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A VARIATION OF KNESER'S THEOREM AND BOUNDARY VALUE PROBLEMS

Dedicated to Professor Tarō Yoshizawa on his sixtieth birthday

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1. Introduction. In order to solve a two point boundary value problem for a nonlinear ordinary differential equation, it seems useful to find a solution curve which lies in a suitable set. What conditions on the set yield such a solution curve then? For this problem, Ważewski [12] obtained a beautiful result called a retract method in 1947. Namely, he has solved the problem by investigating properties of solution curves which pass through boundary points of the set. His method has been developed by Kluczny [8], Hukuhara [3], Jackson and Klaasen [4], Kelley [6] and others.

In this paper, a new approach to the problem will be discussed. Namely, the problem will be solved by finding a condition under which the outside of the given set can be divided into suitable positively or negatively invariant sets. Our method is based on Kneser's theorem (see e.g., [2]), and it is rather close to that of Bebernes and Wilhelmsen [1]. We shall state a fundamental existence theorem (Theorem 1 in this paper) in Section 3. Applications of the theorem to scalar second order ordinary differential equations will be discussed in Section 4.

2. Preliminaries from general topology. The family of all nonempty compact sets in R^n will be denoted by $\mathrm{Comp}(R^n)$. Let F be a mapping from a metric space Λ into $\mathrm{Comp}(R^n)$, that is, F is a multivalued mapping from Λ into R^n. We call F *upper semicontinuous* if for every sequence $\{\lambda_k\}$ in Λ which converges to some point λ_0 in Λ and for every point x_k in $F(\lambda_k)$, there exists a subsequence of $\{x_k\}$ which converges to some point in $F(\lambda_0)$.

LEMMA 1. *Let Λ be a compact and connected metric space, and let $F: \Lambda \to \mathrm{Comp}(R^n)$ be an upper semicontinuous mapping. If $F(\lambda)$ is connected for all λ in Λ, then the set $F(\Lambda)$ defined by $F(\Lambda) = \bigcup \{F(\lambda): \lambda \in \Lambda\}$ is a compact and connected set in R^n.*

PROOF. First it will be verified that $F(\Lambda)$ is compact. Let $\{x_k\}$ be an arbitrary sequence in $F(\Lambda)$. Then there exists a λ_k in Λ such that

x_k belongs to $F(\lambda_k)$ for each $k = 1, 2, \cdots$. Since Λ is compact, we may assume that $\{\lambda_k\}$ converges to some point λ_0 in Λ by taking a subsequence if necessary. The upper semicontinuity of F implies that $\{x_k\}$ has a subsequence which converges to some point in $F(\lambda_0)$. This shows the compactness of $F(\Lambda)$.

Next we shall show the connectedness of $F(\Lambda)$. Suppose that $F(\Lambda)$ is not connected. Then there exist two nonempty and disjoint compact sets S_1 and S_2 such that $S_1 \cup S_2 = F(\Lambda)$, and put $\Lambda_i = \{\lambda \in \Lambda: F(\lambda) \cap S_i \neq \emptyset\}$, $i = 1, 2$. It is easy to see that Λ_1 and Λ_2 are nonempty compact sets and satisfy $\Lambda_1 \cup \Lambda_2 = \Lambda$. Since Λ is connected, the set $\Lambda_1 \cap \Lambda_2$ contains at least one point λ. Put $X_i = F(\lambda) \cap S_i$, $i = 1, 2$. Then X_1 and X_2 are nonempty compact sets and satisfy $X_1 \cup X_2 = F(\lambda)$. Therefore we have $X_1 \cap X_2 \neq \emptyset$ because $F(\lambda)$ is connected. Since $X_1 \cap X_2$ is contained in $S_1 \cap S_2$, we have $S_1 \cap S_2 \neq \emptyset$, a contradiction. q.e.d.

Let H and K be closed sets in a topological space X. A subset A of X will be called a *bridge* from H to K if the following two conditions are satisfied.

(B1) The set A intersects both H and K.

(B2) The set $A \setminus (H \cup K)$ is connected and its closure coincides with A.

REMARK 1. It follows from (B2) that a bridge is a continuum (i.e., closed and connected set). When $H \cap K = \emptyset$, a minimal bridge is an irreducible continuum from H to K (see [10, p. 14]).

LEMMA 2. *Let H and K be disjoint closed sets and M a compact and connected set in a Hausdorff topological space. If M intersects both H and K, then M contains a bridge from H to K.*

This lemma is a direct corollary to Theorems 41 and 47 in [10, pp. 15-17].

3. An existence theorem. We denote a compact interval $[a, b]$ by I. Let Ω be an open subset of $I \times R^n$ in the relative topology of $I \times R^n$ and $f: \Omega \to R^n$ a continuous mapping. We consider the ordinary differential equation

$$x' = f(t, x) \qquad (\,' = d/dt)\,. \tag{1}$$

For a subset A of $I \times R^n$ and a $\tau \in I$, the cross section of A by the hyperplane $t = \tau$ will be denoted by $A|_\tau$, that is, $A|_\tau = \{(t, x) \in A: t = \tau\}$. The closure of A will be denoted by $\bar{A}$. For a subset A of Ω, we define the right emission zone $Z^+(A)$ from A of (1) by $Z^+(A) = \{(t, x(t)) \in \Omega: x$ is a solution of (1) defined on $[t_0, t_1]$ such that $(t_0, x(t_0)) \in A,\ \ t_0 \leqq t \leqq t_1\}$.

