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ON THE ANALYTICITY OF THE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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1. Introduction. Consider the Navier-Stokes equation:

$$(1.1)\qquad \begin{cases} D_t u - \Delta u + \nabla p = f - \operatorname{div} N(u) & \text{in } D^+ \times (0, T)\,, \\ \operatorname{div} u = 0 & \text{in } D^+ \times (0, T)\,, \\ u|_{t=0} = u_0 \quad (\operatorname{div} u_0 = 0)\,, \quad u|_{x_3=0} = 0\,. \end{cases}$$

Here $N(u) = \{u_j u_k\}_{(j,k=1,2,3)}$ and

$$\operatorname{div} N(u) = \begin{pmatrix} \operatorname{div} N_1(u) \\ \operatorname{div} N_2(u) \\ \operatorname{div} N_3(u) \end{pmatrix}, \qquad N_j(u) = \begin{pmatrix} u_1 u_j \\ u_2 u_j \\ u_3 u_j \end{pmatrix}.$$

The set D is a neighborhood of the origin in the three dimensional Euclidean space E_3 and $D^+ = D \cap E_3^+$ with $E_3^+ = \{x = (x_1, x_2, x_3) \in E_3; x_3 > 0\}$. Let Ω and $\mathscr{D}$ be some complex neighborhoods of $(0, T)$ and D, respectively. Let $C^{r,r/2}(D^+ \times \Omega)$ be a weighted Hölder space. Now our result is as follows:

THEOREM 1.1. *Let f and u_0 be analytically extended from $D \times (0, T)$ and D to $\mathscr{D} \times \Omega$ and $\mathscr{D}$, respectively. Let $u \in C^{2+\mu,(2+\mu)/2}(D^+ \times \Omega)$ and $p \in C^{1+\mu,(1+\mu)/2}(D^+ \times \Omega)$ satisfy the equation* (1.1) *which are analytic in $\omega \in \Omega$ for each $x \in D^+$ $(0 < \mu < 1)$. Then $u(x, t)$ and $p(x, t)$ are analytic near $(0, t_0)$ for any t_0 $(0 < t_0 < T)$.*

The analyticity of the solutions was proved in Kahane [3] and Masuda [7], but they only proved the interior analyticity.

Many authors have proved the analyticity of the solutions of elliptic and parabolic equations, for example, Friedman [1], Morrey [8], etc. There are several methods to prove the analyticity. We will here use the method of Morrey. First, by Morrey [8], we shall show that there exists a complex analytic extension of the solution of the associated Stokes equation in a half space. Next, we will decompose the solution (u, p) of (1.1) into $u = u' + u''$, $p = p' + p''$, respectively. Here (u', p') is the solution of some integral equation and (u'', p'') is the solution of some Stokes equation. We will prove that they and their first spatial derivatives have complex analytic extensions in (z_1, z_2, ω) for $(z, \omega) \in \mathscr{D}_{0\delta} = \{(z, \omega) \in D \times \Omega; z = x + iy, \omega = t + is, x, y \in E_3, t, s \in E_1, y_3 = 0, |(z_1, z_2)| < \delta, |\omega - t_0| < \delta, 0 < x_3 < \delta\}$. Hence we see that u, $D_{x_3}u$, p are analytic in (z_1, z_2, ω) for $(z, \omega) \in \mathscr{D}_{0\delta}$. Moreover, we will see that

$$|u(z, \omega)| + |D_{x_3}u(z, \omega)| + |p(z, \omega)| < M$$

for $(z, \omega) \in \mathscr{D}_{0\delta}$, where δ and M are independent of x_3. Therefore, by the Cauchy-Kowalewsky Theorem, there exists a neighborhood of $(0, t_0)$ in $D \times (0, T)$ in which u and p are analytic.

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2. The Stokes equation. We consider the following Stokes equation in a half space:

$$\text{(2.1)} \quad \begin{cases} D_t v - \Delta v + \nabla q = f & \text{in} \quad E_3^+ \times (-T, T) , \\ \quad \operatorname{div} v = \operatorname{div} \phi & \text{in} \quad E_3^+ \times (-T, T) , \\ v|_{t=-T} = v_0 \quad (\operatorname{div} v_0 = \operatorname{div} \phi|_{t=-T}) , \quad v|_{x_3=0} = 0 , \\ v_\infty = \lim_{|x| \to \infty} u(x, t) = 0 , \quad q_\infty = 0 . \end{cases}$$

If the vector valued functions f, ϕ and v_0 are smooth and decrease fast enough as $|x| \to \infty$, then it is well known that the system (2.1) has a unique classical solution, which one can write explicitly in terms of the given data.

First, let $\bar{v}$ be the solution of the equation:

$$\text{(2.2)} \quad \begin{cases} \Delta \bar{v} = \operatorname{div} \phi & \text{in} \quad E_3^+ \times (-T, T) , \\ D_{x_3} \bar{v}|_{x_3=0} = 0 , \quad \bar{v}_\infty = 0 . \end{cases}$$

Next, let (v', p') be the solution of the equation:

$$\text{(2.3)} \quad \begin{cases} D_t v' - \Delta v' + \nabla p' = \tilde{f} & \text{in} \quad E_3 \times (-T, T) , \\ \quad \operatorname{div} v' = 0 & \text{in} \quad E_3 \times (-T, T) , \\ v'|_{t=-T} = \tilde{v}_0 , \quad v'_\infty = p'_\infty = 0 , \end{cases}$$

where $\tilde{f}$, $\tilde{v}_0$ denote smooth extensions, vanishing as $|x| \to \infty$, of the functions f, $v_0 - \nabla \bar{v}|_{t=-T}$ to the spaces $E_3 \times (-\infty, T)$, E_3, respectively. Finally, let (v'', p'') be the solution of the equation:

$$\text{(2.4)} \quad \begin{cases} D_t v'' - \Delta v'' + \nabla p'' = 0 & \text{in} \quad E_3^+ \times (-T, T) , \\ \quad \operatorname{div} v'' = 0 & \text{in} \quad E_3^+ \times (-T, T) , \\ v''|_{t=-T} = 0 , \quad v''|_{x_3=0} = -v'|_{x_3=0} - \nabla \bar{v}|_{x_3=0} = b , \quad v''_\infty = p''_\infty = 0 . \end{cases}$$

Then one can easily verify that

$$(2.5)\qquad \begin{cases} v = U(\tilde{f}, \phi, \tilde{v}_0) = v' + v'' + \nabla\bar{v} , \\ q = P(\tilde{f}, \phi, \tilde{v}_0) = p' + p'' - D_t\bar{v} + \Delta\bar{v} \end{cases}$$

is an actual solution of (2.1).

The integral representation of the solution (2.5) is also known. Let

$$K(x) = -1/4\pi|x| , \qquad Q = \nabla_x K \otimes \delta_t ,$$

$$\Gamma(x, t) = \begin{cases} (4\pi t)^{-3/2} \exp(-|x|^2/4t) & \text{for } t > 0 , \\ 0 & \text{for } t < 0 , \end{cases}$$

$$\Gamma'(x, t) = \Gamma(x, t + T) , \qquad T = \Gamma I - \text{Hess}\,\{(K \otimes \delta_t) * \Gamma\} ,$$

where δ_t is Dirac's delta function on the real line, I is the 3×3 unit matrix, $\text{Hess}\,(F) = \{D_{x_j} D_{x_k} F\}_{(j,k=1,2,3)}$ is the Hessian and $f * g$ is the convolution of functions (or distributions) f and g on $E_3 \times (-\infty, \infty)$. Then we may represent the solution of (2.3) as

$$(2.6)\qquad \begin{cases} v'(x, t) = T*\tilde{f}(x, t) + \Gamma' I*(\tilde{v}_0 \otimes \delta_t)(x, t) - \Gamma' I*(T*\tilde{f}\,|_{t=-T} \otimes \delta_t)(x, t) , \\ p'(x, t) = Q*\tilde{f}(x, t) . \end{cases}$$

