \bar{D})$ has only normal crossings,

(2) there is a decomposition $D^1 = g^{-1}(D) = D^* + E$ such that

(i)* $\kappa(K(\bar{V}^1) + D^{\#} + f^{-1}(B), \bar{V}^1) \geqq 0$,

(ii) $f^*(B + \bar{D}) \leqq f^{-1}(B) + D^{\#} + NE^{\#} + (R_g)^{\#}$ *for some* $N > 0$.

Then $\kappa(K(\bar{V}^1) + D^{\#} + f^{-1}(B)) = \bar{\kappa}(f^{-1}(V) - D^*) = \kappa(K(\bar{V}) + B + \bar{D}, \bar{V})$.

When B has only normal crossings, (i)* is equivalent to (1). Hence, Lemma 2 is a generalization of Lemma 1 and so it suffices to prove Lemma 2.

PROOF OF LEMMA 2. By making use of κ-calculus (see [2]), we have

$$\kappa(K(\bar{V}^1) + f^{-1}(B + \bar{D}), \bar{V}^1) = \kappa(K(\bar{V}^1) + f^{-1}(B) + D^{\#} + E^{\#}, \bar{V}^1)$$
$$= \kappa(K(\bar{V}^1) + f^{-1}(B) + D^{\#} + NE^{\#}, \bar{V}^1)$$

for any $N > 0$, because of (i)*. Then by (ii), we have $f^*(B + \bar{D}) \leqq f^*(B) + D^{\#} + NE^{\#} + (R_f)$. Hence,

$$\begin{aligned}\kappa(K(\bar{V}^1) &+ f^{-1}(B) + D^{\#} + NE^{\#}, \bar{V}^1) \\ &= \kappa(f^*(K(\bar{V})) + R_f + f^{-1}(B) + D^{\#} + NE^{\#}, \bar{V}^1) \\ &\geqq \kappa(f^*(K(\bar{V})) + f^*(B + \bar{D}), \bar{V}^1) = \kappa(K(\bar{V}) + B + \bar{D}, \bar{V}) .\end{aligned}$$ q.e.d.

LEMMA 3. *Let* $\mathscr{D} = \mathscr{D}_1 + \mathscr{D}_2$ *be a sum of two reduced divisors on* $\bar{V}$, μ: $\bar{V} \to \bar{V}_1$ *a proper birational morphism and* D *a reduced divisor on* $\bar{V}_1$ *such that*

(i) $\bar{V}_1$ *is non-singular,*

(ii) $\mu^{-1}(D) = \mathscr{D}$.

Suppose that $\kappa(K(\bar{V}) + \mathscr{D}_1, \bar{V}) \geqq 0$ *and* $\kappa(K(\bar{V}) + \mathscr{D}_2, \bar{V}) \geqq 0$. *Then* $\kappa(K(\bar{V}) + \mathscr{D}, \bar{V}) = \kappa(K(\bar{V}_1) + ND, \bar{V}_1)$ *for any* $N \geqq 1$.

PROOF. $\kappa(K(\bar{V}) + \mathscr{D}_1 + \mathscr{D}_2, \bar{V}) = \kappa(K(\bar{V}) + \mathscr{D}_1 + N_2\mathscr{D}_2, \bar{V})$ for any $N_2 > 0$, since $\kappa(K(\bar{V}) + \mathscr{D}_1, \bar{V}) \geqq 0$. Moreover, $\kappa(K(\bar{V}) + \mathscr{D}_2 + (N_2 - 1)\mathscr{D}_2 + \mathscr{D}_1, \bar{V}) = \kappa(K(\bar{V}) + \mathscr{D}_2 + (N_2 - 1)\mathscr{D}_2 + N_1\mathscr{D}_1, \bar{V}) = \kappa(K(\bar{V}) + N_2\mathscr{D}_2 + N_1\mathscr{D}_1, \bar{V})$ for any $N_1 > 0$. On the other hand, we have $N \gg 0$ such that $\mu^*D \leqq N\mathscr{D}$. Hence, for any $m \geqq 1$,

$$\begin{aligned}\kappa(K(\bar{V}) + \mathscr{D}, \bar{V}) &= \kappa(K(\bar{V}) + mN\mathscr{D}, \bar{V}) \\ &\geqq \kappa(K(\bar{V}) + m\mu^*D, \bar{V}) = \kappa(\mu^*(K(\bar{V}_1) + mD) + R_\mu, \bar{V}) \\ &= \kappa(K(\bar{V}_1) + mD, \bar{V}_1) \geqq \kappa(K(\bar{V}_1) + D, \bar{V}_1) \geqq \kappa(K(\bar{V}) + \mathscr{D}, \bar{V}) .\end{aligned}$$

Thus, $\kappa(K(\bar{V}) + \mathscr{D}, \bar{V}) = \kappa(K(\bar{V}_1) + mD, \bar{V}_1)$ for any $m \geqq 1$. q.e.d.

