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REAL ANALYTIC $SL(n, R)$ ACTIONS ON SPHERES

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0. Introduction. Let $SL(n, R)$ denote the group of all $n \times n$ real matrices of determinant 1. In the previous paper [12], we classified real analytic $SL(n, R)$ actions on the standard n-sphere for each $n \geqq 3$. In this paper we study real analytic $SL(n, R)$ actions on the standard m-sphere for $5 \leqq n \leqq m \leqq 2n-2$. We shall show that such an action is characterized by a certain real analytic $R^{\times}$ action on a homotopy $(m-n+1)$-sphere. Here $R^{\times}$ is the multiplicative group of all non-zero real numbers.

In Section 1 we construct a real analytic $SL(n, R)$ action on the standard $(n+k-1)$-sphere from a real analytic $R^{\times}$ action on a homotopy k-sphere satisfying a certain condition for each $n+k \geqq 6$. In Section 3 we state a structure theorem for a real analytic $SL(n, R)$ action which satisfies a certain condition on the restricted $SO(n)$ action, and in Section 5 we state a decomposition theorem and a classification theorem. In Section 6 we construct real analytic $R^{\times}$ actions on the standard k-sphere. It can be seen that there are infinitely many (at least the cardinality of the real numbers) mutually distinct real analytic $SL(n, R)$ actions on the standard m-sphere.

1. Construction. Let $\psi: R^{\times} \times \Sigma \to \Sigma$ be a real analytic $R^{\times}$ action on a real analytic closed manifold Σ which is homotopy equivalent to the k-sphere. Define a real analytic involution T of Σ by $T(x) = \psi(-1, x)$ for $x \in \Sigma$. Put $F = F(R^{\times}, \Sigma)$, the fixed point set. We say that the action ψ satisfies the condition (P) if

(i) there exists a compact contractible k-dimensional submanifold X of Σ such that $X \cup TX = \Sigma$ and $X \cap TX = F$,

(ii) there exists a real analytic $R^{\times}$ equivariant isomorphism j of $R \times F$ onto an open set of Σ such that $j(0, x) = x$ for $x \in F$. Here $R^{\times}$ acts on R by the scalar multiplication.

Notice that $F = F(T, \Sigma)$, the fixed point set of the involution T by the condition (i), and hence F is a real analytic $(k-1)$-dimensional closed submanifold of Σ. Define a map

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$$f\colon (\boldsymbol{R}^n - 0)\times F \to (\boldsymbol{R}^n - 0)\underset{\boldsymbol{R}^\times}{\times}(\Sigma - F)$$

by $f(u, x) = (u, j(1, x))$ for $u \in \boldsymbol{R}^n - 0, x \in F$. Then the map f is a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ equivariant isomorphism of $(\boldsymbol{R}^n - 0)\times F$ onto an open set of $(\boldsymbol{R}^n - 0)\times_{\boldsymbol{R}^\times}(\Sigma - F)$, where $\boldsymbol{SL}(n, \boldsymbol{R})$ acts naturally on $\boldsymbol{R}^n$, $\boldsymbol{R}^\times$ acts on $\boldsymbol{R}^n$ by the scalar multiplication and $\boldsymbol{R}^\times$ acts on Σ by the given action ψ. Here $(\boldsymbol{R}^n - 0)\times_{\boldsymbol{R}^\times}(\Sigma - F)$ is the quotient of $(\boldsymbol{R}^n - 0)\times(\Sigma - F)$ obtained by identifying (u, y) with $(t^{-1}u, \psi(t, y))$ for $u \in \boldsymbol{R}^n - 0, y \in \Sigma - F$, $t \in \boldsymbol{R}^\times$. Put

$$M(\psi, j) = \boldsymbol{R}^n \times F \underset{f}{\cup} (\boldsymbol{R}^n - 0)\underset{\boldsymbol{R}^\times}{\times}(\Sigma - F)\ ,$$

which is the space formed from the disjoint union of $\boldsymbol{R}^n\times F$ and $(\boldsymbol{R}^n - 0)\times_{\boldsymbol{R}^\times}(\Sigma - F)$ by identifying (u, x) with $f(u, x)$ for $u \in \boldsymbol{R}^n - 0, x \in F$. By the construction, it can be seen that the space $M(\psi, j)$ is a compact Hausdorff space with $\boldsymbol{SL}(n, \boldsymbol{R})$ action, and $M(\psi, j)$ admits a real analytic structure so that the $\boldsymbol{SL}(n, \boldsymbol{R})$ action is real analytic.