Let $\tilde{b}(x', t) = b(x', t)$ for $t \geqq -T$, and $\tilde{b}(x', t) = 0$ for $t \leqq -T$, $G = \{G_{jk}\}_{(j,k=1,2,3)}$, $x' = (x_1, x_2)$,

$$G_{jk}(x, t) = -2\delta_{jk} D_{x_3} \Gamma(x, t) + 2\delta_{k3} D_{x_j} K(x) \otimes \delta_t$$
$$\qquad - 4D_{x_k} \int_{E_2} dy' \int_0^{x_3} D_{y_3} \Gamma(y, t) D_{x_j} K(x - y) dy_3 ,$$

$$A(x, t) = \int_{E_2} \Gamma(y', 0, t)|x - y'|^{-1} dy' ,$$

where δ_{jk} is Kronecker's delta. The solution of (2.4) is written as

$$(2.7)\qquad \begin{cases} v''(x, t) = G * (\tilde{b} \otimes \delta_{x_3})(x, t) , \\ p''(x, t) = -2 \operatorname{div} \{Q_3 I * (\tilde{b} \otimes \delta_{x_3})\}(x, t) - 2(KI \otimes \delta_t) * (D_t \tilde{b}_3 \otimes \delta_{x_3})(x, t) \\ \qquad\qquad - \pi^{-1} \operatorname{div} \left(D_t - \sum_{k=1}^{2} D_{x_k}\right)\{AI * (b \otimes \delta_{x_3})\}(x, t) . \end{cases}$$

Let $N(x, y) = K(x - y) + K(x - \bar{y})$, $\bar{y} = (y_1, y_2, -y_3)$. The solution of (2.2) is written as

$$(2.8)\qquad \bar{v}(x, t) = \int_{E_3^+} N(x, y) \operatorname{div} \phi(y, t) dy .$$

Now, we introduce some function spaces. Let $B(x_0, R)$ denote the ball $|x - x_0| < R$ in E_3 and $B_R = B(0, R)$. Let

$$\sigma = \{x \in E_3; x_3 = 0\} , \qquad \sigma_R = B_R \cap \sigma , \qquad G_R = \{x \in B_R; x_3 > 0\} ,$$
$$I_T = (-T, T) , \qquad I_{hT} = \{\omega = t + is; |s| < h(t + T), t \in I_T\} ,$$

where $0 < h < 1$. Let $C^{k+\mu}(G_R)$, (k; an integer, $0 < \mu < 1$), be the usual Hölder space. For $f \in C^{k+\mu}(G_R)$, we define

$$(2.9)\qquad |f|_{k+\mu} = \sup_{x_1, x_2, \alpha} |D_x^\alpha f(x_1) - D_x^\alpha f(x_2)|/|x_1 - x_2|^\mu\,,\quad (x_1, x_2 \in G_R,\ x_1 \neq x_2,\ |\alpha| = k)\,.$$

For $f \in C^{k+\mu,(k+\mu)/2}(G_R \times I_T)$, we define

$$(2.10)\qquad \begin{aligned} \|f\|_{k+\mu} = &\sup_{x_1, x_2, t, \alpha, a} |D_t^a D_x^\alpha f(x_1, t) - D_t^a D_x^\alpha f(x_2, t)|/|x_1 - x_2|^\mu \\ &+ \sup_{x, t_1, t_2, \alpha, a} |D_t^a D_x^\alpha f(x, t_1) - D_t^a D_x^\alpha f(x, t_2)|/|t_1 - t_2|^{\mu/2} \\ &+ \sup_{x, t_1, t_2, \beta, b} |D_t^b D_x^\beta f(x, t_1) - D_t^b D_x^\beta f(x, t_2)|/|t_1 - t_2|^{(k+\mu-2b-|\beta|)/2}\,, \\ &((x_j, t), (x, t_j) \in G_R \times I_T,\ j = 1, 2,\ x_1 \neq x_2,\ t_1 \neq t_2, \\ &|\alpha| + 2a = k,\ 0 < k + \mu - 2b - |\beta| < 2)\,. \end{aligned}$$

$$(2.11)\qquad C_0^{k+\mu,(k+\mu)/2}(G_R \times I_T) = \{f \in C^{k+\mu,(k+\mu)/2}(G_R \times I_T);\ D_t^a D_x^\alpha f(0, 0) = 0,\ |\alpha| + 2a \leqq k\}\,.$$

Assume f to be of the form $\operatorname{div} F$. Then we have the following by Solonnikov [11, p. 76] and McCracken [6, p. 49].

PROPOSITION 2.1. *Let $F \in C_0^{1+\mu,(1+\mu)/2}(E_3^+ \times I_T)$, $\phi \in C_0^{2+\mu+\varepsilon,(2+\mu+\varepsilon)/2}(E_3^+ \times I_T)$, $v_0 \in C^{2+\mu}(E_3^+)$ and suppose they decrease fast enough as $|x| \to \infty$. Then the solution* (2.5) *satisfies the following properties.*

(1)

$$(2.12)\qquad \begin{aligned} &\|v\|_{2+\mu} + \|\nabla q\|_\mu + \|q\|_{1+\mu-\varepsilon} \\ &\leqq C\{\|F\|_{1+\mu(E_3^+ \times I_T)} + \|\phi\|_{2+\mu+\varepsilon(E_3^+ \times I_T)} + |v_0|_{2+\mu(E_3^+)}\}\,, \\ &(0 < \mu - \varepsilon < 1,\ \varepsilon > 0)\,. \end{aligned}$$

(2) *If $F_j \in C^{1+\mu,(1+\mu)/2}(E_3^+ \times I_T)$ $(j = 1, 2)$ decrease fast enough as $|x| \to \infty$, then*

$$\|U(\operatorname{div} \tilde{F}_1, \phi, \tilde{v}_0) - U(\operatorname{div} \tilde{F}_2, \phi, \tilde{v}_0)\|_{2+\mu} \leqq C\|F_1 - F_2\|_{1+\mu(E_3^+ \times I_T)}\,,$$

where $\tilde{F}_j$ denotes a smooth extension of F_j similar to $\tilde{f}$ for f.

(3) *If $F \in C_0^{1+\mu,(1+\mu)/2}(E_3^+ \times I_{hT})$ and $\phi \in C_0^{2+\mu+\varepsilon,(2+\mu+\varepsilon)/2}(E_3^+ \times I_{hT})$ are analytic in $\omega \in I_{hT}$ for each $x \in E_3^+$, then v and q are analytic in $\omega \in I_{hT}$ for each $x \in E_3^+$ and the inequality* (2.12) *holds for I_{hT}.*

3. The integral equation. In this section we consider the functions v_R, q_R which are determined by u, p of (1.1).

First, we notice the following. There exists R_0 such that G_R is contained in D^+ for $0 < R < R_0$. We regard u, p to be restricted onto such G_R. Since f in (1.1) is analytic in $\mathscr{D} \times \Omega$, there exists $F = \{F_{jk}\}_{(j,k=1,2,3)}$

analytic in $\mathscr{D}_0 \times \Omega$ and satisfying $\operatorname{div} F = f$. Here $\mathscr{D}_0$ is a complex neighborhood of the origin which is contained in $\mathscr{D}$. We choose one such F and fix it in what follows. It is easy to see that we can assume $u_0 = 0$ in (1.1). We put

$$\mathscr{Q}(x, t) = \sum_{2k+|\alpha|\leqq 2} (k!\,\alpha!)^{-1} D_t^k D_x^\alpha u(0, 0) x^\alpha t^k ,$$
$$\mathscr{R}(x, t) = \sum_{|\beta|\leqq 1} D_x^\beta p(0, 0) x^\beta .$$

We define $\Psi(w) = \{\Psi_{jk}(w)\}_{(j,k=1,2,3)}$ by

$$\begin{aligned}\Psi_{jk}(w)(x, t) = &-\{w_j(x, t) + \mathscr{Q}_j(x, t)\}\{w_k(x, t) + \mathscr{Q}_k(x, t)\} + u_j(0, 0)u_k(0, 0)\\ &+ F_{jk}(x, t) - F_{jk}(0, 0) + \delta_{jk}\{D_{x_3}^2 u_k(0, 0) - D_{x_k} p(0, 0)\} x_k .\end{aligned}$$

LEMMA 3.1. *There exists an extension operator* Φ: $C_0^{k+\mu,(k+\mu)/2}(G_R \times I_T) \to C_0^{k+\mu,(k+\mu)/2}(E_3 \times (-\infty, T))$ *such that* $\Phi(f)|_{G_R \times I_T} = f$ *and that*

$$\|\Phi(f)\|_{k+\mu(E_3\times(-\infty,T))} \leqq C\|f\|_{k+\mu(G_R\times I_T)} , \quad (0 < \mu < 1) ,$$

with a constant C *independent of* R, T.

