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SMOOTHING FRACTIONAL GROWTH

JOHN CANTWELL* AND LAWRENCE CONLON*

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Introduction. Let $s > 2$ be a real number and let T be a closed, orientable surface of genus at least 2. In [C-C2, Section 7] we produced C^1-diffeomorphisms h, l of the compact interval $I = [-1, 1]$ so that the foliation $\mathscr{F}(h, l)$ of $T \times I$, produced by "suspension", contains a continuum of leaves, each having growth type exactly that of the function x^s (growth exactly of degree s). This foliation is C^1-trivial at the boundary, hence, as in the proof of [C-C1, (3.5)] it can be imbedded as a component in C^1-foliations of suitable 3-manifolds. This gives the following.

THEOREM. *If M is a closed, orientable 3-manifold with $H^1(M; \boldsymbol{R}) \neq 0$, then M admits a C^1-foliation having a leaf with growth exactly of degree s.*

Many growth types properly between polynomials have also been obtained by Tsuchiya [Ts], who does not claim any smoothness for the construction. In addition, Hector knows a C^1 construction of foliations with similar growth properties (unpublished). Further smoothing of these foliations has seemed difficult. Here we show how to carry out the construction in [C-C2] so that the foliation $\mathscr{F}(h, l)$ is of class C^∞ and is C^∞-trivial at the boundary.

THEOREM $(*)$. *Let $s > 2$ be a real number and let M be a closed, orientable 3-manifold such that $H^1(M; \boldsymbol{R}) \neq 0$. Then there is a C^∞-foliation of M having a leaf with growth exactly of degree s.*

REMARKS. (1) If $s > 3$, then, as in [C-C2, (5.2)], the condition that $H^1(M; \boldsymbol{R}) \neq 0$ can be dropped.

(2) By the theory of [C-C3], one can show that growth exactly of degree $s \neq 1$, $0 < s < 2$, is impossible in C^2-foliations (cf. [Ts, Theorem A]).

(3) If $n + 1 < s < n + 2$, it is relatively easy to obtain differentiability of class C^n (3.3), but we will see (3.5) that the most natural choices of the contraction h will make it impossible for the differentiability to be of class C^{n+1}. In Section 4, an appropriately "unnatural" construction of h will complete the proof of $(*)$.

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(4) Smooth examples of fractional growth in higher codimensions and/or higher dimensions are obtained via Cartesian products $M^3 \times M^m \times M^q$.

1. The basic construction. Let $f(x)(d/dx)$ be a C^∞ vector field on $I = [-1, 1]$ such that $f(x) < 0$, $-1 < x < 1$, and such that f is C^∞-tangent to 0 at $x = \pm 1$. Let $h_u(x)$ denote the associated flow, $-1 \leqq x \leqq 1$, $u \in \boldsymbol{R}$. For each positive value of u, h_u is a C^∞ contraction of $[-1, 1)$ to -1 and it is C^∞-tangent to the identity at $x = \pm 1$. Let $h = h_1$.

Fix $s > 2$ and write $s = 2 + r^{-1}$. Let $p > 0$ be an integer such that $2^{rp} > p$ and $[2^{r(k+1)}] > [2^{rk}]$ for $k \geqq p$, where $[\cdot]$ denotes the greatest integer function. Set

$$N(k) = \begin{cases} k - 1\,, & 1 \leqq k \leqq p\,, \\ [2^{rk}]\,, & k > p\,, \end{cases}$$

a strictly increasing sequence of non-negative integers. Also, set $c(k) = 2^{1-k}$, $k \geqq 1$.

Let $\varphi(x)(d/dx)$ be a C^∞ vector field on I, identically 0 outside of $J = [h(0), 0]$, but nonsingular on $\operatorname{int}(J)$. Let $\Phi_t(x)$ denote the associated flow on $[-1, 1]$. For each integer $k \geqq 1$, define a "bump on the identity" by

$$l_k = h^{N(k)} \circ \Phi_{c(k)} \circ h^{-N(k)}\,,$$

supported on $h^{N(k)}(J)$, and note that the infinite composition $l = \cdots \circ l_k \circ l_{k-1} \circ \cdots \circ l_1$ makes sense and defines a homeomorphism $l\colon I \to I$. In fact, $l\,|\,(-1, 1]$ is a C^∞-diffeomorphism, C^∞-tangent to the identity at $x = 1$, and in [C-C2, (7.2)] we proved

LEMMA (1.1). *l is a C^1-diffeomorphism, C^1-tangent to the identity at $x = -1$.*

Finally, in [C-C2, (7.6)], we proved

THEOREM (1.2). *The C^1-foliation $\mathscr{F}(h, l)$ of $T \times I$ has a continuum of dense leaves without holonomy, each having growth exactly of degree s.*

Here we will show that a careful choice of the vector field $f(x)(d/dx)$ will guarantee that l is a C^∞-diffeomorphism and is C^∞-tangent to the identity at $x = -1$. This will complete the proof of $(*)$.

2. Preliminary lemmas. Let $u > 0$ be fixed, but arbitrary. Define $g_u\colon [-1, 1] \to \boldsymbol{R}$ by

$$g_u(h_u(x)) = h'_u(x)\varphi(x)\,.$$

Thus, g_u is supported on $h_u(J)$ and an elementary computation gives the following.

LEMMA (2.1). *The conjugated flow* $h_u \circ \Phi_t \circ h_{-u}$ *has velocity field* $g_u(x)(d/dx)$.

Set $Q_0(x) = \varphi(x)/f(x)$, $h(0) \leqq x \leqq 0$, and define $Q_n(x) = f(x)Q'_{n-1}(x)$, $n \geqq 1$, $x \in J$.