PROPOSITION 1.1. (a) *Let $j_1\colon \boldsymbol{R}\times F \to \Sigma$ be a real analytic $\boldsymbol{R}^\times$ equivariant isomorphism of $\boldsymbol{R}\times F$ onto an open set of Σ such that $j_1(0, x) = x$ for $x \in F$. Then $M(\psi, j_1)$ is real analytically isomorphic to $M(\psi, j)$ as $\boldsymbol{SL}(n, \boldsymbol{R})$ manifolds.*

(b) *Suppose $n \geqq 1$ and $n + k \geqq 6$. Then $M(\psi, j)$ is real analytically isomorphic to the standard $(n + k - 1)$-sphere.*

PROOF. It is easy to see that there is a real analytic function $s\colon F \to \boldsymbol{R}^\times$ such that $j_1(t, x) = j(s(x)t, x)$ for $t \in \boldsymbol{R}, x \in F$. Let g be a real analytic automorphism of the disjoint union of $\boldsymbol{R}^n\times F$ and $(\boldsymbol{R}^n-0)\times_{\boldsymbol{R}^\times}(\Sigma-F)$ defined by

$$g(u, x) = (s(x)u, x) \quad \text{for} \quad u \in \boldsymbol{R}^n, \quad x \in F\ ,$$
$$g(v, y) = (v, y) \quad \text{for} \quad v \in \boldsymbol{R}^n - 0\ , \quad y \in \Sigma - F\ .$$

Then it is easy to see that g induces a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ equivariant isomorphism of $M(\psi, j_1)$ onto $M(\psi, j)$.

To show (b), we consider the restricted $\boldsymbol{SO}(n)$ action on $M(\psi, j)$. We can assume $j([0, \infty)\times F)\subset X$ by the condition (P). Put $X_1 = X - j([0, 1)\times F)$. Let $\boldsymbol{D}^n$ denote the closed unit disk of $\boldsymbol{R}^n$. Let ∂Y denote the boundary of a given manifold Y. Then it can be seen that there exists an equivariant diffeomorphism

$$M(\psi, j) = \boldsymbol{D}^n \times F \underset{h}{\cup} \partial\boldsymbol{D}^n \times X_1$$

as smooth $\boldsymbol{SO}(n)$ manifolds, where $h\colon \partial\boldsymbol{D}^n\times F \to \partial\boldsymbol{D}^n \times \partial X_1$ is a C^∞ diffeomorphism defined by $h(u, x) = (u, j(1, x))$ for $u \in \partial\boldsymbol{D}^n, x \in F$. Hence

$M(\psi, j)$ is C^∞ diffeomorphic to $\partial(\boldsymbol{D}^n \times X_1)$. Here X_1 is a compact contractible k-manifold; hence $\partial(\boldsymbol{D}^n \times X_1)$ is simply connected for $n \geqq 1$. Therefore $M(\psi, j)$ is C^∞ diffeomorphic to the standard $(n+k-1)$-sphere for $n+k \geqq 6$ by the h-cobordism theorem (cf. Milnor [8, Theorem 9.1]). It is known by Grauert [3] and Whitney [13, Part III] that two real analytic paracompact manifolds are real analytically isomorphic if they are C^∞ diffeomorphic. Consequently, $M(\psi, j)$ is real analytically isomorphic to the standard $(n+k-1)$-sphere for $n+k \geqq 6$. q.e.d.

REMARK. By the condition (P), it is shown that Σ is real analytically isomorphic to the standard k-sphere for $k \geqq 5$ by the h-cobordism theorem.

2. Certain subgroups of $SL(n, R)$. As usual we regard $M_n(\boldsymbol{R})$ with the bracket operation $[A, B] = AB - BA$ as the Lie algebra of $\boldsymbol{GL}(n, \boldsymbol{R})$. Let $\mathfrak{sl}(n, \boldsymbol{R})$ and $\mathfrak{so}(n)$ denote the Lie subalgebras of $M_n(\boldsymbol{R})$ corresponding to the subgroups $\boldsymbol{SL}(n, \boldsymbol{R})$ and $\boldsymbol{SO}(n)$ respectively. Then

$$\mathfrak{sl}(n, \boldsymbol{R}) = \{X \in M_n(\boldsymbol{R}) \colon \text{trace } X = 0\} ,$$
$$\mathfrak{so}(n) = \{X \in M_n(\boldsymbol{R}) \colon X \text{ is skew symmetric}\} .$$

Define certain linear subspaces of $\mathfrak{sl}(n, \boldsymbol{R})$ as follows:

$$\mathfrak{sl}(n-r, \boldsymbol{R}) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \colon A \text{ is } (n-r)\times(n-r) \text{ matrix of trace } 0\right\} ,$$
$$\mathfrak{so}(n-r) = \mathfrak{so}(n) \cap \mathrm{sl}(n-r, \boldsymbol{R}) ,$$
$$\mathfrak{sym}(n-1) = \{X \in \mathfrak{sl}(n-1, \boldsymbol{R}) \colon X \text{ is symmetric}\} ,$$
$$\mathfrak{a} = \{(a_{ij}) \in \mathfrak{sl}(n, \boldsymbol{R}) \colon a_{ij} = 0 \quad \text{for} \quad i \neq 1\} ,$$
$$\mathfrak{a}^* = \{(a_{ij}) \in \mathfrak{sl}(n, \boldsymbol{R}) \colon a_{ij} = 0 \quad \text{for} \quad j \neq 1\} ,$$
$$\mathfrak{b} = \{(a_{ij}) \in \mathfrak{sl}(n, \boldsymbol{R}) \colon a_{ij} = 0 \quad \text{for} \quad i \neq j,\ a_{22} = a_{33} = \cdots = a_{nn}\} .$$