We give here the sketch of the proof for $k = 1$. Let (r, ϕ_1, ϕ_2), $0 \leqq r \leqq R$, $0 \leqq \phi_1 \leqq \pi$, $0 \leqq \phi_2 < 2\pi$, be polar coordinates in G_R. Let the function $x = x(r, \phi_1, \phi_2)$ be the transformation from the polar coordinate to the orthogonal coordinate. We put $f_0(r, \phi_1, \phi_2, t) = f(x(r, \phi_1, \phi_2), t)$ and set

$$f_0'(r, \phi_1, \phi_2, t) = \begin{cases} f_0(r, \phi_1, \phi_2, t) & (0 \leqq r \leqq R) \\ \sum_{j=1}^{2} C_j f_0(R - j(r - R), \phi_1, \phi_2, t) & (R < r \leqq 2R) , \end{cases}$$

where $\sum_{j=1}^2 C_j(-j/2)^m = 1$ $(m = 0, 1)$. We put $f'(x', x_3, t) = f_0'(r(x), \phi_1(x), \phi_2(x), t)$, where the functions $r = r(x)$, $\phi_1 = \phi_1(x)$, $\phi_2 = \phi_2(x)$ are the transformation from the orthogonal coordinate to the polar coordinate. We define f^* by

$$f^*(x', x_3, t) = \begin{cases} f'(x', x_3, t) & (x_3 \geqq 0) \\ \sum_{j=1}^{2} C_j^* f'(x', -jx_3/2, t) & (x_3 < 0) , \end{cases}$$

where $\sum_{j=1}^2 C_j^*(-j/2)^m = 1$ $(m = 0, 1)$, and also $\bar{f}$ by

$$\bar{f}(x, t) = \begin{cases} f^*(x, t) & -T \leqq t \leqq T \\ f^*(x, -t - 2T) & -3T \leqq t \leqq -T . \end{cases}$$

An extension operator stated in the lemma is then given by

$$\Phi(f)(x, t) = \begin{cases} \psi(|x|/R)\psi((-t - T)/T)\bar{f}(x, t) & \text{in } B_{2R}\times(-3T, T) \\ 0 & \text{outside of } B_{2R}\times(-3T, T) , \end{cases}$$

where ψ is a smooth function on the reals such that $\psi(s)=1,\ 0$ for $s\leqq 1$, $s\geqq 4/3$, respectively.

In what follows we fix one such extension operator Φ. We put

$$(3.1)\quad \begin{cases} U(x,t)=U(\operatorname{div}\Phi(\Psi(u-\mathscr{Q})),\ -\Phi(\mathscr{Q}),\ -\Phi(\mathscr{Q})|_{t=-T})(x,t)\,, \\ P(x,t)=P(\operatorname{div}\Phi(\Psi(u-\mathscr{Q})),\ -\Phi(\mathscr{Q}),\ -\Phi(\mathscr{Q})|_{t=-T})(x,t)\,. \end{cases}$$

We define functions v_R and q_R by

$$(3.2)\quad \begin{cases} v_R(x,t)=K_R[\Psi(u-\mathscr{Q})](x,t) \\ \qquad\qquad =U(x,t)-\sum_{2k+|\alpha|\leqq 2}(k!\,\alpha!)^{-1}D_t^kD_x^\alpha U(0,0)x^\alpha t^k\,, \\ q_R(x,t)=L_R[\Psi(u-\mathscr{Q})](x,t)=P(x,t)-\sum_{|\beta|\leqq 1}D_x^\beta P(0,0)x^\beta\,. \end{cases}$$

Then they satisfy

$$(3.3)\quad \begin{cases} D_tv_R-\Delta v_R+\nabla q_R=\operatorname{div}\Phi(\Psi(u-\mathscr{Q})) & \text{in}\quad E_3^+\times I_T\,, \\ \operatorname{div}v_R=-\operatorname{div}\Phi(\mathscr{Q}) & \text{in}\quad E_3^+\times I_T\,, \\ v_R|_{t=-T}=-\Phi(\mathscr{Q})|_{t=-T}-\sum_{2k+|\alpha|\leqq 2}(k!\,\alpha!)^{-1}D_t^kD_x^\alpha U(0,0)x^\alpha t^k\,, & v_R|_{x_3=0}=0\,. \end{cases}$$

From now on, we regard v_R, q_R, K_R, L_R as restricted onto $G_R\times I_T$. By Proposition 2.1 and Lemma 3.1, we easily obtain the following.

PROPOSITION 3.1. *The operators K_R, L_R satisfy the following properties.*

(1) *K_R is an operator from $C_0^{1+\mu,(1+\mu)/2}(G_R\times I_T)$ into $C_0^{2+\mu,(2+\mu)/2}(G_R\times I_T)$ and satisfies*

$$\|K_R[\Psi]\|_{2+\mu}\leqq C\|\Psi\|_{1+\mu}\quad for\quad \Psi\in C_0^{1+\mu,(1+\mu)/2}(G_R\times I_T)\,,$$

where $C=C(R_0,T_0)$ for $R\leqq R_0$, $T\leqq T_0$.

(2) *If $\Psi_j\in C_0^{1+\mu,(1+\mu)/2}(G_R\times I_T)$ $(j=1,2)$, then*

$$\|K_R[\Psi_1]-K_R[\Psi_2]\|_{2+\mu}\leqq C\|\Psi_1-\Psi_2\|_{1+\mu}\,,$$

where $C=C(R_0,T_0)$ for $R\leqq R_0$, $T\leqq T_0$.

(3) *If $\Psi\in C_0^{1+\mu,(1+\mu)/2}(G_R\times I_T)$, then $\nabla L_R[\Psi]\in C_0^{\mu,\mu/2}(G_R\times I_T)$ and $L_R[\Psi]\in C_0^{1+\mu-\varepsilon,(1+\mu-\varepsilon)/2}(G_R\times I_T)$ for any ε satisfying $0<\mu-\varepsilon<1$, $\varepsilon>0$.*

(4) *If $\Psi\in C_0^{1+\mu,(1+\mu)/2}(G_R\times I_{hT})$ is analytic in $\omega\in I_{hT}$ for each $x\in G_R$, then $K_R[\Psi]\in C_0^{2+\mu,(2+\mu)/2}(G_R\times I_{hT})$, $\nabla L_R[\Psi]\in C_0^{\mu,\mu/2}(G_R\times I_{hT})$, $L_R[\Psi]\in C_0^{1+\mu-\varepsilon,(1+\mu-\varepsilon)/2}(G_R\times I_{hT})$ and they are analytic in $\omega\in I_{hT}$ for each $x\in G_R$.*

Putting $\mu=\theta+\varepsilon$ in the hypotheses of Theorem 1.1, the solution (u,p) of (1.1) is in $C^{2+\theta+\varepsilon,(2+\theta+\varepsilon)/2}(G_R\times I_{hT})$ and $C^{1+\theta+\varepsilon,(1+\theta+\varepsilon)/2}(G_R\times I_{hT})$, respectively, and is analytic in $\omega\in I_{hT}$ for each $x\in G_R$. For the function $f=\operatorname{div}F$ in (1.1), we know that $F\in C^{1+\theta+\varepsilon,(1+\theta+\varepsilon)/2}(G_R\times I_{hT})$ is analytic in

$\omega \in I_{hT}$ for each $x \in G_R$. Then, by Proposition 3.1, we have $v_R \in C_0^{2+\theta+\varepsilon,(2+\theta+\varepsilon)/2}(G_R \times I_{hT})$, $\nabla q_R \in C_0^{\theta+\varepsilon,(\theta+\varepsilon)/2}(G_R \times I_{hT})$, $q_R \in C_0^{1+\theta,(1+\theta)/2}(G_R \times I_{hT})$, and they are analytic in $\omega \in I_{hT}$ for each $x \in G_R$. We define H_R and M_R by

$$u = v_R + H_R + \mathscr{Q}\ , \quad \text{and} \quad p = q_R + M_R + \mathscr{R}\ .$$

Then we see that the functions H_R, ∇M_R, M_R have the same regularity as that of v_R, ∇q_R, q_R, respectively. The functions H_R and M_R satisfy

$$(3.4) \qquad \begin{cases} D_t H_R - \Delta H_R + \nabla M_R = 0 & \text{in} \quad G_R \times I_T\ , \\ \quad \operatorname{div} H_R = 0 & \text{in} \quad G_R \times I_T\ , \\ H_R|_{t=-T} = \mathscr{H}\ , \quad H_R|_{x_3=0} = 0\ , \end{cases}$$

where $\mathscr{H} = \sum_{2k+|\alpha|\leqq 2} (k!\ \alpha!)^{-1} D_t^k D_x^\alpha U(0, 0) x^\alpha t^k$.