LEMMA (2.2). *For each integer* $n \geqq 0$,

$$\lim_{u\to\infty} g_u^{(n)}(h_u(x))(f(h_u(x)))^{n-1} = Q_n(x)\ ,$$

C^∞*-uniformly on* J *(i.e., the convergence is uniform in each derivative taken individually).*

PROOF. Since the flow h_u preserves its own velocity field $f(d/dx)$, and since $f(x) \neq 0$ on $(-1, 0]$, a simple computation yields $h'_u(x) = f(h_u(x))/f(x)$, $-1 < x \leqq 0$, $\forall u$. In particular, $g_u(h_u(x))/f(h_u(x)) = Q_0(x)$, $x \in J$, and the lemma follows for $n = 0$. For the inductive step, write $g_u^{(n+1)}(h_u(x))(f(h_u(x)))^n = f(x)g_u^{(n+1)}(h_u(x))h'_u(x)(f(h_u(x)))^{n-1} = f(x)(d/dx)(g_u^{(n)}(h_u(x)))\times (f(h_u(x)))^{n-1} = f(x)(d/dx)(g_u^{(n)}(h_u(x))(f(h_u(x)))^{n-1}) - (n-1)g_u^{(n)}(h_u(x))(f(h_u(x)))^{n-1}\times f'(h_u(x))$. By the inductive hypothesis and the (easily checked) fact that $\lim_{u\to\infty} f'(h_u(x)) = 0$, C^∞-uniformly on J, the lemma follows. q.e.d.

LEMMA (2.3). Q_n *is not identically* 0, $\forall n \geqq 0$.

PROOF. This is evident for Q_0. Also, note that Q_0 is C^∞-tangent to 0 at $x = 0$ and an easy induction shows the same for every Q_n. Suppose the lemma has been established for some $n \geqq 0$. If $0 \equiv Q_{n+1}(x) = f(x)Q'_n(x)$, it follows that Q_n is constant, hence $Q_n \equiv Q_n(0) = 0$. q.e.d.

LEMMA (2.4). *For each* $n \geqq 0$, *there are positive constants* K_n *and* U_n *such that, for all* $u \geqq U_n$ *and* $x \in J$,

$$|g_u^{(n)}(h_u(x))| < K_n |f(h_u(x))|^{1-n}\ .$$

Indeed, (2.4) is an immediate corollary to (2.2). For the following, choose $x_n \in J$ such that $Q_n(x_n) \neq 0$ (2.3) and again apply (2.2).

LEMMA (2.5). *For each* $n \geqq 0$, *there are positive constants* K_n^* *and* U_n^* *and a point* $x_n \in J$ *such that, for all* $u \geqq U_n^*$,

$$|g_u^{(n)}(h_u(x_n))| \geqq K_n^* |f(h_u(x_n))|^{1-n}\ .$$

3. Initial attempts at smoothing l. We imbed $l|(-1, 1]$ in a C^∞ flow λ_t. Let $l_{k,t} = h^{N(k)} \circ \Phi_{c(k)t} \circ h^{-N(k)}$, a C^∞ flow on $[-1, 1]$ supported on $J_k = h^{N(k)}(J)$. Evidently, $l_{k,1} = l_k$. The intervals J_k have disjoint interiors and cluster only at $x=-1$, so $\lambda_t = \cdots \circ l_{k,t} \circ l_{k-1,t} \circ \cdots \circ l_{1,t}$ defines the desired flow on $(-1, 1]$. By (2.1), the velocity field $g(d/dx)$ of this flow is given by $g = \sum_{k=1}^{\infty} c(k) g_{N(k)}$. We set $g(0) = 0$. The following is clear.

LEMMA (3.1). *If* $\lim_{k\to\infty} c(k) g_{N(k)}^{(j)}(h_{N(k)}(x)) = 0$, *uniformly on* J, $0 \leqq j \leqq n$, *then* $\lim_{y\to -1^+} g^{(j)}(y) = 0$ *and* l *is a* C^n*-diffeomorphism,* C^n*-tangent to the identity at* $x = -1$.

LEMMA (3.2). *The condition in* (3.1) *is fulfilled for* $n = 1$.

PROOF. $c(k) g_{N(k)}(h_{N(k)}(x)) = c(k) h'_{N(k)}(x) \varphi(x) = c(k) f(h_{N(k)}(x)) \varphi(x)/f(x)$, so the assertion is clear for $j = 0$. By (2.2), $\lim_{k\to\infty} c(k) g'_{N(k)}(h_{N(k)}(x)) = 0 \cdot Q_1(x)$ uniformly on J, giving the case $j = 1$. q.e.d.

In order to improve the smoothness, we specify the vector field $f(d/dx)$ and the associated flow h_u. On $[-1, 0]$, these will be

$$f(x) = \begin{cases} -(x+1)^2 e^{-1/(x+1)}, & -1 < x \leqq 0 \\ 0, & x = -1. \end{cases}$$

$$h_u(x) = \begin{cases} -1 + (\log(u + e^{1/(x+1)}))^{-1}, & -1 < x \leqq 0 \\ -1, & x = -1. \end{cases}$$

In standard fashion, extend f to $[-1, 1]$ so that $f < 0$ on $(-1, 1)$ and so that f is C^∞-tangent to 0 at $x = 1$. It is elementary that these choices satisfy the smoothness conditions at $x = -1$. Also

$$f(h_u(x)) = -(u + e^{1/(x+1)})^{-1} (\log(u + e^{1/(x+1)}))^{-2}$$

for $-1 < x \leqq 0$, so, by (2.4),

$$|g_u^{(j)}(h_u(x))| \leqq K_j (u + e^{1/(x+1)})^{j-1} (\log(u + e^{1/(x+1)}))^{2j-2}$$

for $u \geqq U_j$ and $x \in J$. Thus,

$$|c(k) g_{N(k)}^{(j)}(h_{N(k)}(x))| \leqq 2K_j 2^{-k} (2^{rk} + e^{1/(x+1)})^{j-1} (\log(2^{rk} + e^{1/(x+1)}))^{2j-2}$$

for large enough values of k and all $x \in J$. For possibly larger values of k, $2^{rk} + e^{1/(x+1)} \leqq 2^{r(k+1)}$, $\forall x \in J$, so

$$|c(k) g_{N(k)}^{(j)}(h_{N(k)}(x))| \leqq \tilde{K}_j (k+1)^{2j-2} 2^{k(r(j-1)-1)}$$

for a suitable positive constant $\tilde{K}_j$ and all large values of k.

LEMMA (3.3). *If* $s > n + 1$, *then, with the above choices,* l *is a* C^n*-diffeomorphism and is* C^n*-tangent to the identity at* $x = -1$.