We denote the set $Z^+(\{p\})$, $p \in \Omega$, simply by $Z^+(p)$. A subset A of Ω will be called a positively invariant set of (1) if $Z^+(A) \subset A$. Similarly, the left emission zone $Z^-(A)$ from A of (1) and a negatively invariant set of (1) are defined.

Let D be a subset of Ω such that $\bar{D}$ is a compact subset of Ω. Furthermore, we assume that the following condition is satisfied.

(C) The set $\Omega \setminus D$ is a sum of a negatively invariant set N (which is allowed to be empty) of (1) and two positively invariant sets P and Q of (1) such that $P \cap \bar{Q} = \bar{P} \cap Q = \emptyset$.

Our main purpose is to prove the following theorem.

THEOREM 1. *Suppose that the condition* (C) *is satisfied. Let E be a compact and connected set in $D \cup P \cup Q$. If the right emission zone $Z^+(E)$ intersects both $\bar{P}$ and $\bar{Q}$, then the cross section $Z^+(E)|_b$ contains a bridge T from $\bar{P}$ to $\bar{Q}$ such that T is the closure of a connected set $T \cap D$. Furthermore, every solution curve of* (1) *joining a point in E to a point in $T \cap D$ lies in D.*

REMARK 2. The conclusion of this theorem shows that the set $D|_b$ is nonempty. In other words, if $D|_b$ is empty, then the assumptions of the theorem do not hold.

To prove Theorem 1, we need some lemmas below. We always assume that the condition (C) holds.

LEMMA 3. *If a subset A of Ω is positively invariant, then the set $\bar{A} \cap \Omega$ is weakly positively invariant, that is, the equation* (1) *has at least one solution which starts from any given point in $\bar{A} \cap \Omega$ and remains in $\bar{A} \cap \Omega$ on its right maximal interval of existence.*

The proof follows from Kamke's theorem (see e.g., [2, Theorem 3.2, pp. 14–15].

LEMMA 4. *The set $D \cup P \cup Q$ is positively invariant.*

PROOF. It suffices to show that $Z^+(D) \subset D \cup P \cup Q$, but this is clear since $\Omega \setminus (D \cup P \cup Q)$ is contained in the negatively invariant set N and $D \cap N = \emptyset$. q.e.d.

LEMMA 5. *There exists an $\varepsilon > 0$ such that every solution $x(t)$ of* (1) *with initial value $(t_0, x(t_0)) \in \bar{D}$ exists for $t_0 \leqq t \leqq \min\{t_0 + \varepsilon, b\}$.*

This lemma is easily proved by [2, Corollary 2.1, p. 11].

We use the following well-known theorem of Kneser's:

LEMMA 6. *Let $p = (t_0, x) \in \Omega$ and $t_0 < \tau$. If every solution of* (1) *passing through p is continuable to $t = \tau$, then the cross section $Z^+(p)|_\tau$ is compact and connected.*

This lemma is easily proved by [2, Theorem 4.1, pp. 15-17].

LEMMA 7. *Let A be a compact and connected set in $D \cup P \cup Q$. If A intersects both $\bar{P}$ and $\bar{Q}$, then A contains a bridge A_0 from $\bar{P}$ to $\bar{Q}$ such that A_0 is the closure of a connected set $A_0 \cap D$. Therefore A_0 is contained in $\bar{D}$, and $Z^+(A_0)$ intersects both $\bar{P}$ and $\bar{Q}$.*

PROOF. We define two sets H and K by $H = A \cap \bar{P}$ and $K = A \cap \bar{Q}$. Then H and K are nonempty compact sets.

First we consider the case where $H \cap K$ is nonempty. In this case, the set $H \cap K$ is contained in D because $H \cap K$ is contained in $\bar{P} \cap \bar{Q} \cap (D \cup P \cup Q)$ but $\bar{P} \cap \bar{Q} \cap (P \cup Q) = (P \cap \bar{Q}) \cup (\bar{P} \cap Q) = \varnothing$. Therefore an arbitrary point in $H \cap K$ is the desired set A_0.

Next we consider the case where $H \cap K = \varnothing$. It follows from Lemma 2 that A contains a bridge A_0 from H to K. Since A is contained in $D \cup H \cup K$, so is A_0. This implies that $A_0 \setminus (H \cup K) \subset A_0 \cap D \subset A_0$. Since A_0 is the closure of the connected set $A_0 \setminus (H \cup K)$, the set $A_0 \cap D$ is also connected and its closure coincides with A_0. q.e.d.