2. THEOREM 2. *Let* $\bar{W}$ *be a complete non-singular algebraic variety of dimension* $n - 1$ *with* $\kappa(\bar{W}) \geqq 0$. *Suppose that there exists a surjective morphism* f: $\bar{V} \to \bar{W}$ *with* $\dim \bar{V} = n$.

Then for any reduced divisor D *on* $\bar{V}$, *we have*

$$\bar{\kappa}(\bar{V} - D) = \kappa(K(\bar{V}) + D, \bar{V}) .$$

PROOF. We may assume that a general fiber $\bar{V}_w$ is irreducible. If $\kappa(\bar{V}_w) \geqq 0$, then by Viehweg's theorem [9], $\kappa(\bar{V}) \geqq 0$. Hence, the assertion follows easily from Theorem 1. Thus we may assume that $\bar{V}_w \simeq \boldsymbol{P}^1$. If $\sharp(\bar{V}_w \cap D) \leqq 1$, then both the sides equal $-\infty$. Therefore, we assume that $\sharp(\bar{V}_w \cap D) \geqq 2$, i.e., $\bar{\kappa}(\bar{V}_w - D) \geqq 0$. By Kawamata's theorem [7], we have $\bar{\kappa}(\bar{V} - D) \geqq 0$. Let $\mu: \bar{V}^* \to \bar{V}$ be a proper birational morphism such that $\bar{V}^*$ is non-singular and that $\mu^{-1}(D)$ has only normal crossings. Let H be the horizontal component of $\mu^{-1}(D)$ with respect to $f \circ \mu: \bar{V}^* \to W$. Then by Kawamata's theorem [7] again, $\bar{\kappa}(\bar{V}^* - H) \geqq 0$. Hence we can apply Lemma 1 and get

$$\bar{\kappa}(\bar{V} - D) = \kappa(K(\bar{V}) + D, \bar{V}) . \qquad \text{q.e.d.}$$

Similarly, we obtain

THEOREM 1*. *Instead of* $\kappa(\bar{W}) \geqq 0$, *we assume that there exists a reduced divisor* G *on* $\bar{W}$ *such that* $\bar{\kappa}(\bar{W} - G) \geqq 0$ *and* $D \geqq f^{-1}(G)$. *Then,*

$$\bar{\kappa}(\bar{V} - D) = \kappa(K(\bar{V}) + D, \bar{V}) .$$

REMARK. The strong virtual singularity theorem does not hold on a non-rational ruled surface, as will be seen in the next example.

EXAMPLE 1. Let $\bar{S}_1 = \boldsymbol{P}^1 \times E$, E being an elliptic curve, and let $D_1 = E \times p_1$, $D_2 = E \times p_2$, $\Delta = q \times \boldsymbol{P}^1$.

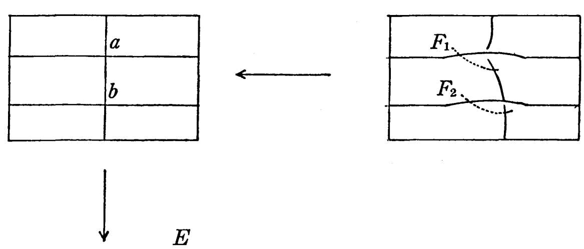

FIGURE 1

Let $\mu: \bar{S} = Q_{a,b}(\bar{S}_1) \to \bar{S}_1$ be a blowing up with centers $a = (q, p_1)$, $b = (q, p_2)$, and $F_1 = \mu^{-1}(a)$, $F_2 = \mu^{-1}(b)$. Denoting by D^* the proper transform of $D = D_1 + D_2 + \Delta$, we define $S = \bar{S} - D^*$. Then $K(\bar{S}) + D^* + F_1 + F_2 \sim \mu^*(K(\bar{S}_1) + D) \sim \mu^*(\Delta)$. Hence $K(\bar{S}) + D^* \sim \Delta^*$. Δ^* is a non-singular rational curve with $(\Delta^*)^2 = -2$. Thus $\bar{P}_m(S) = 1$ for any $m \geqq 1$. Here, $\sim$ denotes the linear equivalence.

3. Let $\bar{V}$ be a complete non-singular algebraic variety and D a reduced divisor. Then define the sets:

$$\mathrm{NC}(D) = \{p \in D\, ;\, D \text{ has only normal crossing at } p\} ,$$
$$\mathrm{NN}(D) = \mathrm{Supp}\, D - \mathrm{NC}(D) .$$

It is clear that $\mathrm{NC}(D) \supset \mathrm{Reg}\, D$, $\mathrm{NN}(D)$ is a closed (proper) subset of D.

We assume $\dim \bar{V} = 2$ and introduce the notion of cusps of D. First assume D to be irreducible and let $\mu: D^* \to D$ be a resolution of singularities. If p is a singular point of D and if $\#\{\mu^{-1}(P)\} = 1$, p is called a *cusp* of D. Next, assume that D consists of irreducible components $C_1, \cdots, C_s$. Let $C_1 \ni p, \cdots, C_r \ni p, C_{r+1} \not\ni p, \cdots, C_s \not\ni p$. If p is a cusp or a simple point of each C_i $(1 \leqq i \leqq r)$ and if $p \in \mathrm{NN}(D)$, then p is called a *cusp* of a reducible curve D. Furthermore, letting p be a cusp of D, we classify cusps as follows (cf. Figure 2).