PROOF. Since $2 + r^{-1} = s > n + 1$, we see that $r(j-1) - 1 < 0$ for $0 \leqq j \leqq n$, so $\lim_{k\to\infty} c(k) g_{N(k)}^{(j)}(h_{N(k)}(x)) = 0$. q.e.d.

For a large class of reasonable choices of $f(d/dx)$, including the above, the differentiability in (3.3) cannot be improved. We make this precise.

PROPOSITION (3.4). *If, for some* $\varepsilon > -1$, $f'(x) \leqq 0$ *on* $[-1, \varepsilon]$, *and if* $s < n + 1$, *then* l *is not* C^n *at* $x = -1$.

PROOF. Set $H_u(x) = 1 + h_u(x) - uf(h_u(x))$, $x \in J$. Then $(\partial/\partial u)H_u(x) = -uf'(h_u(x))f(h_u(x)) \leqq 0$, for u large and $x \in J$. But $\lim_{u\to\infty}(1 + h_u(x)) = 0$ uniformly on J and $H_u(x) > 0$, so $uf(h_u(x))$ is uniformly bounded on J as $u \to \infty$. Thus, for k large and A a suitable positive constant, $|f(h_{N(k)}(x))| < A/N(k)$. Let x_n, K_n^* be as in (2.5) and conclude that

$$|g^{(n)}(h_{N(k)}(x_n))| \geqq K_n^* A^{1-n} 2^{1-k}(N(k))^{n-1} .$$

For k large, $N(k) \geqq 2^{r(k-1)}$, hence

$$|g^{(n)}(h_{N(k)}(x_n))| \geqq K 2^{k(r(n-1)-1)}$$

for a suitable positive constant K. But $r(n-1) - 1 > 0$, so

$$\lim_{k\to\infty} |g^{(n)}(h_{N(k)}(x_n))| = \infty . \qquad \text{q.e.d.}$$

The condition $f' \leqq 0$ on $[-1, \varepsilon]$ implies $h' \leqq 1$ on $[-1, \varepsilon]$, but not conversely. But this weaker condition also implies the conclusion of (3.4).

PROPOSITION (3.5). *If, for some* $\varepsilon > -1$, $h'(x) \leqq 1$ *on* $[-1, \varepsilon]$, *and if* $s < n + 1$, *then* l *is not* C^n *at* $x = -1$.

PROOF. As in the proof of (3.4), it will be enough to produce a constant $A > 0$ such that $|uf(h_u(x))| \leqq A$, for all large u and $x \in J$. For this, we will find u_* so large that, for $u \geqq u_*$,

(a) $$|f(h_u(x))| \leqq 1 ,$$

(b) $$|f(h_u(x))|(u - u_*) \leqq 2 ,$$

uniformly for $x \in J$. It is easy to guarantee (a). Having prove (b), we will take $A = 2 + u_*$.

First, we find u_* so that $|f(h_u(x))|/2 < |f(h_v(x))|$, $u \geqq u_*$, $u - 1 \leqq v \leqq u$, $x \in J$. We can do this for an arbitrary but fixed $x \in J$, since the same u_* works on a neighborhood of x and J is compact. If this cannot be done, there are arbitrarily large values of u and $v \in [u - 1, u]$ such that $|f(h_u(x))|/2 \geqq |f(h_v(x))|$. Let $w_0 \in [v, u]$ maximize $|f(h_w(x))|$ on that interval. Then $f(h_{w_0}(x)) < f(h_{w_0}(x))/2 \leqq f(h_v(x)) < 0$, so

$$\begin{aligned} |f(h_{w_0}(x))|/2 &\leqq |(f(h_{w_0}(x)) - f(h_v(x)))/(w_0 - v)| \\ &= \left|\frac{d}{dt} f(h_t(x))_{t=\xi}\right| = |f'(h_\xi(x))f(h_\xi(x))| , \end{aligned}$$

some $\xi \in (v, w_0)$. That is, there are arbitrarily large values of ξ for which $|f'(h_\xi(x))| \geqq (1/2)|f(h_{w_0}(x))/f(h_\xi(x))| \geqq 1/2$, contradicting the fact that f is C^∞-flat at -1.

Choosing u_* as above, also make it large enough that (a) is satisfied and that $h_v(x) \in [-1, \varepsilon]$, $v \geqq u_*$, $x \in J$. Then, if $v \in [u_*, u]$, choose $p \in \boldsymbol{Z}^+$ such that $v+p \in [u-1, u]$. Then, uniformly for $x \in J$, $|f(h_{v+p}(x))/f(h_v(x))| = h'_p(h_v(x)) = h'(h_{p-1}(h_v(x))) \cdots h'(h_v(x)) \leqq 1$. That is, $|f(h_v(x))| \geqq |f(h_{v+p}(x))| > (1/2)|f(h_u(x))|$, so $(1/2)|f(h_u(x))|(u-u_*) < \int_{u_*}^u |f(h_v(x))|dv = -\int_{u_*}^u f(h_v(x))dv \leqq -\int_0^u f(h_v(x))dv = -h_u(x) + x \leqq -h_u(x) < 1$. This establishes (b). q.e.d.

REMARKS. (1) Thus, for greater smoothness, we will need a very "bumpy" contraction h (Figure 1).

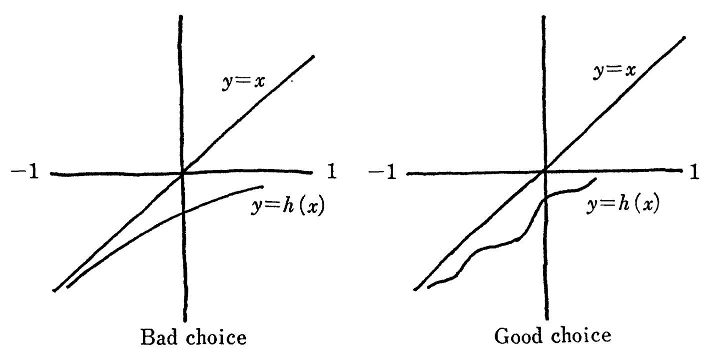

FIGURE 1

(2) The construction in [C-C2, Section 7] allowed a certain latitude in the choices of $c(k)$, $N(k)$. Given integers $s(k) \geqq 2$ for all $k \geqq 2$, we defined $1/c(k) = s(2)s(3) \cdots s(k)$ (and $c(1) = 1$) and required that $N(k) \geqq s(2) + s(3) + \cdots + s(k)$ for all large values of k. In our present construction, all $s(k) = 2$, but many other choices of $s(k)$, $N(k)$, related as above, yield growth types properly between polynomials of consecutive degrees n and $n+1$. In all cases, we can show that (3.5) will hold.