Regarding H_R as given, we shall consider the integral equation:

$$(3.5) \qquad w = K_R[\Psi(w + H_R)]\ .$$

It is obvious that the function v_R is a solution of (3.5) in $C_0^{2+\theta,(2+\theta)/2}(G_R \times I_T)$. Now we show the uniqueness of the solution of (3.5).

PROPOSITION 3.2. *There exists a positive number M such that* (3.5) *has a unique solution in* $\{w \in C_0^{2+\theta,(2+\theta)/2}(G_R \times I_T);\ \|w\|_{2+\theta} \leqq M\}$, *for* $0 < R < R_1$ *and* $T = O(R^2)$, *where* $R_1 = R_1(M)$ *is a sufficiently small number depending on M.*

To prove the above proposition, we need the following.

PROPOSITION 3.3. *Choose R_2 sufficiently small and put $T = O(R^2)$ for $0 < R < R_2$. Suppose $w \in C_0^{2+\mu,(2+\mu)/2}(G_R \times I_T)$ and suppose $\|w\|_{2+\mu}$ is uniformly bounded by some number M for $0 < R < R_2$. Then*

$$(3.6) \qquad \Psi(w) \in C_0^{1+\mu,(1+\mu)/2}(G_R \times I_T)\ ,$$

$$(3.7) \qquad \|\Psi(w)\|_{1+\mu} \leqq C_1 + C_2(M)R^2\ ,$$

where C_1 is independent of M and R, while $C_2(M)$ depends only on M. Moreover, if $\|w_j\|_{2+\mu} \leqq M$ $(j = 1, 2)$, then

$$(3.8) \qquad \|\Psi(w_1) - \Psi(w_2)\|_{1+\mu} \leqq C_3(M)R\|w_1 - w_2\|_{2+\mu}\ ,$$

where $C_3(M)$ depends only on M.

PROOF. In view of the definition of Ψ, we can verify (3.6) immediately. Let $\Psi_{jk}^{(1)}$ be the first and second order terms of w in Ψ_{jk} and let $\Psi_{jk}^{(2)}$ be the remainder. We put $\|\Psi_{jk}^{(2)}\|_{1+\mu} = C_1$. Since

$$\sup_{(x,t)\in G_R \times I_T} |D_t^a D_x^\alpha w(x, t)| \leqq C(R^{2-|\alpha|-a+\mu} + T^{(2-|\alpha|-a+\mu)/2})\|w\|_{2+\mu}\ ,$$

$$(|\alpha| + 2a \leqq 2)\ ,$$

we obtain (3.7) and (3.8) by an easy calculation.

PROOF OF PROPOSITION 3.2. Let w_1 and w_2 be two solutions of (3.5) in $\{w \in C_0^{2+\theta,(2+\theta)/2}(G_R \times I_T);\ \|w\|_{2+\theta} \leqq M\}$ for $0 < R < R_1$. We have $w_1 - w_2 = K_R[\Psi(w_1 + H_R)] - K_R[\Psi(w_2 + H_R)]$. By Proposition 3.1 with $\mu = \theta$, we obtain $\|w_1 - w_2\|_{2+\theta} \leqq C\|\Psi(w_1 + H_R) - \Psi(w_2 + H_R)\|_{1+\theta}$, where $C = C(R_0)$ for $R < R_0$. By Proposition 3.3 with $\mu = \theta$, we see that $\|w_1 - w_2\|_{2+\theta} \leqq C_0(M)R\|w_1 - w_2\|_{2+\theta}$. Hence, choosing R_1 so that $C_0(M)R < 1/2$ for $0 < R < R_1$, we have $\|w_1 - w_2\|_{2+\theta} = 0$.

4. The complex analytic extensions of the operators K_R, L_R. Let

$$\begin{aligned}
&\boldsymbol{B}_{hR} = \{z = x + iy;\ x, y \in E_3,\ |y| < h(R - |x|)\}\ ,\\
&\boldsymbol{B}_{0hR} = \{z \in \boldsymbol{B}_{hR};\ y_3 = 0\}\ , \qquad \boldsymbol{G}_{hR} = \{z \in \boldsymbol{B}_{0hR};\ x_3 > 0\}\ ,\\
&\boldsymbol{\sigma}_{hR} = \{z \in \boldsymbol{B}_{hR};\ z_3 = 0\}\ ,\\
&\boldsymbol{E} = \{\boldsymbol{B}_{0hR} \times \boldsymbol{I}_{hT}\} \cup \{E_3 \times \boldsymbol{I}_{hT}\} \cup \{\boldsymbol{B}_{0hR} \times (-\infty, T)\} \cup \{E_3 \times (-\infty, T)\}\ ,\\
&H^{k+\mu}(\boldsymbol{G}_{hR}) = \{f \in C^{k+\mu}(\boldsymbol{G}_{hR});\ f \text{ is analytic in } z' = (z_1, z_2) \text{ for } z \in \boldsymbol{G}_{hR}\}\ .
\end{aligned}$$

Let the semi-norm $|\cdot|^*_{k+\mu}$ (resp. $\|\cdot\|^*_{k+\mu}$) be an extension of (2.9) (resp. (2.10)) with G_R replaced by $\boldsymbol{G}_{hR}$ (resp. $G_R \times I_T$ by $\boldsymbol{G}_{hR} \times \boldsymbol{I}_{hT}$). We denote by $H^{k+\mu,(k+\mu)/2}(\boldsymbol{G}_{hR} \times \boldsymbol{I}_{hT})$ (resp. $H_0^{k+\mu,(k+\mu)/2}(\boldsymbol{G}_{hR} \times \boldsymbol{I}_{hT})$) the space of functions f in $C^{k+\mu,(k+\mu)/2}(\boldsymbol{G}_{hR} \times \boldsymbol{I}_{hT})$ which are analytic in (z', ω) for $(z, \omega) \in \boldsymbol{G}_{hR} \times \boldsymbol{I}_{hT}$ (resp. the space of functions f in $H^{k+\mu,(k+\mu)/2}(\boldsymbol{G}_{hR} \times \boldsymbol{I}_{hT})$ which satisfy $D_\omega^a D_z^\alpha f(0, 0) = 0$, $|\alpha| + 2a \leqq k$). We define the spaces $H_0^{k+\mu,(k+\mu)/2}(G_R \times \boldsymbol{I}_{hT})$, $H_0^{k+\mu,(k+\mu)/2}(\boldsymbol{E})$ etc. similarly and the norm of the spaces is written as $\|\cdot\|^*_{k+\mu(G_R \times \boldsymbol{I}_{hT})}$, $\|\cdot\|^*_{k+\mu(\boldsymbol{E})}$ etc.