(i) if p is a cusp of some component C_i, then p is called a *cusp of type* I,

(ii) if p is a non-singular point of each component C_j and if at least two tangents of these $C_1, \cdots, C_r$ at p coincide, then p is called a *cusp of type* II,

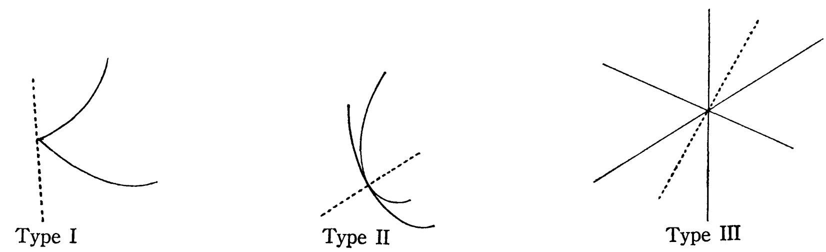

Figure of cusp types
FIGURE 2

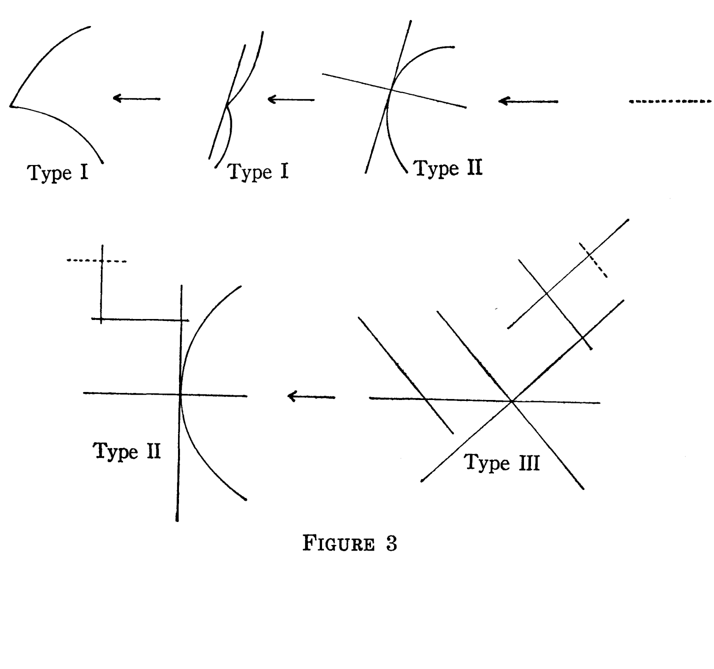

FIGURE 3

(iii) otherwise, p is called a *cusp of type* III.

In general, let D be a reduced divisor on $\bar{V}$ and μ: $\bar{V}^l \to \bar{V}^{l-1} \to \cdots \to \bar{V}^1 \to \bar{V}^0 = \bar{V}$ a composition of blowing ups $\mu_1, \mu_2, \cdots, \mu_l$ such that $D^l = \mu^{-1}(D)$ has only normal crossings. We have reduced transforms of D: D, $D^1 = \mu_1^{-1}(D)$, $D^2 = \mu_2^{-1}(D^2)$ and finally D^l. We say that $\{D, D^1, \cdots, D^l\}$ is the *set of reduced transforms in the process of simplification of the boundary* D.

LEMMA 4. *Let p be a cusp of type* I *of a maybe reducible curve D on a surface $\bar{S}$. Then in a process of simplification of D, there appears a cusp of type* II. *Similarly, in a process of simplification of D which has a cusp of type* II, *there appears a cusp of type* III.

This is obviously seen by the observation of figures as in Figure 3.

4. In this section, we shall study singular curves imbedded in a complete non-singular rational surface $\bar{S}$. First, we recall the $\bar{p}_g$-formula [3, p. 51].

LEMMA 5. *Let $D = \sum C_j$ be a reduced divisor on $\bar{S}$. Then*

$$\bar{p}_g(\bar{S} - D) = \sum g(C_j) + h(\Gamma(D^*)) .$$

Here, by μ: $\bar{S}^* \to \bar{S}$ we denote a composition of blowing ups such that $D^* = \mu^{-1}(D)$ has only normal crossings, $g(C_j)$ is the geometric genus of C_j, $\Gamma(D^*)$ is the graph associated with D^* and $h(\Gamma(D^*))$ is the cyclotomic number of $\Gamma(D^*)$.

By the above formula, we know that when $\bar{p}_g(\bar{S} - D) = 0$, each C_j is a rational curve which has only cusp singularities.

We shall prove the following "Bigenus theorem".

THEOREM 3. *Let D be a reduced divisor on $\bar{S}$. Suppose that* $\# \mathrm{NN}(D) \geqq 2$. *Then* $\bar{P}_2(\bar{S} - D) \geqq 1$.