LEMMA (3.6). *If, for each $\varepsilon > 0$, $N(k)^\varepsilon f(h_{N(k)}(x))$ is bounded away from 0, uniformly on J, as $k \to \infty$, then l will be of class C^∞, C^∞-tangent to the identity at $x = -1$.*

PROOF. Let $B_\varepsilon > 0$ be such that

$$N(k)^\varepsilon |f(h_{N(k)}(x))| \geqq B_\varepsilon\,, \quad \forall k \text{ and } \forall x \in J\,.$$

Then, by (2.4), for k large and $x \in J$,

$$|g^{(n)}(h_{N(k)}(x))| \leqq K_\varepsilon 2^{-k}(N(k))^{\varepsilon(n-1)} \leqq K_\varepsilon 2^{k(r\varepsilon(n-1)-1)}$$

for a suitable constant $K_\varepsilon > 0$. Given n, we choose ε so that $r\varepsilon(n-1) - 1 < 0$. Thus, $\lim_{k\to\infty}|g^{(n)}(h_{N(k)}(x))| = 0$ uniformly on J. q.e.d.

This lemma will guide our construction of f.

4. Achieving infinite smoothness. We outline the basic program. Having fixed $s = 2 + r^{-1}$, hence having fixed the sequence $\{N(k)\}$, we will construct a C^∞ function $F\colon [0, \infty) \to \boldsymbol{R}$ such that

(1) $F(u) > 0$, $0 \leqq u < \infty$;

(2) $\int_0^\infty F = 1$;

(3) $F(u) = c_k$ is constant, $N(k) \leqq u \leqq 1 + N(k)$;

(4) for each $\varepsilon > 0$, $\lim_{k\to\infty}(N(k))^\varepsilon c_k = \infty$.

We will then define $h_u(0) = -\int_0^u F$. Evidently, this is a C^∞ function of u, $h_0(0) = 0$, $h_u(0)$ is strictly decreasing, $0 \leqq u < \infty$, and $\lim_{u\to\infty} h_u(0) = -1$. This defines the (positive) h_u-trajectory of 0 on $(-1, 0]$ with velocity field $f(h_u(0)) = (d/du)h_u(0) = -F(u)$. Of course, we will define $f(-1) = 0$ and, in constructing F, we will have to make sure that

(5) $\lim_{u\to\infty} f^{(n)}(h_u(0)) = 0$, $\forall n \geqq 1$.

CLAIM. *Successfully carrying out the above program will complete the proof of* $(*)$.

PROOF. By (5), $f(d/dx)$ will be a C^∞ vector field on $[-1, 0]$, C^∞-tangent to 0 at $x = -1$, and $f(x) < 0$, $-1 < x \leqq 0$. Since $N(1) = 0$, (3) guarantees that f is constant on $[h_1(0), 0]$, hence $f(d/dx)$ readily extends to a C^∞ vector field on $[-1, 1]$ as required. By (3), (4), and (3.6), the diffeomorphism l will be of class C^∞ and C^∞-tangent to the identity at $x = -1$. q.e.d.

We begin. Choose a sequence of integers $1 < k(1) < \cdots < k(q) < \cdots$ such that

$$N(k) \geqq 2^{rk-1}, \quad k \geqq k(1),$$
$$\sum_{k=k(q)}^{\infty} 2^{-rk/q^2} \leqq 2^{-q}, \quad q \geqq 1.$$

LEMMA (4.1).

$$\sum_{k=k(q)}^{\infty} N(k)^{-1/q} \leqq \sum_{k=k(q)}^{\infty} N(k)^{-1/q^2} \leqq 2\sum_{k=k(q)}^{\infty} 2^{-rk/q^2} < 2^{1-q}, \quad q \geqq 1.$$

LEMMA (4.2). *For a suitable choice of* $\{k(q)\}$ *as above, there is a sequence* $\{\delta(k)\}_{k\geqq 1}$ *such that* $\delta(k) = 1$ *for* $1 \leqq k < k(1)$ *and*

(a) $\{\delta(k)\}$ *decreases weakly monotonically to* 0;

(b) $\delta(k) \geqq 1/q$, $k(q) \leqq k < k(q+1)$;

(c) $\delta(k) \geqq k^{-1/4}$, $k \geqq 1$;

(d) $\delta(k+1) \geqq (k/(k+1))\delta(k)$, $k \geqq 1$.

PROOF. The choice $\delta(k) = 1/q$, $k(q) \leqq k < k(q+1)$, will guarantee (a) and (b). By rechoosing all $k(q)$ sufficiently large, this definition of $\delta(k)$ is also made to satisfy (c). In order to obtain (d), modify this sequence $\{\delta(k)\}$ inductively by

$$\tilde{\delta}(1) = \delta(1)$$
$$\tilde{\delta}(k+1) = \max\{\delta(k+1), (k/(k+1))\tilde{\delta}(k)\} .$$

We must show that (a) still holds for $\{\tilde{\delta}(k)\}$. Evidently, $\tilde{\delta}(k) \geqq \delta(k) \geqq \delta(k+1)$, so $\tilde{\delta}(k+1) \leqq \tilde{\delta}(k)$ for all k. If $\tilde{\delta}(k) = \delta(k)$ infinitely often, we obtain (a). Otherwise, the sequence is ultimately of the form $\tilde{\delta}(k+1) = (k/(k+1))\tilde{\delta}(k)$, and again we obtain (a). q.e.d.

The constants c_k in step (3) of the basic program will be $c_1 = c_2 = \cdots = c_{k_0-1} = 1$, $c_k = N(k)^{-\delta(k)}$, $k \geqq k_0$, where k_0 is so large that $N(k) > N(k-1) + 1$, $k \geqq k_0$.