Then

$$\mathfrak{sl}(n, \boldsymbol{R}) = \mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a} \oplus \mathfrak{a}^* \oplus \mathfrak{b} ,$$
$$\mathfrak{sl}(n-1, \boldsymbol{R}) = \mathfrak{so}(n-1) \oplus \mathfrak{sym}(n-1)$$

as direct sums of vector spaces. Moreover we have

$$\begin{aligned} &[\mathfrak{a}, \mathfrak{a}^*] = \mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{b} , \\ (2.1)\quad &[\mathfrak{a}, \mathfrak{a}] = [\mathfrak{a}^*, \mathfrak{a}^*] = [\mathfrak{b}, \mathfrak{b}] = [\mathfrak{b}, \mathfrak{sl}(n-1, \boldsymbol{R})] = 0 , \\ &[\mathfrak{a}, \mathfrak{b}] = [\mathfrak{a}, \mathfrak{sl}(n-1, \boldsymbol{R})] = \mathfrak{a} , \quad [\mathfrak{a}^*, \mathfrak{b}] = [\mathfrak{a}^*, \mathfrak{sl}(n-1, \boldsymbol{R})] = \mathfrak{a}^* . \end{aligned}$$

Let $\boldsymbol{SL}(n-r, \boldsymbol{R})$ and $\boldsymbol{SO}(n-r)$ denote the connected subgroups of $\boldsymbol{SL}(n, \boldsymbol{R})$ corresponding to the Lie subalgebras $\mathfrak{sl}(n-r, \boldsymbol{R})$ and $\mathfrak{so}(n-r)$, respectively.

Let $Ad: \boldsymbol{SL}(n, \boldsymbol{R}) \to \boldsymbol{GL}(\mathfrak{sl}(n, \boldsymbol{R}))$ be the adjoint representation defined by $Ad(A)X = AXA^{-1}$ for $A \in \boldsymbol{SL}(n, \boldsymbol{R})$, $X \in \mathfrak{sl}(n, \boldsymbol{R})$. Then the linear subspaces $\mathfrak{sl}(n-1, \boldsymbol{R})$, $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{b}$ are $Ad(\boldsymbol{SL}(n-1, \boldsymbol{R}))$ invariant, and the linear subspaces $\mathfrak{so}(n-1)$ and $\mathfrak{sym}(n-1)$ are $Ad(\boldsymbol{SO}(n-1))$ invariant. Moreover, the linear subspaces $\mathfrak{sym}(n-1)$, $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{b}$ are irreducible $Ad(\boldsymbol{SO}(n-1))$ spaces respectively for each $n \geqq 3$. Put

$$\mathfrak{k}(p, q) = \left\{ \begin{pmatrix} 0 & qx_2 & \cdots & qx_n \\ px_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ px_n & 0 & \cdots & 0 \end{pmatrix} : x_i \in \boldsymbol{R} \right\}$$

for $p, q \in \boldsymbol{R}$. Then $\mathfrak{k}(p, q)$ is an $Ad(\boldsymbol{SO}(n-1))$ invariant linear subspace of $\mathfrak{a} \oplus \mathfrak{a}^*$, and we have

$$\begin{aligned} &[\mathfrak{k}(p, q), \mathfrak{sym}(n-1)] = [\mathfrak{k}(p, q), \mathfrak{b}] = \mathfrak{k}(p, -q)\ , \\ &[\mathfrak{k}(p, q), \mathfrak{k}(p, q)] = \begin{cases} 0 & \text{for} \quad pq = 0\ , \\ \mathfrak{so}(n-1) & \text{for} \quad pq \neq 0\ . \end{cases} \end{aligned} \tag{2.2}$$