In order to prove this, we first prove some elementary lemmas. The next result is obvious.

LEMMA 6. *In general, let D be a reduced divisor on a complete non-singular surface $\bar{S}$ and let μ: $\bar{S}^1 = Q_p(\bar{S}) \to \bar{S}$ be the blowing up at p. Letting $m = e(p, D)$ to be the multiplicity of D at p, we have*

$$K(\bar{S}^1) + \mu^{-1}(D) \sim \mu^*(K(\bar{S}) + D) - (m - 2)E .$$

Here $\sim$ indicates the linear equivalence.

Let μ_l: $\bar{S}^l \to \bar{S}^{l-1}$, μ_{l-1}: $\bar{S}^{l-1} \to \bar{S}^{l-2}$, $\cdots$, μ_1: $\bar{S}^1 \to \bar{S}^0 = \bar{S}$ be blowing ups in a process of simplification of D. By p_j we denote the center of μ_{j+1}. Let $D^0 = D$, $D^j = \mu_j^{-1}(D^{j-1})$, $E_j = \mu_j^{-1}(p_{j-1})$ and $m_j = e(p_{j-1}, D^{j-1})$. Then, by using the same symbols to indicate their pullbacks, we have

$$K(\bar{S}^l) + D^l \sim K(\bar{S}) + D - \sum_{j=1}^{l} (m_j - 2)E_j .$$

LEMMA 7. *In the above situation, we further assume $\bar{S}$ to be rational. Let $X = K(\bar{S}) + D - \sum r_j E_j$ $(r_j \geqq 0)$. Then, putting $2\pi(D) - 2 = D(D + K(\bar{S}))$,*

$$\dim |X| + 1 \geqq \pi(D) - \sum r_j(r_j + 1)/2 .$$

Moreover, if there is a reduced connected divisor Y in $|D - \sum (r_j + 1)E_j|$, we have

$$\dim |X| + 1 = \pi(D) - \sum r_j(r_j + 1)/2 .$$

In particular,

$$\bar{p}_g(\bar{S} - D) = \pi(D) - \sum (m_j - 2)(m_j - 1)/2 .$$

PROOF. Using $K(\bar{S}^l) \sim K(\bar{S}) + \sum E_j$ and $X - K(\bar{S}^l) = D - \sum (r_j + 1)E_j$, the assertion follows from the Riemann-Roch theorem on $\bar{S}^l$.

LEMMA 8. *Let $C_1, \cdots, C_r$ be non-singular rational curves on $\bar{S}$ such that* $\mathrm{Sing}(C_1 + \cdots + C_r) = \{p\}$ *with p a cusp of type* III *of $C_1 + \cdots + C_r$. Then $\pi(C_1 + \cdots + C_r) = (r - 2)(r - 1)/2$.*

PROOF. By the adjunction formula, we have

$$2\pi\left(\sum_{j=1}^{r} C_j\right) - 2 = \left(\sum_{j=1}^{r} C_j, \sum_{j=1}^{r} C_j + K(\bar{S})\right) ,$$

hence

$$\pi\left(\sum_{j=1}^{r} C_j\right) = (r - 1)(r - 2)/2 .$$

In the process of simplification of the boundary, we shall use $X^{(i)}$ to indicate the proper transform of the divisor $X^{(i-1)}$ and the same symbol Y to denote the total transform (with suitable coefficients) of the divisor Y. Further, we shall use the symbol $D_1 \wedge D_2$ to denote the *greatest common divisor* of the two effective divisors D_1 and D_2.

LEMMA 9. *Let $D = C_1 + C_2 + L + \Gamma_1 + \Gamma_2$ be a reduced divisor on $\bar{S}$ each component of which is a non-singular rational curve such that D has two triple points p and q as in Figure 4:*
Then $\bar{P}_2(\bar{S} - D) \geqq 1$.

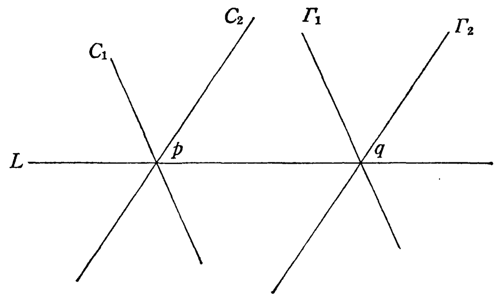

FIGURE 4

PROOF. Letting $\mu\colon \bar{S}^{\#} = Q_{p,q}(\bar{S}) \to \bar{S}$ be the blowing up at p, q and putting $E = \mu^{-1}(p)$, $F = \mu^{-1}(q)$, we have by Lemma 6

$$K(\bar{S}^{\#}) + \mu^{-1}(D) = K + D - E - F ,$$

K being $K(\bar{S})$. Since $\pi(C_1 + C_2 + L) = \pi(L_1 + L_2 + \Gamma) = 1$, it follows that $|K + C_1 + C_2 + L| \neq \varnothing$ and $|K + \Gamma_1 + \Gamma_2 + L| \neq \varnothing$. Hence there is an effective divisor $Z \in |2K + C_1 + C_2 + 2L + \Gamma_1 + \Gamma_2| = |2K + D + L|$. Furthermore,

$$(K(\bar{S}^{\#}) + \mu^{-1}(D)) = 2K + 2D - 2E - 2F \sim Z + D - L - 2E - 2F .$$