LEMMA (4.3). *For each* $\varepsilon > 0$, $\lim_{k\to\infty}(N(k))^{\varepsilon}c_k = \infty$.

PROOF. $\lim_{k\to\infty}\delta(k) = 0$. q.e.d.

LEMMA (4.4). $\sum_{k=1}^{\infty} c_k < \infty$.

PROOF. By (4.1) and (b) in (4.2), $\sum_{k=1}^{\infty} c_k < \sum_{q=1}^{\infty}\sum_{k=k(q)}^{\infty} N(k)^{-1/q} \leqq \sum_{q=1}^{\infty} 2^{1-q}$. q.e.d.

Remark that (4.4) is a necessary condition that $\int_0^\infty F < \infty$. The choice $c_1 = 1$ will force $\int_0^\infty F = \eta > 1$, but ulimately we will normalize by replacing F with F/η, giving step (2) of the basic program.

A similar application of (4.1) and (4.2) gives

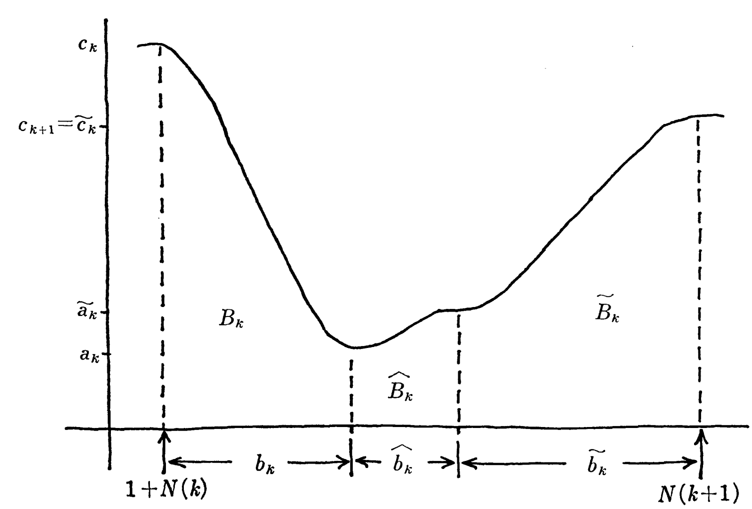

FIGURE 2

LEMMA (4.5). $\sum_{k=1}^{\infty} N(k)^{-\delta(k)^2} < \infty$.

For k_0 sufficiently large and for each $k \geqq k_0$, we will construct the graph of $F(u)$ on $[1 + N(k), N(k+1)]$ as indicated in Figure 2. There will be a major drop from c_k to a value a_k and a major rise from a value $\tilde{a}_k$ to $\tilde{c}_k = c_{k+1}$. Although we have pictured $a_k \leqq \tilde{a}_k$, this is not necessary.

We will have to guarantee that the areas B_k, $\hat{B}_k$, and $\tilde{B}_k$ in Figure 2 satisfy $\sum_{k=k_0}^{\infty}(B_k + \hat{B}_k + \tilde{B}_k) < \infty$. Together with (4.4), this will guarantee that $\int_0^{\infty} F < \infty$.

For the three segments of this graph we will construct "model" functions λ_0, $\hat{\lambda}_0$, $\tilde{\lambda}_0$ as in Figure 3 (where the numbers d_k, $\tilde{d}_k$ remain to

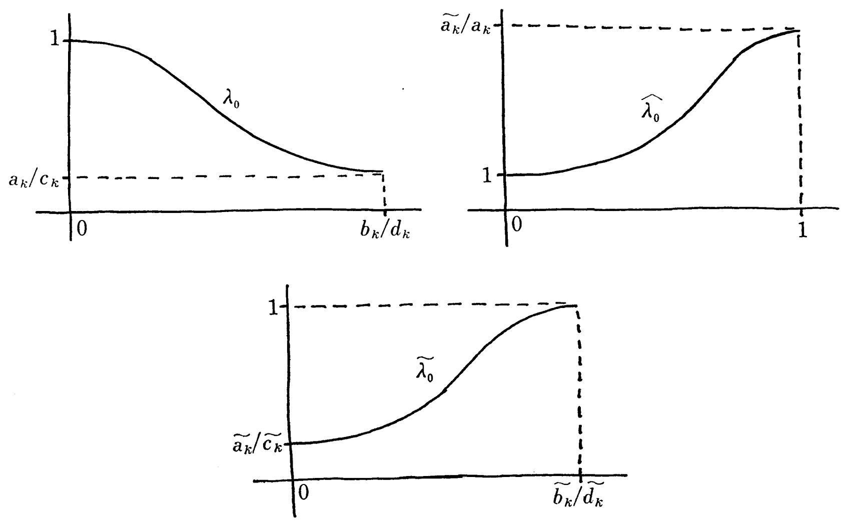

FIGURE 3

be specified). These functions will be C^{∞}-flat at the endpoints of their domains. In each case, the respective segments of $F(u)$ will be given (after a suitable translation in u) by

$$F(u) = c_k\lambda_0(u/d_k) = c_k\lambda_0(u^*)$$
$$F(u) = a_k\hat{\lambda}_0(u/\hat{b}_k) = a_k\hat{\lambda}_0(\hat{u})$$
$$F(u) = \tilde{c}_k\tilde{\lambda}_0(u/\tilde{d}_k) = \tilde{c}_k\tilde{\lambda}_0(\tilde{u}) .$$

These functions will depend on k, but certain of their essential features will not (e.g., there will be uniform upper bounds on their integrals).

In order to handle step (5) in the general program, we define inductively

$$\lambda_n = \lambda_{n-1}'/\lambda_0 , \quad n \geqq 1 ,$$

and similarly $\widehat{\lambda}_n$ and $\widetilde{\lambda}_n$. The usefulness of these functions comes from the following.

LEMMA (4.6). *After a suitable translation in u, the derivatives $f^{(n)}(h_u(0))$, $n \geqq 0$, will be given in the three respective domains by*

(a) $(-1)^{n+1} c_k^{1-n} d_k^{-n} \lambda_n(u^*)$

(b) $(-1)^{n+1} a_k^{1-n} \widehat{b}_k^{-n} \widehat{\lambda}_n(\widehat{u})$

(c) $(-1)^{n+1} \widetilde{c}_k^{1-n} \widetilde{d}_k^{-n} \widetilde{\lambda}_n(\widetilde{u})$.