LEMMA 8. *Let A be a compact and connected set in $D \cup P \cup Q$. Suppose that $Z^+(A)$ intersects both $\bar{P}$ and $\bar{Q}$. Then A contains a subcontinuum A_0 such that $A_0 \subset \bar{D}$ and that $Z^+(A_0)$ intersects both $\bar{P}$ and $\bar{Q}$.*

PROOF. When A does not intersect $\bar{P} \cup \bar{Q}$, we can set $A_0 = A$, while if A intersects both $\bar{P}$ and $\bar{Q}$, Lemma 7 assures the existence of the set A_0.

Therefore it remains only to prove the assertion in the case where A intersects only one of $\bar{P}$ or $\bar{Q}$, say $\bar{P}$. Since $Z^+(A)$ intersects $\bar{Q}$, the set A contains a point p such that $Z^+(p)$ intersects $\bar{Q}$. It will be verified that p belongs to D. Since A is contained in $D \cup P$, we see that p belongs to $D \cup P$. Suppose $p \in P$. Then the positive invariance of P and the assumption that $P \cap \bar{Q} = \varnothing$ imply that $Z^+(p) \cap \bar{Q} = \varnothing$. This is a contradiction, and hence p belongs to D.

When p belongs to $\bar{P}$, we can set $A_0 = \{p\}$. So we consider the case where p does not belong to $\bar{P}$. In this case, $\bar{P}$ and $\{p\}$ are disjoint closed sets and each of them intersects A. It then follows from Lemma 2 that A contains a bridge A_0 from $\bar{P}$ to $\{p\}$. Clearly $A_0 \setminus (\bar{P} \cup \{p\})$ is contained in D, and hence we have $A_0 \subset \bar{D}$. Since p belongs to A_0, we see that $Z^+(A_0)$ intersects $\bar{Q}$. Of course $Z^+(A_0)$ intersects $\bar{P}$ because $Z^+(A_0)$ contains A_0 and A_0 intersects $\bar{P}$. q.e.d.

PROOF OF THEOREM 1. We denote by 2^Ω the family of all subsets of

Ω, and define a mapping $F: I \times \Omega \to 2^{\Omega}$ in the following way. For $\tau \in I$ and $p = (t, x) \in \Omega$, the set $F(\tau, p)$ in Ω is defined by

$$F(\tau, p) = \begin{cases} Z^+(p)|_\tau & \text{for} \quad \tau > t\ , \\ \{p\} & \text{for} \quad \tau \leqq t\ . \end{cases}$$

For a subset A of Ω, the set $F(\tau, A)$ is defined by $F(\tau, A) = \bigcup \{F(\tau, p): p \in A\}$. Take a partition $a = t_0 < t_1 < \cdots < t_m = b$ of the interval I such that $\max\{t_i - t_{i-1}: 1 \leqq i \leqq m\} < \varepsilon$, where ε is the number given in Lemma 5.

By Lemma 8, the set E contains a compact and connected set E_0 such that $E_0 \subset \bar{D}$ and that $Z^+(E_0)$ intersects both $\bar{P}$ and $\bar{Q}$. It follows from Lemmas 5 and 6 that the set $F(t_1, p)$ is compact and connected for all $p \in E_0$. Furthermore by Kamke's theorem, it is easy to see that the mapping $F(t_1, \cdot)$ is upper semicontinuous on E_0. Therefore it follows from Lemma 1 that $F(t_1, E_0)$ is a compact and connected set.

By Lemma 3, the set $\bar{P} \cap \Omega$ is weakly positively invariant. Therefore we easily observe $Z^+(F(t_1, E_0))$ must intersect $\bar{P}$. Similarly, $Z^+(F(t_1, E_0))$ intersects $\bar{Q}$. It follows from Lemma 4 that $F(t_1, E_0)$ is contained in $D \cup P \cup Q$. Therefore by Lemma 8, the set $F(t_1, E_0)$ contains a subcontinuum E_1 such that $E_1 \subset \bar{D}$ and that $Z^+(E_1)$ intersects both $\bar{P}$ and $\bar{Q}$. Here, we note that $Z^+(E_1) \subset Z^+(E_0) \subset Z^+(E)$.

By repeating this procedure, we obtain a compact and connected set E_m in $F(t_m, E_{m-1})$ such that $E_m \subset \bar{D}$ and that $Z^+(E_m)$ intersects both $\bar{P}$ and $\bar{Q}$. Clearly, $Z^+(E_m) \subset Z^+(E) \subset D \cup P \cup Q$. Since $t_m = b$, we have $Z^+(E_m) = E_m$. Therefore by Lemma 7, we can find the desired continuum T in E_m, that is, in $Z^+(E)|_b$.

The last assertion of the theorem follows from Lemma 4, the positive invariance of $P \cup Q$ and the assumption that $D \cap (P \cup Q) = \varnothing$. q.e.d.