Then,

$$D - L = C_1 + C_2 + \Gamma_1 + \Gamma_2 \sim C_1' + C_2' + \Gamma_1' + \Gamma_2' + 2E + 2F .$$

Hence

$$2(K(\bar{S}^{\#}) + \mu^{-1}(D)) \sim Z + C_1' + C_2' + \Gamma_1' + \Gamma_2' . \qquad \text{q.e.d.}$$

LEMMA 10. *Let* $D = C_1 + C_2 + C_3 + \Gamma_1 + \Gamma_2 + \Gamma_3$ *be a reduced divisor on* $\bar{S}$ *consisting of non-singular rational curves* $C_1, \cdots, \Gamma_3$. *Suppose that* D *has two triple points* p *and* q *as in Figure* 5. *Then*

$$\bar{P}_2(\bar{S} - D) \geqq 1,\ \bar{P}_3(\bar{S} - D) \geqq 2 \ \textit{and}$$
$$\bar{\kappa}(\bar{S} - D) = \kappa(K(\bar{S}) + ND, \bar{S}) \ \textit{for any}\ N \geqq 1 .$$

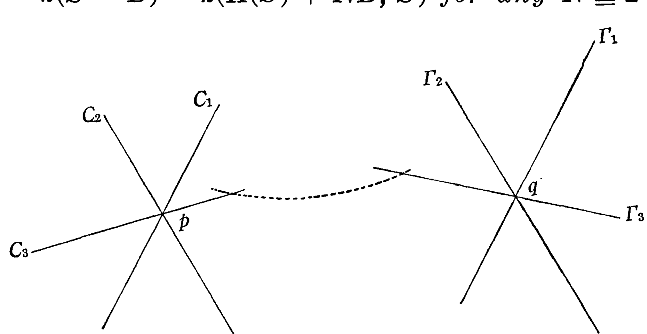

FIGURE 5

PROOF. Let $\mu\colon \bar{S}^{\#}=Q_{p,q}(\bar{S})\to\bar{S}$ be the blowing up at p and q. Then $\mu^{-1}(D)+K(\bar{S}^{\#})=K(\bar{S})+D-E-F$ by Lemma 6. Since $\pi(\tilde{C})=\pi(\tilde{\Gamma})=1$ (where $\tilde{C}=C_1+C_2+C_3$, $\tilde{\Gamma}=\Gamma_1+\Gamma_2+\Gamma_3$), there exist $X\in|K(\bar{S})+\tilde{C}|$ and $Y\in|K(\bar{S})+\tilde{\Gamma}|$. Hence $X+Y\in|2K(\bar{S})+\tilde{C}+\tilde{\Gamma}|$. Thus

$$\begin{aligned}2(K(\bar{S}^{\#})+\mu^{-1}(D)) &\sim X+Y+\tilde{C}+\tilde{\Gamma}-2E-2F\\ &\sim X+Y+\tilde{C}'+\tilde{\Gamma}'+E+F\,.\end{aligned}$$

This implies $\bar{P}_2(\bar{S}-D)\geqq 1$. Since $2X+Y\in|3K(\bar{S})+2\tilde{C}+\tilde{\Gamma}|$, we have

$$\begin{aligned}3(K(\bar{S}^{\#})+\mu^{-1}(D)) &\sim 2X+Y+\tilde{C}+2\tilde{\Gamma}-3E-3F\\ &\sim 2X+Y+\tilde{C}'+\tilde{\Gamma}+\tilde{\Gamma}'\,.\end{aligned}$$

Similarly,

$$3(K(\bar{S}^{\#})+\mu^{-1}(D))\sim 2Y+X+\tilde{\Gamma}'+\tilde{C}+\tilde{C}'\,.$$

If $\bar{P}_3(\bar{S}-D)=1$, we would have

$$2X+Y+\tilde{C}'+\tilde{\Gamma}+\tilde{\Gamma}'=2Y+X+\tilde{\Gamma}'+\tilde{C}+\tilde{C}'\,.$$

Hence, $X+\tilde{\Gamma}=Y+\tilde{C}$. But since $\tilde{C}\wedge\tilde{\Gamma}=0$, $X-\tilde{C}=Y-\tilde{\Gamma}$ would then be effective. By $X-\tilde{C}\sim K(\bar{S})$, we would have $\kappa(\bar{S})\geqq 0$, a contradiction.