PROOF. Consider only (a), the other cases being entirely similar. The case $n = 0$ is trivial. For the inductive step, remark that $(d/du)h_u(0) = -F(u) = -c_k\lambda_0(u^*)$, so $-c_k f^{(n+1)}(h_u(0))\lambda_0(u^*) = (d/du)f^{(n)}(h_u(0)) = (-1)^{n+1} c_k^{1-n} d_k^{-n} \lambda_n'(u^*)(du^*/du)$. Since $du^*/du = d_k^{-1}$, the assertion follows. q.e.d.

In the next lemma, a monomial $\lambda_1^{(m_1)} \lambda_1^{(m_2)} \cdots \lambda_1^{(m_q)}$ of derivatives of λ_1 of orders $m_i \geqq 0$ is said to have *total degree* $q + \sum_{i=1}^q m_i$.

LEMMA (4.7). *For each $n \geqq 1$, and independently of k, there exist polynomials P_n such that*

$$\lambda_n = \lambda_0^{1-n} P_n(\lambda_1^{(n-1)}, \lambda_1^{(n-2)}, \cdots, \lambda_1', \lambda_1)$$

and each monomial in $P_n(\lambda_1^{(n-1)}, \cdots, \lambda_1)$ has total degree n. These same relationships hold with λ_n replaced by $\widehat{\lambda}_n$ (respectively $\widetilde{\lambda}_n$), λ_1 replaced by $\widehat{\lambda}_1$ (respectively $\widetilde{\lambda}_1$), and λ_0 replaced by $\widehat{\lambda}_0$ (respectively $\widetilde{\lambda}_0$).

PROOF. Take $P_1(x) = x$ and, inductively $\lambda_{n+1} = \lambda_n'/\lambda_0 = \lambda_0^{-1}(\lambda_0^{1-n} P_n(\lambda_1^{(n-1)}, \cdots, \lambda_1))' = \lambda_0^{-n}(P_n(\lambda_1^{(n-1)}, \cdots, \lambda_1))' + \lambda_0^{-1}(1-n)\lambda_0^{-n}\lambda_0' P_n(\lambda_1^{(n-1)}, \cdots, \lambda_1) = \lambda_0^{-n}((P_n(\lambda_1^{(n-1)}, \cdots, \lambda_1))' + (1-n)\lambda_1 P_n(\lambda_1^{(n-1)}, \cdots, \lambda_1)) = \lambda_0^{-n} P_{n+1}(\lambda_1^{(n)}, \cdots, \lambda_1)$. q.e.d.

Because of (4.7), we will be able to employ (4.6) effectively in verifying step (5) of the general program.

We begin the construction. The only delicate step will be the definition of λ_0. Indeed, $\widetilde{\lambda}_0$ will be obtained from λ_0 by an orientation reversing change of parameter and the construction of $\widehat{\lambda}_0$ will be easy.

(A) *The construction of* λ_0. The domain $[0, b_k/d_k]$ will be subdivided into several segments. The initial segment will be $[0, l_k]$ where $l_k = 1 + \log(2^{r\sigma(k)})$, $\sigma(k) = [r^{-1}\log_2(k)] \in \boldsymbol{Z}$. In particular, $l_k \leqq 1 + \log(k)$ and $k - \sigma(k) \to \infty$.

On this initial segment, it will be convenient first to define λ_1 and then obtain $\lambda_0(u) = \exp\left(\int_0^u \lambda_1\right)$. The definition is indicated in Figure 4 (where $k \geqq k_0$ and k_0 is large enough that everything fits in). Here λ_1

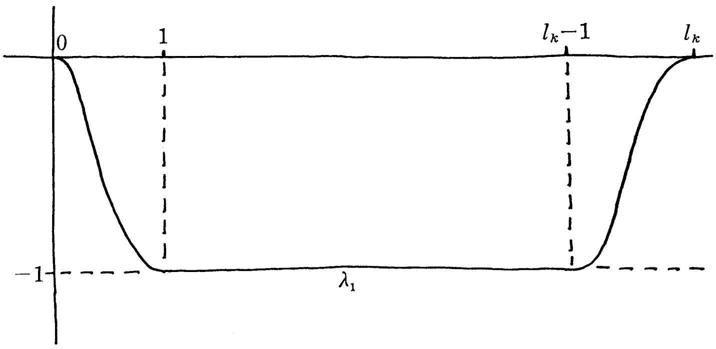

FIGURE 4

is C^∞-tangent to 0 at $u = 0$ and at $u = l_k$, $\lambda_1 < 0$ on $(0, l_k)$, and $\lambda_1 \equiv -1$ on $[1, l_k - 1]$. Furthermore, $\lambda_1 | [0, 1] = \alpha$ is chosen independently of k and $\lambda_1 | [l_k - 1, l_k]$ is obtained from α by the orientation reversing change of parameter $u^* = l_k - u$. Finally, choose α so that $\int_0^1 \alpha = -1/2$.

Remark that, for each $n \geqq 0$, $\lambda_1^{(n)}$ is uniformly bounded independently of k. The function λ_0 is C^∞-flat at $u = 0$ and at $u = l_k$, $\lambda_0(0) = 1$, $\lambda_0(l_k) = e^{1-l_k} = 2^{-r\sigma(k)} \geqq 1/k$, and $\lambda_0'(u) < 0$, $0 < u < l_k$. An easy computation gives

LEMMA (4.8). $\int_0^{l_k} \lambda_0 \leqq 2 + e^{-1/2}$.

LEMMA (4.9). *For each $n \geqq 1$, there exists a constant K_n, independent of k, such that, on $[0, l_k]$, $|\lambda_n| \leqq k^{n-1} K_n$.*

PROOF. Since $\lambda_1^{(j)}$ is uniformly bounded, independently of k, $0 \leqq j \leqq n - 1$, and since $\lambda_0 \geqq 1/k$, the assertion follows from (4.7). q.e.d.