LEMMA 2.3. *Suppose $n \geqq 3$. Let $\mathfrak{g}$ be a proper Lie subalgebra of $\mathfrak{sl}(n, \boldsymbol{R})$ which contains $\mathfrak{so}(n-1)$. Then $\mathfrak{g}$ is one of the following: $\mathfrak{so}(n-1)$, $\mathfrak{so}(n-1) \oplus \mathfrak{b}$, $\mathfrak{so}(n-1) \oplus \mathfrak{a}$, $\mathfrak{so}(n-1) \oplus \mathfrak{a}^*$, $\mathfrak{so}(n-1) \oplus \mathfrak{k}(p, q)$ for $pq \neq 0$, $\mathfrak{so}(n-1) \oplus \mathfrak{a} \oplus \mathfrak{b}$, $\mathfrak{so}(n-1) \oplus \mathfrak{a}^* \oplus \mathfrak{b}$, $\mathfrak{sl}(n-1, \boldsymbol{R})$, $\mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{b}$, $\mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a}$, $\mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a}^*$, $\mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a} \oplus \mathfrak{b}$, $\mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a}^* \oplus \mathfrak{b}$.*

PROOF. Since $\mathfrak{g}$ contains $\mathfrak{so}(n-1)$, $\mathfrak{g}$ is an $Ad(\boldsymbol{SO}(n-1))$ invariant linear subspace of $\mathfrak{sl}(n, \boldsymbol{R})$. Hence we have $\mathfrak{g} = \mathfrak{so}(n-1) \oplus (\mathfrak{g} \cap \mathfrak{sym}(n-1)) \oplus (\mathfrak{g} \cap (\mathfrak{a} \oplus \mathfrak{a}^*)) \oplus (\mathfrak{g} \cap \mathfrak{b})$ as a direct sum of $Ad(\boldsymbol{SO}(n-1))$ invariant linear subspaces. Since $\mathfrak{sym}(n-1)$ is irreducible, we have $\mathfrak{g} \cap \mathfrak{sym}(n-1) = 0$ or $\mathfrak{sym}(n-1)$. Since $\mathfrak{g}$ is a proper Lie subalgebra of $\mathfrak{sl}(n, \boldsymbol{R})$, $\mathfrak{g}$ does not contain $\mathfrak{a} \oplus \mathfrak{a}^*$ by (2.1). Suppose $n \geqq 4$. Then we derive that $\mathfrak{g} \cap (\mathfrak{a} \oplus \mathfrak{a}^*)$ coincides with certain $\mathfrak{k}(p, q)$. If $\mathfrak{g}$ contains $\mathfrak{sym}(n-1)$, then (2.2) implies that $\mathfrak{g} \cap (\mathfrak{a} \oplus \mathfrak{a}^*) = 0$, $\mathfrak{a}$ or $\mathfrak{a}^*$. Now we can prove the lemma for $n \geqq 4$ by a routine work from (2.1) and (2.2). The proof for $n = 3$ is similar, so we omit the detail. q.e.d.

REMARK. Let $G(p, q)$ denote the connected Lie subgroup of $\boldsymbol{SL}(n, \boldsymbol{R})$ corresponding to the Lie subalgebra $\mathfrak{so}(n-1) \oplus \mathfrak{k}(p, q)$ for $pq \neq 0$. If $pq < 0$, then $G(p, q)$ is conjugate to $G(1, -1) = \boldsymbol{SO}(n)$. If $pq > 0$, then $G(p, q)$ is conjugate to $G(1, 1)$, which is non-compact.

Put $X_1 = \left(\begin{array}{cc|c} 1 & 0 & \\ 1 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)$.

LEMMA 2.4. (i) *Assume that* $\mathfrak{g}$ *is one of the following*:

$$\mathfrak{so}(n-1),\ \mathfrak{so}(n-1)\oplus\mathfrak{b},\ \mathfrak{so}(n-1)\oplus\mathfrak{a},\ \mathfrak{so}(n-1)\oplus\mathfrak{a}\oplus\mathfrak{b}\ ,$$
$$\mathfrak{so}(n-1)\oplus\mathfrak{k}(p, q)\quad \text{for}\quad pq\neq 0,\ \mathfrak{sl}(n-1, \boldsymbol{R}),\ \mathfrak{sl}(n-1, \boldsymbol{R})\oplus\mathfrak{b}\ .$$

Then $\mathfrak{so}(n)\cap Ad(X_1)\mathfrak{g}=\mathfrak{so}(n-2)$.

(ii) *Assume that* $\mathfrak{g}$ *is one of the following*:

$$\mathfrak{so}(n-1)\oplus\mathfrak{a}^*,\ \mathfrak{so}(n-1)\oplus\mathfrak{a}^*\oplus\mathfrak{b}\ .$$

Then $\mathfrak{so}(n)\cap Ad({}^tX_1^{-1})\mathfrak{g}=\mathfrak{so}(n-2)$.

PROOF. Since $\mathfrak{so}(n)\cap Ad(X_1)\mathfrak{g}=\{A\in\mathfrak{so}(n)\colon X_1^{-1}AX_1\in\mathfrak{g}\}$, we have the desired equations by a routine work from the following relation:

$$X_1^{-1}(a_{ij})X_1=\begin{pmatrix} a_{11}+a_{12} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21}+a_{22}-a_{11}-a_{12} & a_{22}-a_{12} & a_{23}-a_{13} & \cdots & a_{2n}-a_{1n} \\ a_{31}+a_{32} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1}+a_{n2} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

q.e.d.