We here follow Morrey [8]. Let B be the ball $|x| < R$ in E_n. Let

$$\begin{aligned}
&\boldsymbol{B} = \{z = x + iy;\ x, y \in E_n,\ |y| < h(R - |x|)\}\ ,\\
&\boldsymbol{X} = \{z = x + iy;\ x, y \in E_n,\ |y| < h|x|\}\ .
\end{aligned}$$

Suppose that the kernel $\mathscr{T}(x)$ has an analytic extension onto $\boldsymbol{X}$. For each $z = x + iy \in \boldsymbol{B}$, we define a surface $S(z)$ in $\boldsymbol{B}$ passing through the point z, by the equations $\xi = \xi(r)$ for $r \in \bar{B}$, where ξ satisfies

$$(4.1)\quad \begin{cases}
(1)\ \ \operatorname{Re}\xi(r) = r\ ,\quad r \in B\ ,\qquad \operatorname{Im}\xi(r) = 0\ ,\quad r \in \partial B\ ;\\
(2)\ \ \operatorname{Im}\xi(r) = y\ ,\qquad |\operatorname{Im}\xi(r) - y| < h|r - x|\ ;\\
(3)\ \ |\operatorname{Im}\xi(r)| < h(R - |r|)\ ,\quad r \in B\ ;\\
(4)\ \ \operatorname{Im}\xi(r) \in C(\bar{B})\ ,\ \text{differentiable almost everywhere and its}\\
\qquad \text{derivatives}\ \ D_{r_j}\operatorname{Im}\xi(r)\ \ (j = 1, 2, \cdots, n)\ \ \text{are in}\ \ L^\infty(B)\ .
\end{cases}$$

For $f \in H^\mu(\boldsymbol{B})$ and a kernel $\mathscr{T}(x)$, we define the integral over the surface $S(z)$ by

$$\int_{S(z)} \mathscr{F}(z-\xi)f(\xi)d\xi = \int_B \mathscr{F}(z-\xi(r))f(\xi(r))J(r)dr\,, \tag{4.2}$$

where $J(r) = \partial(\xi_1, \xi_2, \cdots, \xi_n)/\partial(r_1, r_2, \cdots, r_n)$.

PROPOSITION 4.1. *Let $f \in H^\mu(\boldsymbol{B})$, $z \in \boldsymbol{B}$.*

(1) *If both surfaces $S(z)$ and $S^*(z)$ satisfy* (4.1), *then the corresponding integrals defined by* (4.2) *have the same value.*

(2) *The function $F(z) = \int_{S(z)} \mathscr{F}(z-\xi)f(\xi)d\xi$ is analytic on $\boldsymbol{B}$ and we have*

$$D_{z_j}F(z) = \int_{S(z)} D_{z_j}\mathscr{F}(z-\xi)f(\xi)d\xi\,. \tag{4.3}$$

REMARK 4.1. The above proposition holds also if we replace $\boldsymbol{B}$ by $\boldsymbol{B}(k) = \{z \in \boldsymbol{B};\, z_k \in E_1\}$.

PROPOSITION 4.2. *Suppose $f \in C^\mu(E_n)$ and suppose it decreases fast enough as $|x| \to \infty$. If the integral*

$$F(x) = \int_{E_n \setminus B} \mathscr{F}(x-r)f(r)dr$$

is absolutely convergent, then $F(x)$ can be analytically extended to $\boldsymbol{B}$. We have $F(z) = \int_{E_n\setminus B} \mathscr{F}(z-r)f(r)dr$ for $z \in \boldsymbol{B}$.

Now, we return to the Stokes equation (2.1). Notice that the solution (v, q) of (2.1) can be written as follows:

$$\begin{cases} v = U(\tilde{f}, \phi, \tilde{v}_0) = v' + v'' + \nabla\bar{v}\,, \\ q = P(\tilde{f}, \phi, \tilde{v}_0) = p' + p'' - D_t\bar{v} + \Delta\bar{v}\,. \end{cases} \tag{2.5}$$

We have the following proposition.

PROPOSITION 4.3. *Suppose that, for the function f in* (2.1), *there exists $F = \{F_{jk}\}_{(j,k=1,2,3)}$ such that $f = \operatorname{div} F$ and F has an analytic extension $\tilde{F}$ to $\boldsymbol{E}$. Let ϕ, $\tilde{v}_0$ have analytic extensions to $\{E_3 \cup \boldsymbol{B}_{0hR}\} \times \boldsymbol{I}_{hT}$, $E_3 \cup \boldsymbol{B}_{0hR}$, respectively. Then the solution $v(x,t) = U(\operatorname{div}\tilde{F}, \phi, \tilde{v}_0)(x,t)$, $q(x,t) = P(\operatorname{div}\tilde{F}, \phi, \tilde{v}_0)(x,t)$ has an analytic extension to $\boldsymbol{G}_{hR} \times \boldsymbol{I}_{hT}$ and satisfies*

$$\begin{aligned} &\|v\|^*_{2+\mu} + \|\nabla q\|^*_\mu + \|q\|^*_{1+\mu-\varepsilon} \\ &\qquad \leqq C\{\|\tilde{F}\|^*_{1+\mu(E)} + \|\phi\|^*_{2+\mu+\varepsilon(\{E_3\cup B_{0hR}\}\times I_{hT})} + |\tilde{v}_0|^*_{2+\mu(E_3\cup B_{0hR})}\}\,. \end{aligned}$$

To prove this proposition, we first give the complex analytic extensions of the kernels $T(x,t)$, $Q(x,t)$, $A(x,t)$ and $G(x,t)$. The analytic extension of $K(x)$ is well known. So, by using Propositions 4.1 and 4.2, we obtain the following lemmas, where we define

$$I^0 = (0, \infty), \qquad I_h^0 = \{\omega = t + is; |s| < ht, t \in I^0\},$$
$$X_h = \{z = x + iy; x, y \in E_3, |y| < h|x|\},$$
$$X_{0h} = \{z \in X_h; y_3 = 0\}, \qquad Y_h = \{z \in X_{0h}; x_3 > 0\}.$$

LEMMA 4.1. *The kernel $T(x, t)$ (resp. $A(x, t)$, resp. $G_{jk}(x, t)$ $(k \neq 3)$) can be extended to an analytic function in (z, ω) for $(z, \omega) \in X_h \times I_h^0$ (resp. (z', ω) for $(z, \omega) \in Y_h \times I_h^0$ resp. (z', ω) for $(z, \omega) \in Y_h \times I_h^0$), and we have, for $(z, \omega) \in X_h \times I_h^0$,*

$$|D_z^\alpha D_\omega^m T(z, \omega)| \leqq C(|x|^2 + t)^{-(|\alpha|+3)/2-m} \tag{4.4}$$

(resp. for $(z, \omega) \in Y_h \times I_h^0$,

$$|D_z^\alpha D_\omega^m A(z, \omega)| \leqq C(|x|^2 + t)^{-(|\alpha|+1)/2} t^{-m-1/2}, \tag{4.5}$$

resp. for $(z, \omega) \in Y_h \times I_h^0$,

$$|D_{z'}^\alpha D_{z_3}^n D_\omega^m G_{jk}(z, \omega)| \leqq Ct^{-m-1/2}(|x|^2 + t)^{-(|\alpha|+3)/2}(x_3^2 + t)^{-n/2}. \tag{4.6}$$

The real number h depends on the kernels. We choose and fix a sufficiently small positive number h so that the above analytic extensions exist.

We put $\mathscr{F} = \Gamma$, $\boldsymbol{B} = \boldsymbol{B}_{0hR}$ and take $S(z)$ satisfying (4.1). Then we have the following.

LEMMA 4.2. *Suppose that $f \in C_0^{\mu,\mu/2}(E_3 \times (-\infty, T))$ and suppose it decreases fast enough as $|(x, t)| \to \infty$, and has an analytic extension to $H_0^{\mu,\mu/2}(\boldsymbol{E})$. Put*

$$F(x, t) = \int_{-\infty}^t \int_{E_3} \Gamma(x - r, t - \tau) f(r, \tau) dr d\tau .$$

Then $F(x, t)$ can be extended analytically in (z', ω) for $(z, \omega) \in \boldsymbol{B}_{0hR} \times \boldsymbol{I}_{hT}$ and we have

$$\|F\|_{2+\mu}^* \leqq C\|f\|_{\mu(E)}^* .$$

Similarly, the following lemmas hold.