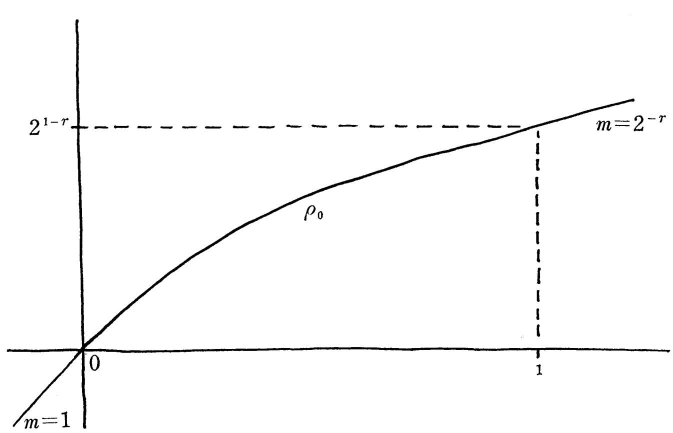

FIGURE 5

These two lemmas give the essential features of $\lambda_0|[0, l_k]$ that are independent of k.

In order to extend λ_0 to $[0, b_k/d_k]$, fix a strictly increasing function $\rho_0: [0, 1] \to [0, 2^{1-r}]$, concave down and C^∞-tangent at $u = 0$ and $u = 1$ to straight lines of respective slopes $m = 1$ and $m = 2^{-r}$ (Figure 5).

Define $\eta(j) = \delta(j - \sigma(j)) \geqq \delta(j)$ and remark that, for j sufficiently large, this sequence becomes weakly monotonic decreasing to 0. Set

$$\rho_j(u) = 2^{-rj\eta(j)}\rho_0(2^{-rj(1-\eta(j))}u) , \quad 0 \leqq u \leqq 2^{rj(1-\eta(j))} .$$

Remark that the total rise of ρ_j is

$$T_0 = 2^{1-r} , \quad j = 0 ,$$
$$T_j = 2^{1-r}2^{-rj\eta(j)} \leqq 2^{1-r}2^{-rj\delta(j)} , \quad j \geqq 1 ,$$

hence, by (4.1) and (4.2), $T = \sum_{j=0}^{\infty} T_j < \infty$. Set

$$l_k^0 = l_k + 2^{r\sigma(k)(1-\eta(\sigma(k)))} ,$$
$$l_k^j = l_k^{j-1} + 2^{r(\sigma(k)+j)(1-\eta(\sigma(k)+j))} , \quad 1 \leqq j \leqq k ,$$
$$J_0 = [l_k, l_k^0] ,$$
$$J_j = [l_k^{j-1}, l_k^j] , \quad 1 \leqq j \leqq k ,$$

and define $\rho: [0, l_k^k] \to [0, \infty)$ by

$$\rho(u) = \int_0^u \lambda_0 , \quad 0 \leqq u \leqq l_k ,$$
$$\rho(u) = \rho(l_k) + T_{\sigma(k)} + \cdots + T_{\sigma(k)+j-1} + \rho_{\sigma(k)+j}(u - l_k^{j-1}) ,$$
$$u \in J_j , \quad 0 \leqq j \leqq k .$$

Thus, ρ is C^∞ and we can define the smooth extension of λ_0 to $[0, l_k^k]$ by $\lambda_0(u) = \rho'(u) > 0$, a function that is C^∞-flat at $u = 0$ and at $u = l_k^k$. We define $b_k/d_k = l_k^k$ and $A = 2 + e^{-1/2} + T$. Then (4.8) gives the following.

LEMMA (4.10). $\int_0^{b_k/d_k} \lambda_0 \leqq A$, *a bound that is independent of* k.

By an elementary application of (4.7) to these definitions, we also obtain

LEMMA (4.11). *For each* $n \geqq 1$, *there is a constant* C_n, *not depending on* k, *such that, for* $\sigma(k) \leqq i \leqq \sigma(k) + k$ *and all* $u \in J_{i-\sigma(k)}$,

$$|\lambda_n(u)| \leqq 2^{-ri(1-n\eta(i))}C_n .$$

In particular, this bound becomes arbitrarily small as $k \to \infty$.

In checking (4.11), the assertion in (4.7) about total degree is essential.

Since λ_0 is monotonic decreasing, its minimum value is $\lambda_0(l_k^k) =$

$\rho'_{\sigma(k)+k}(l_k^k) = 2^{-r(k+\sigma(k)+1)}$. Thus, $a_k = c_k 2^{-r(k+\sigma(k)+1)}$. We specify $b_k = l_k^k d_k$ by defining $d_k = N(k)^{\delta(k)-\delta(k)^2}$. In order to know that, for k_0 sufficiently large and all $k \geqq k_0$, the interval $[1 + N(k), N(k+1)]$ will be big enough for our construction, we need the following.

LEMMA (4.12). $\lim_{k\to\infty} b_k/(N(k+1) - N(k) - 1) = 0$.

PROOF. First remark that $\eta(k + \sigma(k)) = \delta(k + \sigma(k) - \sigma(k + \sigma(k))) \geqq \delta(k)$. Thus, for k sufficiently large and K^* a suitable positive constant, $l_k^k \leqq l_k + \sum_{j=0}^{k+\sigma(k)} (2^{r(1-\delta(k))})^j \leqq 1 + \log(k) + 2^{r(1-\delta(k))}(2^{r/2} - 1)^{-1} 2^{r(1-\delta(k))(k+\sigma(k))} \leqq K^* 2^{r\sigma(k)} 2^{rk(1-\delta(k))} \leqq K^* k 2^{rk(1-\delta(k))}$. Also, $N(k+1) - N(k) - 1 \geqq 2^{r(k+1)} - 2^{rk} - 2 \geqq K^{**} 2^{rk}$, for k sufficiently large and K^{**} a suitable positive constant. Thus, setting $K = K^*/K^{**}$, we obtain $b_k/(N(k+1) - N(k) - 1) \leqq Kk2^{-rk\delta(k)^2} \leqq Kk2^{-r\sqrt{k}}$, where the second inquality is by (4.2), part (c). q.e.d.