Let $L(n)$, $L^*(n)$, $N(n)$ and $N^*(n)$ denote the connected Lie subgroups of $\boldsymbol{SL}(n, \boldsymbol{R})$ corresponding to the Lie subalgebras $\mathfrak{sl}(n-1, \boldsymbol{R})\oplus\mathfrak{a}$, $\mathfrak{sl}(n-1, \boldsymbol{R})\oplus\mathfrak{a}^*$, $\mathfrak{sl}(n-1, \boldsymbol{R})\oplus\mathfrak{a}\oplus\mathfrak{b}$ and $\mathfrak{sl}(n-1, \boldsymbol{R})\oplus\mathfrak{a}^*\oplus\mathfrak{b}$, respectively. Then these are closed subgroups of $\boldsymbol{SL}(n, \boldsymbol{R})$.

PROPOSITION 2.5. *Suppose* $n\geqq 3$. *Let* M *be an* $\boldsymbol{SL}(n, \boldsymbol{R})$ *space. Assume that the restricted* $\boldsymbol{SO}(n)$ *action on* M *has at most two orbit types* $\boldsymbol{SO}(n)/\boldsymbol{SO}(n-1)$ *and* $\boldsymbol{SO}(n)/\boldsymbol{SO}(n)$. *Then the identity component of an isotropy group of the* $\boldsymbol{SL}(n, \boldsymbol{R})$ *action on* M *is conjugate to one of the following*: $L(n)$, $L^*(n)$, $N(n)$ $N^*(n)$ *and* $\boldsymbol{SL}(n, \boldsymbol{R})$.

PROOF. Let $\mathfrak{g}$ be the Lie algebra corresponding to an isotropy group. By the assumption on the restricted $\boldsymbol{SO}(n)$ action, we see that $Ad(x)\mathfrak{g}$ contains $\mathfrak{so}(n-1)$ for some $x\in\boldsymbol{SL}(n, \boldsymbol{R})$. Such a Lie subalgebra is determined by Lemma 2.3. Moreover, we can derive $\mathfrak{so}(n)\cap Ad(y)\mathfrak{g}\neq\mathfrak{so}(n-2)$ for any $y\in\boldsymbol{SL}(n, \boldsymbol{R})$ by the assumption on the restricted $\boldsymbol{SO}(n)$ action. Hence we see that $\mathfrak{g}$ is one of the following up to conjugation: $\mathfrak{sl}(n-1, \boldsymbol{R})\oplus\mathfrak{a}$, $\mathfrak{sl}(n-1, \boldsymbol{R})\oplus\mathfrak{a}^*$, $\mathfrak{sl}(n-1, \boldsymbol{R})\oplus\mathfrak{a}\oplus\mathfrak{b}$, $\mathfrak{sl}(n-1, \boldsymbol{R})\oplus\mathfrak{a}^*\oplus\mathfrak{b}$, $\mathfrak{sl}(n, \boldsymbol{R})$ by Lemma 2.3 and Lemma 2.4. On the other hand, it is easy to see that the restricted $\boldsymbol{SO}(n)$ actions on the homogeneous spaces $\boldsymbol{SL}(n, \boldsymbol{R})/L(n)$, $\boldsymbol{SL}(n, \boldsymbol{R})/L^*(n)$, $\boldsymbol{SL}(n, \boldsymbol{R})/N(n)$ and $\boldsymbol{SL}(n, \boldsymbol{R})/N^*(n)$ have only one orbit type $\boldsymbol{SO}(n)/\boldsymbol{SO}(n-1)$ respectively. q.e.d.

3. Structure theorem. Let $\phi: G \times M \to M$ be a real analytic G action. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G. Let $L(M)$ denote the Lie algebra of all real analytic vector fields on M. Then we can define a Lie algebra homomorphism $\phi^+: \mathfrak{g} \to L(M)$ as follows (cf. Palais [10, Chapter II, Theorem II]):

$$\phi^+(X)_q(f) = \lim_{t \to 0} (f(\phi(\exp(-tX), q)) - f(q))/t$$

for $X \in \mathfrak{g}$, $q \in M$ and a real analytic function f defined on a neighborhood of q. It is easy to see that $\phi^+(X)_q = 0$ iff q is a fixed point of the one-parameter subgroup $\{\exp tX\}$. For each subgroup H of G, let $F(H, M)$ denote the fixed point set of the restricted H action of ϕ. Then $F(H, M)$ is a closed subset of M.