LEMMA 4.3. *Suppose that $g \in C_0^{1+\mu,(1+\mu)/2}(E_3 \times (-\infty, T))$ and suppose it has a compact support in $E_3 \times (-\infty, T)$ and has an analytic extension to $H_0^{1+\mu,(1+\mu)/2}(\boldsymbol{E})$. Then the function $F(x, t)$ defined by*

$$F(x, t) = \int_{E_3} K(x - r) D_r g(r, t) dr ,$$

can be extended analytically in (z', ω) for $(z, \omega) \in \boldsymbol{B}_{0hR} \times \boldsymbol{I}_{hT}$ and satisfies

$$\|D_z^\alpha F\|_\mu^* \leqq C\|g\|_{1+\mu(E)}^* , \quad |\alpha| = 2 , \tag{4.7}$$

(4.8) $$\|F\|^*_{1+\mu-\varepsilon} \leqq C\|g\|^*_{1+\mu(E)} .$$

LEMMA 4.4. *Let $\tilde{v}_0$ be in $H^{2+\mu}(E_3 \cup \boldsymbol{B}_{0hR})$. Then the function $F(x, t)$ defined by*

$$F(x, t) = \int_{E_3} \Gamma(x - r, t + T)\tilde{v}_0(r)dr ,$$

can be extended to $\{E_3 \cup \boldsymbol{B}_{0hR}\} \times \boldsymbol{I}_{hT}$ so that $F \in H^{2+\mu,(2+\mu)/2}(\{E_3 \cup \boldsymbol{B}_{0hR}\} \times \boldsymbol{I}_{hT})$. We have

$$\|F\|^*_{2+\mu} \leqq C|\tilde{v}_0|^*_{2+\mu(E_3 \cup \boldsymbol{B}_{0hR})} .$$

PROOF OF PROPOSITION 4.3. As mentioned above, the functions v' and p' given by (2.6) have analytic extensions and satisfy the desired inequalities. In other words, we obtain the extension and the estimate for p' by Lemma 4.3. Regarding the function v' as the solution of the Cauchy problem for the heat equation with $f - p'$ on the right-hand side, by Lemmas 4.2 and 4.3, we have

$$\|v'\|^*_{2+\mu} + \|p'\|^*_{1+\mu-\varepsilon} + \|\nabla p'\|^*_{\mu} \leqq C\{\|F\|^*_{1+\mu(E)} + |\tilde{v}_0|^*_{2+\mu(E_3 \cup \boldsymbol{B}_{0hR})}\} .$$

In the same way as in the proof of Lemma 4.3, we have

$$\|\bar{v}\|^*_{3+\mu} \leqq C\|\phi\|^*_{2+\theta+\varepsilon(\{E_3 \cup \boldsymbol{B}_{0hR}\} \times \boldsymbol{I}_{hT})} ,$$

where $\bar{v}$ is given by (2.8).

LEMMA 4.5. *Let b_3 be the third component of b in (2.4) and suppose it satisfies the condition $D_t b_3 = \sum_{j=1}^2 D_{x_j} e_j$. Let $b_j \in H_0^{2+\mu,(2+\mu)/2}(\{\sigma \cup \boldsymbol{\sigma}_{hR}\} \times \boldsymbol{I}_{hT})$ and $e_k \in H_0^{1+\mu,(1+\mu)/2}(\{\sigma \cup \boldsymbol{\sigma}_{hR}\} \times \boldsymbol{I}_{hT})$, $j = 1, 2, 3$, $k = 1, 2$. Then the functions v'' and p'' given by (2.4) are extended analytically in (z', ω) for $(z, \omega) \in \boldsymbol{G}_{hR} \times \boldsymbol{I}_{hT}$ and satisfy*

$$\|v''\|^*_{2+\mu} + \|p''\|^*_{1+\theta-\varepsilon} + \|\nabla p''\|^*_{\mu} \leqq C(\|b\|^*_{2+\mu(\{\sigma \cup \boldsymbol{\sigma}_{hR}\} \times \boldsymbol{I}_{hT})} + \|e\|^*_{1+\mu(\{\sigma \cup \boldsymbol{\sigma}_{hR}\} \times \boldsymbol{I}_{hT})}) .$$

For the detail of the proof of each lemma see Morrey [9, pp. 174–179].

In view of the definition of b_3 in (2.4), we get the following by Solonnikov [11, p. 53].

$$D_t b_3(x', t) = \sum_{j=1}^{2} D_{x_j} b_j'(x', t) + \sum_{j=1}^{2} D_{x_j} b_j''(x', t) ,$$

where

$$b_j'(x', t) = (4\pi)^{-1}\Big[D_{x_3}\int_{E_3} \Gamma(r, t)|x - r|^{-1}dr * f_j - D_{x_j}\int_{E_3} \Gamma(r, t)|x - r|^{-1}dr * f_3\Big]\Big|_{x_3=0} ,$$

$$b_j''(x', t) = D_{x_j} \int_{E_3} \Gamma(x - r, t + T)\tilde{v}_{03}(r)dr \Big|_{x_3=0} - D_{x_3} \int_{E_3} \Gamma(x - r, t + T)\tilde{v}_{0j}(r)dr \Big|_{x_3=0} .$$

The above formula is also valid after the analytic extension. Hence it is easy to see that there exist e_k $(k = 1, 2)$ satisfying $D_t b_3 = \sum_{k=1}^{2} D_{x_k} e_k$.

We see that the norms of b and e are bounded by those of $\tilde{f}$ and $\tilde{v}_0$. Therefore the proof of Proposition 4.3 is complete.

Similarly, we have the following.

PROPOSITION 4.4. *Suppose that, for f in* (2.1), *there exists* $F = \{F_{jk}\}_{(j,k=1,2,3)}$ *such that* $f = \operatorname{div} F$ *and* F *is in* $H_0^{1+\mu, (1+\mu)/2}(E_3^+ \times I_{hT})$. *Let* $f = 0$ *and* $\operatorname{div}\phi = 0$ *on* $G_R \times I_{hT}$. *Let* $\phi \in H_0^{2+\mu+\varepsilon, (2+\mu+\varepsilon)/2}(E_3^+ \times I_{hT})$, $v_0 \in H^{2+\mu}(E_3^+ \cup G_{hR})$. *Then the solution* (v, q) *of* (2.1) *has an analytic extension onto* $G_{hR} \times I_{hT}$ *and satisfies*

$$\|v\|_{2+\mu}^* + \|\nabla q\|_{\mu}^* + \|q\|_{1+\mu-\varepsilon}^* \leqq C\{\|F\|_{1+\mu(E_3^+\times I_{hT})}^* + \|\phi\|_{2+\mu+\varepsilon(E_3^+\times I_{hT})}^* + |v_0|_{2+\mu(E_3^+\cup G_{hR})}^*\} .$$

Now we will give here the analytic extensions of H_R and M_R.

PROPOSITION 4.5. *Let* $F \in H^{1+\theta+\varepsilon, (1+\theta+\varepsilon)/2}(G_R \times I_{hT})$, $u \in H^{2+\theta+\varepsilon, (2+\theta+\varepsilon)/2}(G_R \times I_{hT})$, $p \in H^{1+\theta+\varepsilon, (1+\theta+\varepsilon)/2}(G_R \times I_{hT})$. *Then* H_R *and* M_R *have extensions* $\bar{H}_R$ *and* $\bar{M}_R$ *which are analytic in* (z', ω) *for* $(z, \omega) \in G_{hR} \times I_{hT}$ *and satisfy*

$$\|\bar{H}_R\|_{2+\theta}^* + \|\bar{M}_R\|_{1+\theta-\varepsilon}^* \leqq C\{\|H_R\|_{2+\theta+\varepsilon(G_R\times I_{hT})} + \|M_R\|_{1+\theta(G_R\times I_{hT})}\} .$$