4. An application. In this section we consider a scalar second order ordinary differential equation

$$x'' = f(t, x, x')\ . \tag{2}$$

We denote the compact interval $[a, b]$ by I. Let α and β be twice continuously differentiable functions on I satisfying $\alpha(t) \leqq \beta(t)$ on I, and define the (t, x)-set W by

$$W: a \leqq t \leqq b\ , \quad \alpha(t) \leqq x \leqq \beta(t)\ .$$

Let ϕ and ψ be continuously differentiable functions on W, and let W_0 be the compact subset of W defined by

$$W_0: (t, x) \in W\ , \quad \phi(t, x) \leqq \psi(t, x)\ .$$

The function $f = f(t, x, y)$ in the equation (2) is assumed to be continuous on the compact (t, x, y)-set D given by

$$D : (t, x) \in W_0 , \quad \phi(t, x) \leqq y \leqq \psi(t, x) .$$

For a function $\theta = \phi$ or ψ, we define the function $\Gamma(\theta)$ on W_0 by

$$[\Gamma(\theta)](t, x) = f(t, x, \theta(t, x)) - \theta_t(t, x) - \theta_x(t, x)\theta(t, x) ,$$

where θ_t and θ_x denote the partial derivatives of θ with respect to t and x, respectively. Suppose that α and β satisfy

$$(3) \qquad \alpha''(t) \geqq f(t, \alpha(t), \alpha'(t)) \quad \text{for} \quad (t, \alpha(t), \alpha'(t)) \in D$$

and

$$(4) \qquad \beta''(t) \leqq f(t, \beta(t), \beta'(t)) \quad \text{for} \quad (t, \beta(t), \beta'(t)) \in D .$$

Furthermore, we suppose that each of functions $\Gamma(\phi)$ and $\Gamma(\psi)$ does not change the sign on W_0, that is, nonnegative on W_0 or nonpositive on W_0.

Consider the conditions

(i) $\Gamma(\phi) \geqq 0$ on W_0 and $\phi(t, \alpha(t)) \leqq \alpha'(t)$ on I,
(ii) $\Gamma(\phi) \leqq 0$ on W_0 and $\phi(t, \beta(t)) \leqq \beta'(t)$ on I,
(iii) $\Gamma(\psi) \geqq 0$ on W_0 and $\alpha'(t) \leqq \psi(t, \alpha(t))$ on I,
(iv) $\Gamma(\psi) \leqq 0$ on W_0 and $\beta'(t) \leqq \psi(t, \beta(t))$ on I.

Then, by combinations of these conditions, there are four cases. However as will be seen in Section 5, it turns out that there are essentially two cases, and in each of the cases we can state an existence theorem.

THEOREM 2. *Suppose that both of the conditions* (i) *and* (iii) *hold. Then the equation* (2) *has at least one solution* $x(t)$ *satisfying*

$$(5) \qquad \alpha(t) \leqq x(t) \leqq \beta(t) \quad on \quad I$$

and

$$(6) \qquad \phi(t, x(t)) \leqq x'(t) \leqq \psi(t, x(t)) \quad on \quad I .$$

THEOREM 3. *Suppose that both of the conditions* (i) *and* (iv) *hold, and that the inequality* $\phi(t, x) \leqq \psi(t, x)$ *holds on* W, *namely*, $W = W_0$. *Then for an arbitrary number* η *with* $\alpha(b) \leqq \eta \leqq \beta(b)$, *the equation* (2) *has at least one solution* $x(t)$ *satisfying* (5), (6) *and* $x(b) = \eta$.

Theorem 3 is an extension of Nagumo's theorem [11] (also, see [3], [5]), which has already been proved in [5] by means of Hukuhara's theory [3] based on the fact that the family F of all solution curves of the system

$$x' = y\ , \quad y' = f(t, x, y)\ ,$$

which are defined on compact intervals and contained in D, forms a left Kneser family under the assumptions of Theorem 3. However, the proof given here is simpler than that in [5], and it seems impossible to prove that F is a left Kneser family or a right Kneser family under the assumptions of Theorem 2. Theorems 2 and 3 include Lemma 11 in Knobloch [9] (also, see [3]).

We can prove Theorems 2 and 3 in the same manner, but the proof of Theorem 2 is more complicated than that of Theorem 3 because the condition $W = W_0$ is not assumed in Theorem 2. So we give a proof of Theorem 2 and only sketch the proof of Theorem 3.

To prove Theorem 2, the function f will be extended to the domain $\Omega = I \times R \times R$ suitably, and we shall apply Theorem 1 then. f will be extended step by step, and therefore it will be convenient to give the following lemma which can be easily proved by Tietze's extension theorem.