LEMMA 3.1. *Let $\phi: \boldsymbol{SL}(n, \boldsymbol{R}) \times M \to M$ be a real analytic action. Let $p \in F(\boldsymbol{SL}(n, \boldsymbol{R}), M)$. Suppose that there exists an analytic system of coordinates $(U; u_1, \cdots, u_m)$ with origin at p, such that*

$$(*) \qquad \phi^+((x_{ij}))_q = -\sum_{i,j=1}^{n} x_{ij} u_j(q)(\partial/\partial u_i)$$

for $(x_{ij}) \in \mathfrak{sl}(n, \boldsymbol{R})$, $q \in U$. Then, (i) *there exists an open neighborhood V of p in $F(\boldsymbol{SL}(n, \boldsymbol{R}), M)$ and there exists an analytic isomorphism h of $\boldsymbol{R}^n \times V$ onto an open set of M such that*
(a) *$h(0, v) = v$ for $v \in V$,*
(b) *$h(gu, v) = \phi(g, h(u, v))$ for $g \in \boldsymbol{SL}(n, \boldsymbol{R})$, $u \in \boldsymbol{R}^n$, $v \in V$.*
Moreover, (ii) *if pairs (V_1, h_1) and (V_2, h_2) satisfy the conditions* (a), (b), *then*

$$h_1(\boldsymbol{R}^n \times V_1) \cap h_2(\boldsymbol{R}^n \times V_2) = h_1(\boldsymbol{R}^n \times (V_1 \cap V_2)) ,$$

and there exists a unique real analytic real valued function f on $V_1 \cap V_2$ such that $h_1(u, v) = h_2(f(v)u, v)$ for $u \in \boldsymbol{R}^n$, $v \in V_1 \cap V_2$.

PROOF. The assumption $(*)$ implies $F(\boldsymbol{SL}(n, \boldsymbol{R}), M) \cap U = \{q \in U: u_1(q) = \cdots = u_n(q) = 0\}$. Define a real analytic isomorphism k of U onto an open set of $\boldsymbol{R}^m$ by $k(q) = (u_1(q), \cdots, u_m(q))$. There is a positive real number r such that $\boldsymbol{D}_r^n \times \boldsymbol{D}_r^{m-n} \subset k(U)$, namely $(u_1, \cdots, u_m) \in k(U)$ for $(u_1, \cdots, u_n) \in \boldsymbol{D}_r^n$, $(u_{n+1}, \cdots, u_m) \in \boldsymbol{D}_r^{m-n}$. Here we denote $\boldsymbol{D}_r^n = \{(v_1, \cdots, v_n) \in \boldsymbol{R}^n: v_1^2 + \cdots + v_n^2 < r^2\}$. Consider the following curves

$$a(t) = a(t; X, u, v) = k(\phi(\exp tX, k^{-1}(u, v))) ,$$
$$b(t) = b(t; X, u, v) = ((\exp tX)u, v)$$

for $X \in \mathfrak{sl}(n, \boldsymbol{R})$, $u \in \boldsymbol{D}_r^n$, $v \in \boldsymbol{D}_r^{m-n}$. The curve $b(t)$ is defined for each $t \in \boldsymbol{R}$, the curve $a(t)$ is defined on an interval $(-t_1, t_2)$ for some positive real numbers t_1, t_2. Put $X = (x_{ij})$, $a(t) = (a_1(t), \cdots, a_m(t))$ and $b(t) = (b_1(t), \cdots, b_m(t))$. Then it follows from the assumption $(*)$ that

$$(d/dt)a_i(t) = \sum_{j=1}^{n} x_{ij}a_j(t) \quad \text{for} \quad 1 \leqq i \leqq n \,,$$
$$(d/dt)a_i(t) = 0 \quad \text{for} \quad n < i \leqq m \,.$$

On the other hand,

$$(d/dt)b_i(t) = \sum_{j=1}^{n} x_{ij}b_j(t) \quad \text{for} \quad 1 \leqq i \leqq n \,,$$
$$(d/dt)b_i(t) = 0 \quad \text{for} \quad n < i \leqq m$$

by the definition of $b(t)$. Since $a(0) = b(0)$, we can derive that

$$(**) \qquad a(t; X, u, v) = b(t; X, u, v)$$

on the interval $(-t_1, t_2)$. Put $u_0 = (r/2, 0, \cdots, 0) \in \boldsymbol{D}_r^n$. Then it follows from the equation $(**)$ that the identity component of an isotropy group at $k^{-1}(u_0, v)$ coincides with $L(n)$ for each $v \in \boldsymbol{D}_r^{m-n}$. Hence we can define a map $h'\colon \boldsymbol{R}^n \times \boldsymbol{D}_r^{m-n} \to M$ by

$$h'(u, v) = \begin{cases} k^{-1}(0, v) & \text{for} \quad u = 0 \,, \\ \phi(g, k^{-1}(u_0, v)) & \text{for} \quad u = gu_0 \,, \quad g \in \boldsymbol{SL}(n, \boldsymbol{R}) \,. \end{cases}$$