PROOF. We already know that H_R and M_R satisfy

$$\begin{cases} D_t H_R - \Delta H_R + \nabla M_R = 0 & \text{in } G_R \times I_T , \\ \operatorname{div} H_R = 0 & \text{in } G_R \times I_T , \\ H_R|_{t=-T} = \sum_{2k+|\alpha|\leqq 2} (k!\,\alpha!)^{-1} D_t^k D_x^\alpha U(0, 0) x^\alpha t^k , & H_R|_{x_3=0} = 0 , \end{cases}$$

and $H_R \in H_0^{2+\theta+\varepsilon, (2+\theta+\varepsilon)/2}(G_R \times I_{hT})$, $M_R \in H_0^{1+\theta, (1+\theta)/2}(G_R \times I_{hT})$, and $\nabla M_R \in H_0^{\theta+\varepsilon, (\theta+\varepsilon)/2}(G_R \times I_{hT})$. Let $\tilde{H}_R$ and $\tilde{M}_R$ be extensions of H_R and M_R to $E_3^+ \times (-\infty, T)$ as in Lemma 3.1. Let $f^* = D_t\tilde{H}_R - \Delta\tilde{H}_R + \nabla\tilde{M}_R$, $\phi^* = \tilde{H}_R$, $v_0^* = \tilde{H}_R|_{t=-T}$. It is easy to see that v_0^* is a polynomial on G_R, $f^* = 0$ and $\operatorname{div}\phi^* = 0$ on $G_R \times I_{hT}$. Since $D_t H_R = \Delta H_R - \nabla M_R$ in $G_R \times I_{hT}$, we see by the construction of H_R, M_R that there exists F^* such that $f^* = \operatorname{div} F^*$ and $F^* \in H_0^{1+\theta, (1+\theta)/2}(E_3^+ \times I_{hT})$. Moreover, we know that $\phi^* \in H_0^{2+\theta+\varepsilon, (2+\theta+\varepsilon)/2}(E_3^+ \times I_{hT})$ and $v_0^* \in H^{2+\theta, (2+\theta)/2}(E_3^+ \cup G_{hR})$. Then the functions $\tilde{H}_R$ and $\tilde{M}_R$ satisfy

$$(4.9)\qquad \begin{cases} D_t\tilde{H}_R - \Delta\tilde{H}_R + \nabla\tilde{M}_R = f^* & \text{in} \quad E_3^+\times \boldsymbol{I}_{hT}\,, \\ \operatorname{div}\tilde{H}_R = \operatorname{div}\phi^* & \text{in} \quad E_3^+\times \boldsymbol{I}_{hT}\,, \\ \tilde{H}_R|_{t=-T} = v_0^*\,, \quad \tilde{H}_R|_{x_3=0} = 0\,, \end{cases}$$

and have compact supports. Regarding f^*, ϕ^*, v_0^*, as given data, and $\tilde{H}_R$, $\tilde{M}_R$ as solutions of (4.9), by Proposition 4.4 with $\mu=\theta$, we see that $\tilde{H}_R$ and $\tilde{M}_R$ have analytic extensions $\bar{H}_R$, $\bar{M}_R$ to $\boldsymbol{G}_{hR}\times\boldsymbol{I}_{hT}$ and satisfy

$$\begin{aligned} &\|\bar{H}_R\|_{2+\theta}^* + \|\bar{M}_R\|_{1+\theta-\varepsilon}^* \\ &\quad \leqq C\{\|F^*\|_{1+\theta(E_3^+\times \boldsymbol{I}_{hT})}^* + \|\phi^*\|_{2+\theta+\varepsilon(E_3^+\times\boldsymbol{I}_{hT})}^* + |v_0^*|_{2+\theta(E_3^+\cup\boldsymbol{G}_{hR})}^*\}\,. \end{aligned}$$

By using Lemma 3.1, we obtain Proposition 4.5.

REMARK 4.2. The extension operator Φ defined in Lemma 3.1 also satisfies the following. If $f\in H_0^{k+\mu,(k+\mu)/2}(\boldsymbol{G}_{hR}\times\boldsymbol{I}_{hT})$, then $\Phi(f)\in H_0^{k+\mu,(k+\mu)/2}(\boldsymbol{E})$ for $k=1, 2$, $0<\mu<1$. We have

$$\|\Phi(f)\|_{k+\mu(E)}^* \leqq C\|f\|_{k+\mu}^*\,,$$

where C is independent of R and T.

By the above we obtain the complex analytic extensions of the operators K_R, L_R.

PROPOSITION 4.6. *The operators K_R and L_R satisfy the following properties.*

(1) *K_R is an operator from $H_0^{1+\mu,(1+\mu)/2}(\boldsymbol{G}_{hR}\times\boldsymbol{I}_{hT})$ into $H_0^{2+\mu,(2+\mu)/2}(\boldsymbol{G}_{hR}\times\boldsymbol{I}_{hT})$ and we have*

$$\|K_R[\Psi]\|_{2+\mu}^* \leqq C\|\Psi\|_{1+\mu}^* \quad \textit{for} \quad \Psi\in H_0^{1+\mu,(1+\mu)/2}(\boldsymbol{G}_{hR}\times\boldsymbol{I}_{hT})\,,$$

where $C=C(R_0, T_0)$ for $R<R_0$, $T<T_0$.

(2) *If $\Psi_j\in H_0^{1+\mu,(1+\mu)/2}(\boldsymbol{G}_{hR}\times\boldsymbol{I}_{hT})$ $(j=1, 2)$, then*

$$\|K_R[\Psi_1]-K_R[\Psi_2]\|_{2+\mu}^* \leqq C\|\Psi_1-\Psi_2\|_{1+\mu}^*\,,$$

where $C=C(R_0, T_0)$ for $R<R_0$, $T<T_0$.

(3) *If $\Psi\in H_0^{1+\mu,(1+\mu)/2}(\boldsymbol{G}_{hR}\times\boldsymbol{I}_{hT})$, then $\nabla L_R[\Psi]\in H_0^{\mu,\mu/2}(\boldsymbol{G}_{hR}\times\boldsymbol{I}_{hT})$ and $L_R[\Psi]\in H_0^{1+\mu-\varepsilon,(1+\mu-\varepsilon)/2}(\boldsymbol{G}_{hR}\times\boldsymbol{I}_{hT})$ for any ε with $0<\mu-\varepsilon<1$, $\varepsilon>0$.*

PROOF. We know that

$$\begin{aligned} K_R[\Psi](x,t) &= U(\operatorname{div}\Phi(\Psi), -\Phi(\mathscr{Q}), -\Phi(\mathscr{Q})|_{t=-T})|_{G_R\times I_T}(x,t) \\ &\quad - \sum_{2k+|\alpha|\leqq 2}(k!\,\alpha!)^{-1}D_t^kD_x^\alpha U(0,0)x^\alpha t^k|_{G_R\times I_T}\,, \\ L_R[\Psi](x,t) &= P(\operatorname{div}\Phi(\Psi), -\Phi(\mathscr{Q}), -\Phi(\mathscr{Q})|_{t=-T})|_{G_R\times I_T}(x,t) \\ &\quad - \sum_{|\beta|\leqq 1} D_x^\beta P(0,0)x^\beta|_{G_R\times I_T}\,. \end{aligned}$$

In view of the definition of $\mathscr{Q}$ and Remark 4.2, it is easy to see that the function $\Phi(\mathscr{Q})$ is in $H_0^{2+\mu+\varepsilon,(2+\mu+\varepsilon)/2}(E)$. Putting $\tilde{F}=\Phi(\Psi)$, $\phi=-\Phi(\mathscr{Q})$, and $\tilde{v}_0=-\Phi(\mathscr{Q})$, we use Proposition 4.3. Then, noticing Remark 4.2, we have Proposition 4.6.

5. Proof of Theorem 1.1. First, we prove the following proposition for small R, T.

PROPOSITION 5.1. *Let* $u_0\in H^{2+\theta+\varepsilon}(G_{hR})$. *Suppose that* $u\in H^{2+\theta+\varepsilon,(2+\theta+\varepsilon)/2}(G_R\times I_{hT})$ *and* $p\in H^{1+\theta+\varepsilon,(1+\theta+\varepsilon)/2}(G_R\times I_{hT})$ *satisfy*

$$(5.1)\qquad \begin{cases} D_t u-\Delta u+\nabla p=f-\operatorname{div}N(u) & in\quad G_R\times I_T\,,\\ \operatorname{div}u=0 & in\quad G_R\times I_T\,,\\ u|_{t=-T}=u_0\ (\operatorname{div}u_0=0)\,,\quad u|_{x_3=0}=0\,. \end{cases}$$

Suppose that there exists $F=\{F_{jk}\}_{(j,k=1,2,3)}$ *such that* $f=\operatorname{div}F$ *and* $F\in H^{1+\theta+\varepsilon,(1+\theta+\varepsilon)/2}(G_{hR}\times I_{hT})$, $(0<\theta+\varepsilon<1,\ \varepsilon>0)$. *Then we can extend* u *and* p *so that* $u\in H^{2+\theta,(2+\theta)/2}(G_{hR}\times I_{hT})$, $D_{x_3}u\in H^{1+\theta,(1+\theta)/2}(G_{hR}\times I_{hT})$, $p\in H^{1+\theta,(1+\theta)/2}(G_{hR}\times I_{hT})$.