LEMMA 9. *Let X be a metric space and Y a compact subset of X. Then we have the following.*

(a) *Suppose that Y is the sum of compact sets D and A_i and that $u\colon D \to R$ and $u_i\colon A_i \to R$ are continuous functions, where i runs through $1, \cdots, m$. If $u(x) \geqq u_i(x)$ on $D \cap A_i$, then there exists a continuous function $U\colon Y \to R$ such that $U(x) = u(x)$ on D and that $U(x) \geqq u_i(x)$ on A_i.*

(b) *Suppose that $g\colon X \to R$ is continuous and that A is a compact subset of X. If a continuous function $v\colon A \to R$ satisfies $v(x) \leqq g(x)$ on $Y \cap A$, then there exists a continuous function $G\colon X \to R$ such that $G(x) = g(x)$ on Y, $G(x) \geqq g(x)$ on X and that $G(x) \geqq v(x)$ on A.*

PROOF OF THEOREM 2. We denote the surface $\{(t, x, \phi(t, x))\colon (t, x) \in W\}$ by S. Put $M(t, x) = \max\{\phi(t, x), \psi(t, x)\}$ for $(t, x) \in W$. The boundary of W_0 in W will be denoted by ∂W_0, that is, ∂W_0 is the intersection of W_0 with the closure of $W \setminus W_0$.

Set $Y = D \cup S$, $A_1 = Y \cap \{(t, \beta(t), \beta'(t))\colon t \in I\}$, $A_2 = S$ and $A_3 = \{(t, x, \phi(t, x))\colon (t, x) \in \text{closure of } W \setminus W_0\}$. Then A_1, A_2 and A_3 are compact sets and $Y = D \cup A_1 \cup A_2 \cup A_3$. If $u(t, x, y) = f(t, x, y)$ and if $u_i(t, x, y)$ are defined by $u_1 = \beta''(t)$, $u_2 = \phi_t(t, x) + \phi_x(t, x)\phi(t, x)$ and $u_3 = \psi_t(t, x) + \psi_x(t, x)\psi(t, x)$, then the assumptions in Lemma 9 (a) are satisfied, where we note that $D \cap A_3$ coincides with the set $\{(t, x, \phi(t, x))\colon (t, x) \in \partial W_0\}$ and $\phi = \psi$ on ∂W_0. Therefore there exists a continuous extension $f_1(t, x, y)$ of $f(t, x, y)$ to Y satisfying $f_1(t, x, y) \geqq u_i(t, x, y)$ on $D \cap A_i$, that is,

$$f_1(t, \beta(t), \beta'(t)) \geqq \beta''(t) \qquad \text{for} \quad (t, \beta(t), \beta'(t)) \in Y\ , \tag{7}$$

$$(8)\qquad f_1(t, x, \phi(t, x)) \geqq \phi_t(t, x) + \phi_x(t, x)\phi(t, x) \qquad \text{for} \quad (t, x) \in W$$

and

$$(9)\qquad f_1(t, x, \psi(t, x)) \geqq \psi_t(t, x) + \psi_x(t, x)\psi(t, x)$$
$$\text{for} \quad (t, x) \in \text{closure of} \quad W \setminus W_0 .$$

It follows from (iii), (9) and the definition of $M(t, x)$ that

$$(10)\qquad f_1(t, x, M(t, x)) \geqq \psi_t(t, x) + \psi_x(t, x)\psi(t, x) \qquad \text{for} \quad (t, x) \in W .$$

In the inequality (8) we can assume the strict inequality

$$(11)\qquad f_1(t, x, \phi(t, x)) > \phi_t(t, x) + \phi_x(t, x)\phi(t, x) \quad \text{for} \quad (t, x) \in W \setminus W_0$$

by replacing f_1 by the function $f_1(t, x, y) + \text{dist}((t, x), W_0)$, where $\text{dist}((t, x), W_0)$ denotes the distance from the point (t, x) to the set W_0.

Next it will be shown that f_1 has a continuous extension f_2 to $X \equiv W \times R$ which satisfies

$$(12)\qquad f_2(t, \beta(t), \beta'(t)) \geqq \beta''(t) \qquad \text{for} \quad t \in I ,$$

$$(13)\qquad f_2(t, x, y) \geqq f_1(t, x, M(t, x)) \qquad \text{for} \quad (t, x) \in W , \quad y > M(t, x)$$

and

$$(14)\qquad f_2(t, x, y) \geqq f_1(t, x, \phi(t, x)) \qquad \text{for} \quad (t, x) \in W , \quad y < \phi(t, x) .$$

To prove this, define the function $g: X \to R$ by

$$g(t, x, y) = \begin{cases} f_1(t, x, M(t, x)) & \text{for} \quad (t, x) \in W , \quad y > M(t, x) , \\ f_1(t, x, y) & \text{for} \quad (t, x, y) \in Y , \\ f_1(t, x, \phi(t, x)) & \text{for} \quad (t, x) \in W , \quad y < \phi(t, x) . \end{cases}$$

Then clearly g is a continuous extension of f_1. We denote the compact set $\{(t, \beta(t), \beta'(t)) : t \in I\}$ by A. It follows from (7) that $g(t, x, y) \geqq \beta''(t)$ on $Y \cap A$. Therefore by Lemma 9 (b), we have a continuous extension f_2 of f_1 to X which satisfies (12), (13) and (14) by the definition of g.