(B) *The construction of* $\tilde{\lambda}_0$. Here we take $\tilde{b}_k = b_k$, $\tilde{d}_k = d_k$, $\tilde{c}_k = c_{k+1}$, and define $\tilde{a}_k$ by $\tilde{a}_k/\tilde{c}_k = a_k/c_k$. We set $\tilde{\lambda}_0(u) = \lambda_0((b_k/d_k) - u)$ and obtain

$$\int_0^{\tilde{b}_k/\tilde{d}_k} \tilde{\lambda}_0 = \int_0^{b_k/d_k} \lambda_0 , \quad |\tilde{\lambda}_n(u)| = |\lambda_n((b_k/d_k) - u)| ,$$

so (4.9), (4.10), and (4.11) hold for $\tilde{\lambda}_0$ and, of course, (4.12) holds for $\tilde{b}_k$.

(C) *The construction of* $\hat{\lambda}_0$.

LEMMA (4.13). $\lim_{k\to\infty} \tilde{a}_k/a_k = \lim_{k\to\infty} c_{k+1}/c_k = 1$.

PROOF. We have $\tilde{a}_k/a_k = c_{k+1}/c_k = N(k)^{\delta(k)}/N(k+1)^{\delta(k+1)}$. By (4.2), part (d), $\tilde{a}_k/a_k \leqq N(k+1)^{\delta(k)}/N(k+1)^{\delta(k)k/(k+1)} = (N(k+1)^{1/(k+1)})^{\delta(k)} \leqq (2^r)^{\delta(k)}$. Also, since $\{\delta(k)\}$ is weakly monotonically decreasing, $\tilde{a}_k/a_k \geqq (N(k)/N(k+1))^{\delta(k)} \geqq ((2^{rk} - 1)/2^{r(k+1)})^{\delta(k)}$. q.e.d.

Fix a bump function $\gamma \geqq 0$ on $[0, 1]$, identically 0 near the endpoints, such that $\int_0^1 \gamma = 1$. For $0 \leqq u \leqq 1$, set

$$\hat{\lambda}_1(u) = \log(\tilde{a}_k/a_k)\gamma(u) , \quad \hat{\lambda}_0(u) = \exp\left(\int_0^u \hat{\lambda}_1\right) .$$

Thus, $\hat{\lambda}_0(u)$ is identically 1 for u near 0 and identically $\tilde{a}_k/a_k$ for u near 1.

LEMMA (4.14). *For k_0 sufficiently large and $k \geqq k_0$, $\int_0^1 \hat{\lambda}_0 \leqq B$, a bound that is independent of k.*

PROOF. This is a corollary of (4.13). q.e.d.

LEMMA (4.15). *For k_0 sufficiently large and $k \geqq k_0$, there are constants D_n, depending only on $n \geqq 1$, such that $|\hat{\lambda}_n| \leqq D_n$.*

PROOF. Use (4.7) and (4.13). q.e.d.

(D) *Completion of the basic program.* By (4.12), choose k_0 so large that, for $k \geqq k_0$, $b_k = \tilde{b}_k \leqq (1/3)(N(k+1) - N(k) - 1)$. This gives enough room to carry out the construction of Figure 2.

Finitely often we have had to choose k_0 sufficiently large, so we fix the largest of these. Thus, F has been constructed on $[N(k_0), \infty)$ and on each interval $[N(k), 1 + N(k)]$, $1 \leqq k < k_0$. Fill in the finitely many gaps in any convenient way.

PROPOSITION (4.16). $\int_0^\infty F = \eta < \infty$.

PROOF. Referring to (4.4) and Figure 2, we see that we must prove (a) $\sum_{k=k_0}^\infty \hat{B}_k < \infty$, (b) $\sum_{k=k_0}^\infty B_k < \infty$, and (c) $\sum_{k=k_0}^\infty \tilde{B}_k < \infty$. By (4.14), $\hat{B}_k \leqq a_k \hat{b}_k B$. But $a_k = c_k 2^{-r(k+\sigma(k)+1)}$ and $\hat{b}_k \leqq N(k+1) - N(k) - 1 \leqq 2^{r(k+1)}$. Thus, $a_k \hat{b}_k \leqq c_k$ and (4.4) implies (a). By (4.10), $B_k \leqq c_k d_k A$. But $c_k d_k = N(k)^{-\delta(k)^2}$, so (4.5) implies (b). Finally, $\tilde{B}_k < \tilde{c}_k \tilde{d}_k A = c_{k+1} d_k A$ and, by (4.13), $c_{k+1} d_k / c_k d_k \to 1$, so $\sum c_{k+1} d_k$ also converges. q.e.d.

PROPOSITION (4.17). *For each* $n \geqq 1$, $\lim_{u\to\infty} f^{(n)}(h_u(0)) = 0$.

PROOF. By (4.6), (4.9), (4.11) and (4.15), we are reduced to proving (a) $\lim_{k\to\infty} c_k^{1-n} d_k^{-n} k^{n-1} = 0$, (b) $\lim_{k\to\infty} \tilde{c}_k^{1-n} \tilde{d}_k^{-n} k^{n-1} = 0$, and (c) $\lim_{k\to\infty} a_k^{1-n} \hat{b}_k^{-n} = 0$. By (4.13), (a) and (b) are equivalent. Furthermore, $c_k^{1-n} d_k^{-n} = N(k)^{-\delta(k)+n\delta(k)^2} \leqq 2^{-rk\delta(k)/2}$ (for k large) $\leqq 2^{-rk^{3/4}/2}$ ((4.2), part (c)). This gives (a) and (b). Finally, for k large and a suitable constant $K > 0$, $\hat{b}_k \geqq K 2^{rk}$ by (4.12), so $a_k^{1-n} \hat{b}_k^{-n} \leqq K^{-n} 2^{r(n-1)} 2^{r\sigma(k)(n-1)} 2^{rk((n-1)\delta(k)-1)} \leqq K^{-n} 2^{r(n-1)} k^{n-1} 2^{-rk/2}$ for k large. This gives (c). q.e.d.

We have carried out our basic program, except that $\int_0^\infty F = \eta > 1$. As earlier remarked, we normalize by replacing F with F/η.

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DEPARTMENT OF MATHEMATICS
ST. LOUIS UNIVERSITY
ST. LOUIS, MISSOURI 63103
U.S.A.

AND

DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY
CAMPUS BOX 1146
ST. LOUIS, MISSOURI 63130
U. S. A.