As is well known, we may assume that $u_0=0$. To prove this proposition, we consider the integral equation:

$$(3.5)\qquad w=K_R[\Psi(w+H_R)]$$

in $H_0^{2+\theta,(2+\theta)/2}(G_{hR}\times I_{hT})$. Since the solution (u,p) of (5.1) is written as $u=v_R+H_R+\mathscr{Q}$, $p=q_R+M_R+\mathscr{R}$, it is sufficient to prove that v_R, $H_R\in H_0^{2+\theta,(2+\theta)/2}(G_{hR}\times I_{hT})$, q_R, M_R, $D_{x_3}v_R$, $D_{x_3}H_R\in H_0^{1+\theta,(1+\theta)/2}(G_{hR}\times I_{hT})$. By Proposition 4.5, we see that H_R and M_R satisfy the above properties. Regarding $H_R\in H_0^{2+\theta,(2+\theta)/2}(G_{hR}\times I_{hT})$ as known, we seek the solution w of (3.5) in $H_0^{2+\theta,(2+\theta)/2}(G_{hR}\times I_{hT})$. To continue the proof, we need the following Propositions 5.2 and 5.3, the first of which can be proved in the same way as Proposition 3.3.

PROPOSITION 5.2. *Choose* R_3 *sufficiently small and put* $T=O(R^2)$ *for* $0<R<R_3$. *Suppose that* $w\in H_0^{2+\mu,(2+\mu)/2}(G_{hR}\times I_{hT})$ *and* $\|w\|_{2+\mu}^*$ *is uniformly bounded by some number* M *for* $0<R<R_3$. *Then*

$$(5.2)\qquad \Psi(w)\in H_0^{1+\mu,(1+\mu)/2}(G_{hR}\times I_{hT})\,,$$

$$(5.3)\qquad \|\Psi(w)\|_{1+\mu}^*\leqq C_1+C_2(M)R^2\,,$$

where C_1 *is independent of* M *and* R, *while* $C_2(M)$ *depends on* M. *Moreover, if* $\|w_j\|_{2+\mu}^*\leqq M$ $(j=1,2)$, *then*

$$(5.4)\qquad \|\Psi(w_1)-\Psi(w_2)\|_{1+\mu}^*\leqq C_3(M)R\|w_1-w_2\|_{2+\mu}^*\,,$$

where $C_3(M)$ *depends on* M.

PROPOSITION 5.3. *Choose R_4 sufficiently small and put $T = O(R^2)$ for $0 < R < R_4$. Suppose that H_R is in $H_0^{2+\theta,(2+\theta)/2}(G_{hR} \times I_{hT})$. Then there exists a solution w of* (3.5) *which is in $H_0^{2+\theta,(2+\theta)/2}(G_{hR} \times I_{hT})$.*

PROOF. We define the sequence $\{w^k\}$ by $w^0 = 0$ and $w^{k+1} = K_R[\Psi(w^k + H_R)]$. By Proposition 4.6 with $\mu = \theta$ and Proposition 5.2, we see that there exists a positive constant M such that $w^k \in H_0^{2+\theta,(2+\theta)/2}(G_{hR} \times I_{hT})$ and $\| w^k \|_{2+\theta}^* \leqq M$. Choosing R_4 sufficiently small, we then have $\| w^{k+1} - w^k \|_{2+\theta}^* \leqq 2^{-1} \| w^k - w^{k-1} \|_{2+\theta}^*$ for $R < R_4$. This shows that the sequence $\{w^k\}$ is a Cauchy sequence in $H_0^{2+\theta,(2+\theta)/2}(G_{hR} \times I_{hT})$, whose limit w' is the solution of (3.5).

We now continue the proof of Proposition 5.1. The solution $w \in H_0^{2+\theta,(2+\theta)/2}(G_{hR} \times I_{hT})$ is also in $C_0^{2+\theta,(2+\theta)/2}(G_R \times I_T)$ and $\| w \|_{2+\theta}$ is uniformly bounded by some number M for $0 < R < R_4$. On the other hand, v_R is the solution of (3.5) and is in $C_0^{2+\theta,(2+\theta)/2}(G_R \times I_T)$ and $\| v_R \|_{2+\theta}$ is also uniformly bounded by M for $0 < R < R_4$. So, by Proposition 3.2, we obtain $w = v_R$ in $G_R \times I_T$. In other words, there exists an analytic extension of v_R to $H_0^{2+\theta,(2+\theta)/2}(G_{hR} \times I_{hT})$. The same properties are true for q_R and $D_{x_3} v_R$. Therefore the proof of Proposition 5.1 is complete.

Now, we prove Theorem 1.1. We return to the solution (u, p) of (1.1). It is easy to see that the hypotheses of Proposition 5.1 follow from those of Theorem 1.1 with $\mu = \theta + \varepsilon$. By Proposition 5.1, there is a constant $\delta > 0$ such that u, $D_{x_3} u$, p are analytic in (z', ω) for $(z, \omega) \in \mathscr{D}_{0\delta} = \{(z, \omega) \in \mathscr{D} \times \Omega;\ |z'| < \delta,\ |\omega - t_0| < \delta,\ y_3 = 0,\ 0 < x_3 < \delta\}$, and $|u(z', x_3, \omega)| + |D_{x_3} u(z', x_3, \omega)| + |p(z', x_3, \omega)| < M$ for $(z, \omega) \in \mathscr{D}_{0\delta}$, where δ and M are independent of x_3. We consider the Cauchy problem:

$$
(5.5) \qquad \begin{cases}
D_{z_3}^2 \bar{u}_1 = D_\omega \bar{u}_1 - \sum_{j=1}^{2} D_{z_j}^2 \bar{u}_1 + D_{z_1} \bar{p} - f_1 + \sum_{j=1}^{3} \bar{u}_j D_{z_j} \bar{u}_1 \,, \\
D_{z_3}^2 \bar{u}_2 = D_\omega \bar{u}_2 - \sum_{j=1}^{2} D_{z_j}^2 \bar{u}_2 + D_{z_2} \bar{p} - f_2 + \sum_{j=1}^{3} \bar{u}_j D_{z_j} \bar{u}_2 \,, \\
D_{z_3}^2 \bar{u}_3 = -D_{z_1} D_{z_3} \bar{u}_1 - D_{z_2} D_{z_3} \bar{u}_2 \,, \\
D_{z_3} \bar{p} = -D_\omega \bar{u}_3 + \sum_{j=1}^{2} (D_{z_j}^2 \bar{u}_3 + D_{z_j} D_{z_3} \bar{u}_j) + f_3 - \sum_{j=1}^{3} \bar{u}_j D_{z_j} \bar{u}_3 \,, \\
\bar{u}|_{z_3=\delta'} = u(z', \delta', \omega) \,, \quad D_{z_3} \bar{u}|_{z_3=\delta'} = D_{x_3} u(z', \delta', \omega) \,, \\
\bar{p}|_{z_3=\delta'} = p(z', \delta', \omega) \,, \quad (0 < \delta' < \delta) \,.
\end{cases}
$$

By the Cauchy-Kowalewsky Theorem, there exists a unique analytic solution of (5.5) in $\mathscr{D}_{\delta''} = \{(z, \omega) \in \mathscr{D} \times \Omega; |z'| < \delta'', |\omega - t_0| < \delta'', |z_3 - \delta'| < \delta''\}$, where δ'' depends on δ and M, but δ'' is independent of δ'. Choosing δ' sufficiently small, we see that $(0, t_0)$ is in $\mathscr{D}_{\delta''}$.

On the other hand, by Kahane [3], we know that, under the same assumption as in Theorem 1.1, the solution (u, p) of (1.1) is analytic near $(0, 0, \delta', t_0)$. The functions u and p satisfy (5.5). Then we have $(u, p) = (\bar{u}, \bar{p})$ near $(0, 0, \delta', t_0)$. Therefore, u and p are analytic near $(0, t_0)$.

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