Finally we define a continuous extension h of f_2 to $\Omega = I \times R \times R$ by

$$h(t, x, y) = \begin{cases} f_2(t, \beta(t), y) + x - \beta(t) & \text{for} \quad x > \beta(t) , \\ f_2(t, x, y) & \text{for} \quad \alpha(t) \leqq x \leqq \beta(t) , \\ f_2(t, \alpha(t), y) + x - \alpha(t) & \text{for} \quad x < \alpha(t) . \end{cases}$$

Now instead of the equation (2), we consider the equation

$$(15)\qquad x'' = h(t, x, x')$$

or the equivalent system

$$(16)\qquad x' = y , \quad y' = h(t, x, y) ,$$

and examine the properties of solution curves of (16) in Ω.

Let x be a solution of (15) satisfying $x(\tau) < \alpha(\tau)$ and $x'(\tau) = \alpha'(\tau)$ for some τ in I. Here notice that by the conditions (i) and (iii), we have $(t, \alpha(t), \alpha'(t)) \in D$ for all t in I, and hence the inequality (3) holds for all t in I. Then we obtain $x''(\tau) = h(\tau, x(\tau), x'(\tau)) = f_2(\tau, \alpha(\tau), \alpha'(\tau)) + x(\tau) - \alpha(\tau) < f(\tau, \alpha(\tau), \alpha'(\tau)) \leqq \alpha''(\tau)$. Therefore we easily observe that the (t, x, y)-set P_1 defined by

$$P_1 : a \leqq t \leqq b\ , \quad x < \alpha(t)\ , \quad y \leqq \alpha'(t)$$

is a positively invariant set of (16) and the (t, x, y)-set N_1 defined by

$$N_1 : a \leqq t \leqq b\ , \quad x < \alpha(t)\ , \quad y \geqq \alpha'(t)$$

is a negatively invariant set of (16). Similarly, by (12), the following two (t, x, y)-sets

$$P_2 : a \leqq t \leqq b\ , \quad x > \beta(t)\ , \quad y \geqq \beta'(t)$$

and

$$N_2 : a \leqq t \leqq b\ , \quad x > \beta(t)\ , \quad y \leqq \beta'(t)$$

are, respectively, a positively invariant set and a negatively invariant set of (16).

Let P_3 be the (t, x, y)-set defined by

$$P_3 : a \leqq t \leqq b\ , \quad \alpha(t) \leqq x \leqq \beta(t)\ , \quad y > M(t, x)\ .$$

It will be verified that every solution $(x(t), y(t))$ of (16) starting from a point in P_3 to the right remains in P_3 as long as the solution satisfies the inequality $\alpha(t) \leqq x(t) \leqq \beta(t)$. Indeed otherwise, there exists a solution $x(t)$ of (15) defined on $[t_0, t_1]$ such that $(t, x(t), x'(t)) \in P_3$ on $[t_0, t_1)$ and $x'(t_1) = M(t_1, x(t_1))$. Define the function $V(t)$ by

$$V(t) = [x'(t) - \psi(t, x(t))] \exp \int_{t_0}^{t} \psi_x(s, x(s))ds \qquad \text{for} \quad t_0 \leqq t \leqq t_1\ .$$

Then we have for $t \in [t_0, t_1]$,

$$\begin{aligned} V'(t) \exp\left[-\int_{t_0}^{t} \psi_x(s, x(s))\,ds \right] & \\ &= x''(t) - \psi_t(t, x(t)) - \psi_x(t, x(t))\psi(t, x(t)) \\ &= f_2(t, x(t), x'(t)) - \psi_t(t, x(t)) - \psi_x(t, x(t))\psi(t, x(t)) \\ &\geqq f_2(t, x(t), M(t, x(t))) - \psi_t(t, x(t)) - \psi_x(t, x(t))\psi(t, x(t)) \quad \text{[by (13)]} \\ &\geqq 0 \quad \text{[by (10)]}\ , \end{aligned}$$

that is, $V'(t) \geqq 0$ on $[t_0, t_1]$. Since $V(t_0) > 0$, we have $V(t_1) > 0$, that is,

$x'(t_1) > \psi(t_1, x(t_1))$. By the definition of $M(t, x)$ and the assumption that $x'(t_1) = M(t_1, x(t_1))$, we have $x'(t_1) = \phi(t_1, x(t_1))$ and $(t_1, x(t_1)) \in W \setminus W_0$. On the other hand, the inequality $x'(t) \geqq M(t, x(t)) \geqq \phi(t, x(t))$ holds on $[t_0, t_1]$. Therefore it follows that

$$x''(t_1) \leqq \frac{d}{dt}\phi(t, x(t))\,|_{t=t_1}\,,$$

namely, $h(t_1, x(t_1), x'(t_1)) \leqq \phi_t(t_1, x(t_1)) + \phi_x(t_1, x(t_1))\phi(t_1, x(t_1))$. This contradicts (11) because $h(t_1, x(t_1), x'(t_1)) = f_1(t_1, x(t_1), \phi(t_1, x(t_1)))$ and $(t_1, x(t_1)) \in W \setminus W_0$. Therefore we are done.