Furthermore,

$$\begin{aligned}6(K(\bar{S}^{\#})+\mu^{-1}(D)) &\sim 3X+3Y+3\tilde{C}'+3\tilde{\Gamma}'+3E+3F\\ &\sim 3X+3Y+2\tilde{C}'+2\tilde{\Gamma}'+\tilde{C}+\tilde{\Gamma}\geqq D\,.\end{aligned}$$

Hence

$$\begin{aligned}\bar{\kappa}(S) &= \kappa(K(\bar{S}^{\#})+\mu^{-1}(D)+6N(K(\bar{S}^{\#})+\mu^{-1}(D)),\bar{S}^{\#})\\ &\geqq \kappa(K(\bar{S}^{\#})+\mu^{-1}(D)+N\mu^{*}(D),\bar{S}^{\#})\\ &\geqq \kappa(\mu^{*}(K(\bar{S}))+R_{\mu}+\mu^{-1}(D)+\mu^{*}(D)+(N-1)\mu^{*}D,\bar{S}^{\#})\\ &\geqq \kappa(\mu^{*}(K(\bar{S})+(N-1)D),\bar{S}^{\#})=\kappa(K(\bar{S})+(N-1)D,\bar{S})\end{aligned}$$

for any $N\geqq 2$. Thus $\bar{\kappa}(S)=\kappa(K(\bar{S})+ND,\bar{S})$, for any $N>0$. q.e.d.

LEMMA 11. *Let D be a reduced divisor on $\bar{S}$. Suppose that $\bar{\kappa}(\bar{S}-D)=\kappa(K(\bar{S})+ND,\bar{S})\geqq 0$ for any $N>1$. Then $\bar{S}-D$ is an elliptic surface or $\bar{\kappa}(\bar{S}-D)=0$ or 2.*

PROOF. Assume $\bar{\kappa}(\bar{S}-D)=1$ and fix $N\geqq 3$. There exists a $(K(\bar{S})+ND)$-canonical fibered surface $\psi\colon\bar{S}\to J$ such that $\kappa((K(\bar{S})+ND)\psi^{-1}(u),\psi^{-1}(u))=0$ for a general point $u\in J$. Hence, when $(D,\psi^{-1}(u))\neq 0$, it follows that

$$-(K(\bar{S}),\psi^{-1}(u))=N(D,\psi^{-1}(u))\geqq 3(D,\psi^{-1}(u))\geqq 3\,.$$

On the other hand,

$$(K(\bar{S}), \psi^{-1}(u)) = 2g(\psi^{-1}(u)) - 2 \geqq -2 ,$$

where $g(\psi^{-1}(u))$ denotes the genus of $\psi^{-1}(u)$. Thus we arrive at a contradiction.

When $(D, \psi^{-1}(u)) = 0$, D is vertical with respect to ψ and $K|\psi^{-1}(u) \sim 0$, i.e., $\psi^{-1}(u)$ is an elliptic curve. q.e.d.

REMARK. Under the hypothesis of Lemma 11, suppose $\bar{\kappa}(\bar{S} - D) = 1$. Then D is contained in a finite union of fibers of the elliptic surface.

Now, we prove Theorem 3. It is no loss of generality to assume $\bar{p}_g(\bar{S} - D) = 0$. Then D consists of rational curves which have only cusp singular points. By hypothesis, there are at least two cusps. After suitable blowing ups, we may assume that p and q are cusps of type III by Lemma 5. Applying Lemmas 9 and 10, we complete the proof. q.e.d.

THEOREM 4 (The virtual singularity theorem). *Let D be a reduced divisor on a complete non-singular rational surface $\bar{S}$. Assume one of the following*:

(1) *There is a non-rational component of D,*

(2) $\sharp(\mathrm{NN}(D)) \geqq 3$,

(3) $\sharp(\mathrm{NN}(D)) = 2$ *and one of* $\mathrm{NN}(D)$ *is not a cusp,*

(3)′ $\sharp(\mathrm{NN}(D)) = 1$ *and there is an effective divisor D_0 contained in D such that $h(\Gamma(D_0)) = 1$ and $D_0 \cap \mathrm{NN}(D) = \varnothing$. Then, the virtual singularity theorem holds for $(\bar{S}, D)$.*

PROOF. Assume (1). Let $\mu: \bar{S}^* \to \bar{S}$ be a birational morphism such that $(\bar{S}^*, \mu^{-1}D)$ is a ∂-surface. Take a non-rational component C of $\mu^{-1}D$. Then $\bar{\kappa}(\bar{S}^* - C) \geqq 0$. Hence, by Lemma 1, we get the assertion.