First we shall show that $kh' =$ identity on $\boldsymbol{D}_r^n \times \boldsymbol{D}_r^{m-n}$. Let $u \in \boldsymbol{D}_r^n$ and $u \neq 0$. Then u can be expressed as follows: $u = (\exp X_1 \cdot \exp X_2)u_0$ for $X_1 \in \mathfrak{so}(n)$, and X_2 is a diagonal matrix with diagonal components $c, -c, 0, \cdots, 0$ for $c \in \boldsymbol{R}$. The equation $(**)$ implies that $k(\phi(\exp tX_2, k^{-1}(u_0, v)) = ((\exp tX_2)u_0, v)$ for $|t| \leqq 1$ and $k(\phi(\exp tX_1, k^{-1}((\exp X_2)u_0, v))) = ((\exp tX_1)(\exp X_2)u_0, v)$ for $t \in \boldsymbol{R}$. Then we have $kh' =$ identity on $\boldsymbol{D}_r^n \times \boldsymbol{D}_r^{m-n}$. Since $k\colon U \to k(U)$ is a real analytic isomorphism, it follows that the restriction of h' to $\boldsymbol{D}_r^n \times \boldsymbol{D}_r^{m-n}$ is a real analytic isomorphism of $\boldsymbol{D}_r^n \times \boldsymbol{D}_r^{m-n}$ onto an open set of M. On the other hand, the restriction of h' to $(\boldsymbol{R}^n - 0) \times \boldsymbol{D}_r^{m-n}$ is real analytic by definition. Moreover, the map h' is $\boldsymbol{SL}(n, \boldsymbol{R})$ equivariant by definition. Hence the map h' is a real analytic local isomorphism at each point of $\boldsymbol{R}^n \times \boldsymbol{D}_r^{m-n}$.

Now we shall show that h' is an injection. Assume that $h'(g_1u_0, v_1) = h'(g_2u_0, v_2)$ for some $g_i \in \boldsymbol{SL}(n \in \boldsymbol{R})$, $v_i \in \boldsymbol{D}_r^{m-n}$. Since h' is equivariant, we have $k^{-1}(u_0, v_1) = \phi(g_1^{-1}g_2, k^{-1}(u_0, v_2))$. Put $g = g_1^{-1}g_2$. Let L_i be the identity component of the isotropy group at $k^{-1}(u_0, v_i)$. Then $L_1 = gL_2g^{-1}$ and $L_i = L(n)$ by the assumption $(*)$. Hence $g \in NL(n)$, the normalizer of $L(n)$ in $\boldsymbol{SL}(n, \boldsymbol{R})$. The equation $(**)$ implies that $k(\phi((x_{ij}), k^{-1}(u_0, v))) =$

$(x_{11}u_0, v)$ for $v \in \boldsymbol{D}_r^{m-n}$, $(x_{ij}) \in NL(n)$, $0 < |x_{11}| < 2$. We can choose g or g^{-1} as (x_{ij}) such that $0 < |x_{11}| < 2$. It follows that $v_1 = v_2$ and $g = g_1^{-1}g_2 \in L(n)$. Hence $g_1u_0 = g_2u_0$. Therefore h' is an injection. The map $v \to h'(0, v)$ is a real analytic isomorphism of $\boldsymbol{D}_r^{m-n}$ onto an open neighborhood V of p in $F(\boldsymbol{SL}(n, \boldsymbol{R}), M)$.

Define a map $h: \boldsymbol{R}^n \times V \to M$ by $h(u, v) = h'(u, k(v))$ for $u \in \boldsymbol{R}^n$, $v \in V$. Then it is easy to see that h is a real analytic isomorphism of $\boldsymbol{R}^n \times V$ onto an open set of M satisfying the conditions (a), (b).