Thus, it turns out that every solution curve of (16) which leaves P_3 must enter $P_1 \cup P_2 \cup N_1 \cup N_2$. However, such a solution curve must enter P_2 because P_3 does not intersect the negatively invariant set $N_1 \cup N_2$ of (16) and the inequality $\alpha'(t) \leqq \psi(t, \alpha(t))$ on I prevents the curve from entering P_1. Consequently, $P_2 \cup P_3$ is a positively invariant set of (16).

Now let N_3 be the set of points (t, x, y) such that $a \leqq t \leqq b$, $\alpha(t) \leqq x \leqq \beta(t)$, $y < \phi(t, x)$ or that $(t, x) \in W \setminus W_0$, $y = \phi(t, x)$. Then we observe that $N_2 \cup N_3$ is negatively invariant by (8), (11), (14) and an argument similar to the one above.

Thus $\Omega \setminus D$ is expressed as the sum of a negatively invariant set $N = N_1 \cup N_2 \cup N_3$ and two positively invariant sets $P = P_1$ and $Q = P_2 \cup P_3$ (see Figure 1, where the shaded areas are positively invariant sets). It is clear that $P \cap \bar{Q} = \bar{P} \cap Q = \varnothing$.

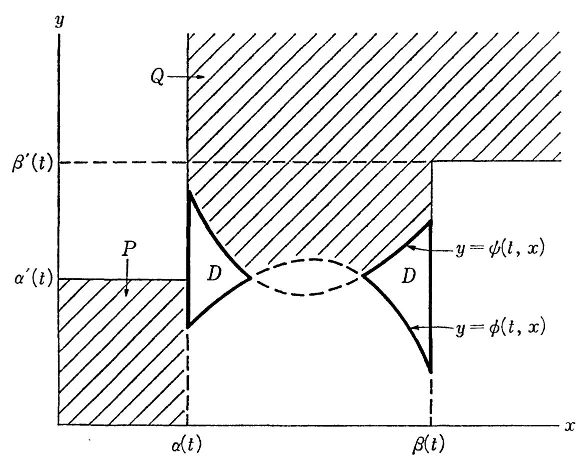

FIGURE 1, $\Omega|_t$, $t \in I$.

Let E be a line segment whose endpoints are $(a, \alpha(a), \alpha'(a)) \in \bar{P}$ and $(a, \alpha(a), \psi(a, \alpha(a))) \in \bar{Q}$. Then E satisfies all the assumptions in Theorem

1, and hence we obtain a solution curve of (16) which lies in D on I. Namely, the equation (15) has a solution $x(t)$ satisfying (5) and (6). Since $f = h$ on D, this solution $x(t)$ satisfies (2). q.e.d.

SKETCH OF THE PROOF OF THEOREM 3. Let P_1, P_2, P_3, N_1, N_2 and N_3 be the sets defined in the proof of Theorem 2. We can find a continuous extension h of f to the domain $\Omega = I \times R \times R$ so that $P = P_1$ and $Q = P_2$ are positively invariant sets of (16) and $N = N_1 \cup N_2 \cup N_3 \cup P_3$ is a negatively invariant set of (16) (see Figure 2, where the shaded areas are positively invariant sets). Clearly, $P \cap \bar{Q} = \bar{P} \cap Q = \varnothing$.

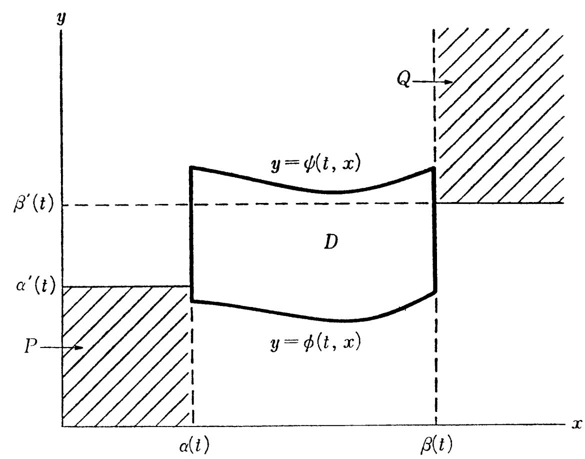

FIGURE 2, $\Omega|_t$, $t \in I$.

Let E be a continuous curve in $D|_a$ joining the point $(a, \alpha(a), \phi(a, \alpha(a))) \in \bar{P}$ to the point $(a, \beta(a), \psi(a, \beta(a))) \in \bar{Q}$. Then E satisfies all the assumptions in Theorem 1, and hence $Z^+(E)|_b$ contains a bridge T from $\bar{P}$ to $\bar{Q}$ such that $T =$ closure of $T \cap D \subset \bar{D} = D$. It then follows that T must intersect the line segment whose endpoints are $(b, \eta, \phi(b, \eta))$ and $(b, \eta, \psi(b, \eta))$ for any η, $\alpha(b) \leqq \eta \leqq \beta(b)$. Therefore the equation (2) has a solution $x(t)$ satisfying (5), (6) and $x(b) = \eta$.

5. Several remarks. For ordinary differential equations, the replacement of t by $-t$ does not yield any essential difference. Therefore by such a replacement, we obtain corresponding theorems in the dual position.