Next, assume (2). Choose two points p and q from $\mathrm{NN}(D)$. Performing blowing ups with centers which are points over p and q, we have a proper birational morphism $\rho: \bar{S}^* \to \bar{S}$ such that ρ is isomorphic except around p and q and that $\rho^{-1}(D)$ has only normal crossings at all points over p and q. Then take a proper birational morphism $\mu: \bar{S}^\sharp \to \bar{S}^*$ such that $\mu^{-1}\rho^{-1}(D)$ has only normal crossings. Now, let D^* be the proper transform of D by ρ^{-1}. We have an effective divisor $\mathscr{E}$ such that $\mathscr{D} = \rho^{-1}(D) = D^* + \mathscr{E}$. There is $N_1 > 0$ such that $N_1\mathscr{E} + D^* \geqq \rho^*(D)$. Next, let $\mathscr{D}^*$ be the proper transform of $\mathscr{D}$ by μ^{-1}. By Theorem 3, $\bar{\kappa}(\bar{S}^\sharp - \mathscr{D}^*) \geqq 0$. Hence, in view of Lemma 1,

$$\bar{\kappa}(\bar{S} - D) = \bar{\kappa}(\bar{S}^* - \mathscr{D}) = \kappa(K(\bar{S}^*) + \mathscr{D}, \bar{S}^*) .$$

Recalling the hypothesis, we get $\dim |K(\bar{S}^*) + D^*| \geqq 0$. From this, it follows that $\kappa(K(\bar{S}^*)+D^*, \bar{S}^*) \geqq 0$. Applying Lemma 2, we obtain

$$\kappa(K(\bar{S}^*)+\mathscr{D}, \bar{S}^*) = \kappa(K(\bar{S})+D, \bar{S}) .$$

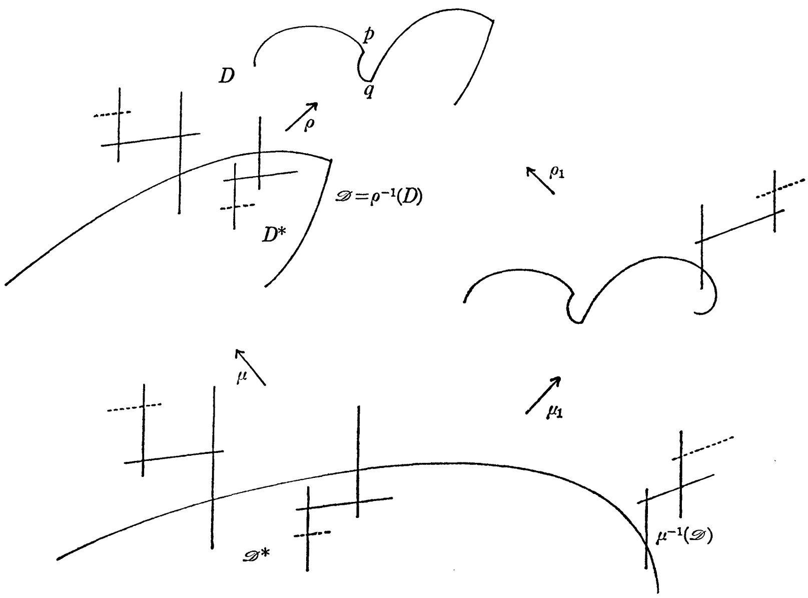

FIGURE 6

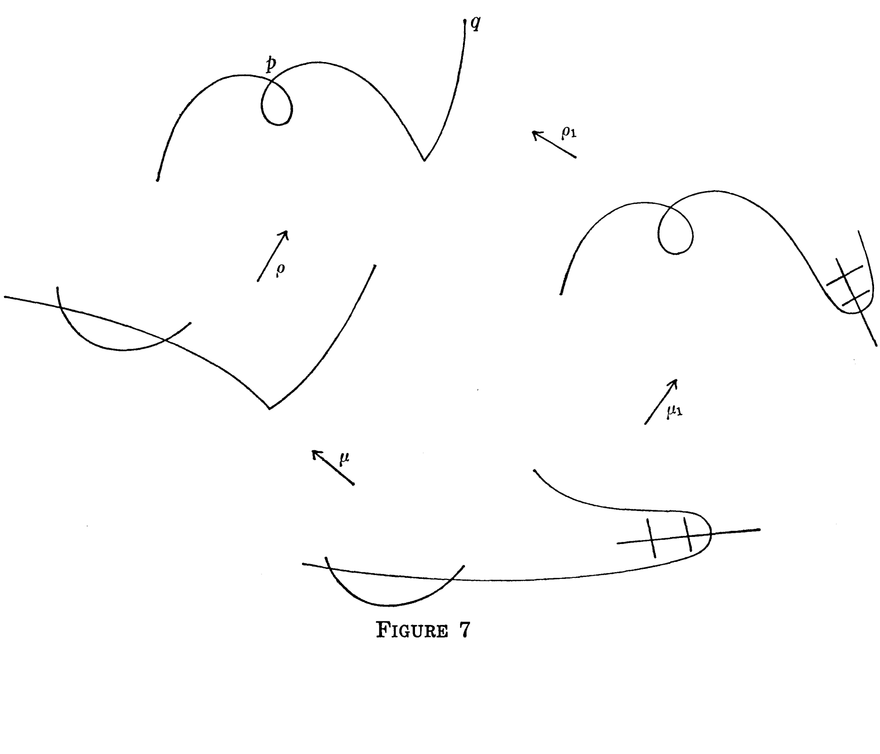

FIGURE 7

Similarly, one can show that (3) or (3)′ implies the virtual singularity theorem. Actually, it suffices to use the pictures illustrated in Figure 7 for the condition (3).