We consider the following condition (C′) corresponding to the condition (C).

(C′) The set $\Omega \setminus D$ is the sum of a positively invariant set P and two negatively invariant sets N and L such that $N \cap \bar{L} = \bar{N} \cap L = \varnothing$.

THEOREM 1′. *Suppose that the condition* (C′) *holds and that E is a* *compact and connected set in* $D \cup N \cup L$. *If* $Z^-(E)$ *intersects both* $\bar{N}$ *and* $\bar{L}$, *then* $Z^-(E)|_a$ *contains a bridge* T *from* $\bar{N}$ *to* $\bar{L}$ *such that* T *is the closure of a connected set* $T \cap D$. *Furthermore, every solution curve of* (1) *joining a point in* E *to a point in* $T \cap D$ *lies in* D.

Similarly, by substituting $-t$ for t and $-x$ for x in (2), the conditions (i), (ii), (iii) and (iv) are transformed, respectively, into (ii), (i), (iv) and (iii) provided that $[-b, -a]$, $-f(-t, -x, y)$, $-\beta(-t)$, $-\alpha(-t)$, $\phi(-t, -x)$ and $\psi(-t, -x)$ stand for $[a, b]$, $f(t, x, y)$, $\alpha(t)$, $\beta(t)$, $\phi(t, x)$ and $\psi(t, x)$, respectively. Here, notice that by this replacement, the conditions (3) and (4) are preserved. Thus, corresponding to Theorems 2 and 3, we have the following theorems.

THEOREM 2$'$. *Suppose that both of the conditions* (ii) *and* (iv) *hold. Then the equation* (2) *has at least one solution* $x(t)$ *satisfying* (5) *and* (6).

THEOREM 3$'$. *Suppose that both of the conditions* (ii) *and* (iii) *hold, and that the inequality* $\phi(t, x) \leqq \psi(t, x)$ *holds on* W. *Then for an arbitrary number* ξ *with* $\alpha(a) \leqq \xi \leqq \beta(a)$, *the equation* (2) *has at least one solution* $x(t)$ *satisfying* (5), (6) *and* $x(a) = \xi$.

We can prove Theorems 2$'$ and 3$'$ by using Theorem 1, too.

In [7], we considered the boundary layer problem

$$x''' + 2xx'' + 2\lambda(1 - x'^2) = 0\,,$$
$$x(0) = x'(0) = 0\,, \quad x'(\infty) = 1\,,$$
$$0 < x(t) < 1 \quad \text{for} \quad 0 < t < \infty\,,$$

by reducing it to a singular boundary value problem for a second order ordinary differential equation. To solve the problem we applied an existence theorem under the conditions (ii) and (iv), but here the equality sign could not be allowed for $\Gamma(\phi) \leqq 0$ and $\Gamma(\psi) \leqq 0$. Therefore by utilizing Theorem 1 and the same argument as in the proof of Theorem 2 for the existence theorem used in [7], we can allow the equality sign for $\lambda > -0.19880$ in [7], which is a range to assure the existence of a solution of the problem.

Until now the interval I was assumed compact. However even in the case where I is not compact, we can have an existence theorem for a solution of (1) to remain in a given set. All notations and terminologies given in Section 3 will remain except that $\bar{A}$ denotes the closure of $A \subset I \times R^n$ in the relative topology of $I \times R^n$ and that the compactness of $\bar{D} \subset \Omega$ is replaced by the compactness of $\bar{D} \cap (J \times R^n) \subset \Omega$ for each compact set $J \subset I$. Then under the condition (C), we have the following corollaries to Theorem 1 which are easily proved by Theorem 1 and the standard diagonal process.

COROLLARY 1. *Let* $I = [a, b)$, $-\infty < a < b \leqq +\infty$, *and let* E *be a compact and connected set in* $D \cup P \cup Q$. *If* $Z^+(E)$ *intersects both* $\bar{P}$ *and* $\bar{Q}$, *then the equation* (1) *has at least one solution* $x(t)$ *such that* $(t_0, x(t_0)) \in E$ *for some* $t_0 \in I$ *and that* $(t, x(t)) \in \bar{D}$ *for* $t_0 \leqq t < b$.

COROLLARY 2. *Let* $I = (a, b)$, $-\infty \leqq a < b \leqq +\infty$. *If there exists a sequence* $\{t_k\}$ *in* I *and a sequence* $\{E_k\}$ *of compact and connected sets in* $(D \cup P \cup Q)|_{t_k}$ *such that* $\lim_{k\to\infty} t_k = a$ *and that* $Z^+(E_k)$ *intersects both* $\bar{P}$ *and* $\bar{Q}$ *for each* $k = 1, 2, \cdots$, *then the equation* (1) *has at least one solution which lies in* $\bar{D}$ *on the interval* I.

We leave it to the reader to state, under the condition (C′), Corollaries 1′ and 2′ to Theorem 1′ analogous to Corollaries 1 and 2 to Theorem 1.

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