THEOREM 5. *Let D be a reduced divisor on $\bar{S}$. If there is a proper birational morphism $\rho\colon \bar{S}^* \to \bar{S}$ such that $\rho^{-1}(D)$ contains an effective divisor $\mathscr{D}$ which satisfies the hypothesis of Lemma* 10. *Then $\kappa(\bar{S} - D) = \kappa(K(\bar{S}) + ND, \bar{S})$ for any $N \geqq 1$.*

PROOF. We use the proof of Lemma 10. Then

$$\bar{\kappa}(\bar{S} - D) = \kappa(K(\bar{S}^*) + N\rho^{-1}(D), \bar{S}^*)\,, \quad \text{for any} \quad N \geqq 1.$$

From this, we readily infer that

$$\bar{\kappa}(\bar{S} - D) = \kappa(K(\bar{S}) + N_1 D, \bar{S}) \quad \text{for any} \quad N_1 \geqq 1\,.$$

q.e.d.

REMARKS (1) The pictures in Figure 8 are examples of D satisfying the hypothesis of Theorem 5.

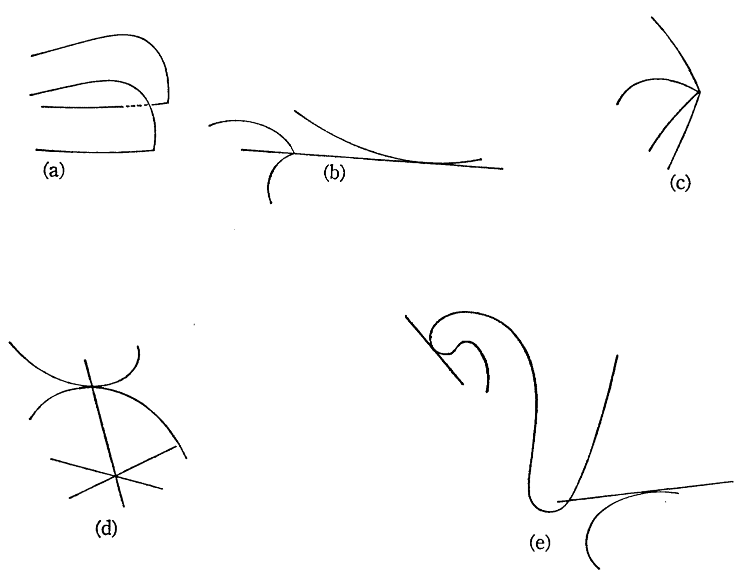

FIGURE 8

(2) In these cases, from Lemmas 10 and 12, it follows that $\bar{\kappa}(S - D) = 1$ or 2. The first case occurs only when $S - D$ is an elliptic surface (cf. Lemma 11 and Remark).

EXAMPLE 2. Let C be a non-singular cubic curve in $\boldsymbol{P}^2$. Attaching ten 1/2-points to $\boldsymbol{P}^2 - C$, we have a surface S and its completion $\bar{S}$ with smooth boundary C. Then $C^2=-1$ and $K(\bar{S})+C\sim 0$ (cf. [6], [12]). Thus

$$\bar{\kappa}(S)=\kappa(K(\bar{S})+NC,\bar{S})=\kappa(C,\bar{S})=0\,.$$

EXAMPLE 3. Let C_1, C_2, C_3 be three lines on $\boldsymbol{P}^2$ such that $C_1\cap C_2\cap C_3=\{p\}$. Attaching several 1/2-points to $\boldsymbol{P}^2-(C_1\cup C_2\cup C_3)$, we have a surface S and its completion $\bar{S}$ with smooth boundary $D=C_1^*+C_2^*+C_3^*$, C_i^* being the proper transforms of the C_i, such that the matrix $[(C_i^*, C_j^*)]$ is negative-definite. Then

$$\kappa(K(\bar{S})+ND,\bar{S})=\kappa(D,\bar{S})=\kappa(K(\boldsymbol{P}^2)+C_1+C_2+C_3,\boldsymbol{P}^2)=0\,.$$

Appendix. Let C denote an irreducible curve on $\boldsymbol{P}^2$ of degree d. Suppose that there is a point p on C such that $C-\{0\}$ is isomorphic to the affine line $\boldsymbol{A}^1$. Let e indicate the multiplicity of C at p.

PROPOSITION A (H. Yoshihara). *If $n\geqq 3e$, then $\bar{\kappa}(\boldsymbol{P}^2-C)=2$.*

PROPOSITION B (H. Yoshihara). *If $n=6$, then $e\geqq 3$.*

COROLLARY. *If $n\leqq 6$, then $\bar{\kappa}(\boldsymbol{P}^2-C)=-\infty$.*

Proof of Proposition A follows immediately from Lemma 6. But the proof of Proposition B depends on a laborious and long computation (see [11]).

REMARK (Y. Yoshihara). There exists a sextic curve Γ on $\boldsymbol{P}^2$ which has a singular point p with $\Gamma-\{p\}\cong G_m$. In this case, $\bar{\kappa}(\boldsymbol{P}^2-\Gamma)=2$.

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