Next, let $h_i: \boldsymbol{R}^n \times V_i \to M$ be a real analytic into isomorphism satisfying the conditions (a), (b) for $i = 1, 2$. Put $\boldsymbol{e} = (1, 0, \cdots, 0) \in \boldsymbol{R}^n$. Assume that $\phi(g_1, h_1(\boldsymbol{e}, v_1)) = \phi(g_2, h_2(\boldsymbol{e}, v_2))$ for some $g_i \in \boldsymbol{SL}(n, \boldsymbol{R})$, $v_i \in V_i$. Then $h_1(\boldsymbol{e}, v_1) = \phi(g_1^{-1}g_2, h_2(\boldsymbol{e}, v_2))$, and hence $g_1^{-1}g_2 \in NL(n)$, because the isotropy group at $h_i(\boldsymbol{e}, v_i)$ coincides with $L(n)$. Put x_t the diagonal matrix with diagonal components $t, t^{-1}, 1, \cdots, 1$. Then $x_t \in NL(n)$. Since $h_i(t\boldsymbol{e}, v_i) = \phi(x_t, h_i(\boldsymbol{e}, v_i))$ and $NL(n)/L(n)$ is abelian, it follows that $h_1(t\boldsymbol{e}, v_1) = \phi(g_1^{-1}g_2, h_2(t\boldsymbol{e}, v_2))$ for $t \neq 0$. Let $t \to 0$. Then $v_1 = \phi(g_1^{-1}g_2, v_2) = v_2$. It follows that $h_1(\boldsymbol{R}^n \times V_1) \cap h_2(\boldsymbol{R}^n \times V_2)$ is contained in $h_i(\boldsymbol{R}^n \times V)$ for $V = V_1 \cap V_2$. Since $h_i(\boldsymbol{R}^n \times V)$ is a smallest open $\boldsymbol{SL}(n, \boldsymbol{R})$ invariant neighborhood of $V = h_i(0 \times V)$, we can derive that $h_1(\boldsymbol{R}^n \times V) = h_2(\boldsymbol{R}^n \times V)$, and hence $h_1(\boldsymbol{R}^n \times V_1) \cap h_2(\boldsymbol{R}^n \times V_2) = h_1(\boldsymbol{R}^n \times V)$.

From the above argument, there exists a unique real analytic function $f: V \to \boldsymbol{R}$ such that $h_1(\boldsymbol{e}, v) = h_2(f(v)\boldsymbol{e}, v)$ for $v \in V$. Then $h_1(u, v) = h_2(f(v)u, v)$ for $u \in \boldsymbol{R}^n$, $v \in V$, because h_1 and h_2 are $\boldsymbol{SL}(n, \boldsymbol{R})$ equivariant.
q.e.d.

REMARK 3.2. Let M be a real analytic paracompact manifold. Then M admits a real analytic Riemannian metric, because M is real analytically isomorphic to a real analytic closed submanifold of $\boldsymbol{R}^N$ (cf. Grauert [3, Theorem 3]). Suppose that M admits a real analytic action of a compact Lie group H. Then M admits a real analytic H invariant Riemannian metric, by averaging a given real analytic Riemannian metric. In particular, each connected component of $F(H, M)$ is a real analytic closed submanifold of M.

LEMMA 3.3. *Suppose* $n \geqq 3$. *Let* $\phi: \boldsymbol{SL}(n, \boldsymbol{R}) \times M \to M$ *be a real analytic* $\boldsymbol{SL}(n, \boldsymbol{R})$ *action on a connected paracompact m-manifold. Suppose that the restricted* $\boldsymbol{SO}(n)$ *action of* ϕ *has just two orbit types* $\boldsymbol{SO}(n)/\boldsymbol{SO}(n-1)$ *and* $\boldsymbol{SO}(n)/\boldsymbol{SO}(n)$. *Then*

(a) *each connected component of* $F(\boldsymbol{SO}(n), M)$ *is* $(m-n)$*-dimensional,*

(b) $F(\boldsymbol{SO}(n-1), M)$ *is connected and* $(m-n+1)$*-dimensional,*

(c) $F(\boldsymbol{SO}(n-1), M)$ *coincides with either* $F(L(n), M)$ *or* $F(L^*(n), M)$.

Moreover, if $F(SO(n-1), M) = F(L(n), M)$, *then there is an equivariant decomposition*:

$$M - F = SL(n, R) \underset{NL(n)}{\times} F(L(n), M - F)\ ,$$

where $F = F(SL(n, R), M) = F(SO(n), M)$.

PROOF. It follows from the assumption that the isotropy representation at a point of $F(SO(n), M)$ is equivalent to $\rho_n \oplus$ trivial. Here ρ_n is the canonical representation of $SO(n)$. Hence (a) follows. Put $X = F(SO(n-1), M) - F(SO(n), M)$. There is an equivariant decomposition:

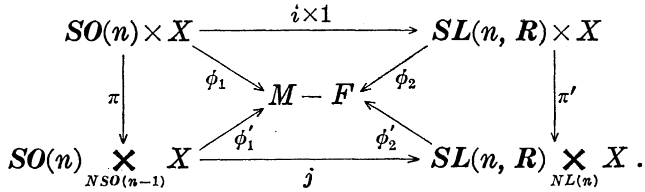
$$M - F = SO(n)/SO(n-1) \underset{W}{\times} X\ ,$$

where $W = NSO(n-1)/SO(n-1) = Z_2$. In particular, $\dim X = m - n + 1$. Let $\pi\colon M \to M^* = SO(n)\backslash M$ be the canonical projection to the orbit space M^*. Then $M^* = \pi(F(SO(n-1), M))$ by the assumption. Put g_0 the diagonal matrix with diagonal components $-1, -1, 1, \cdots, 1$. Define a map $T\colon F(SO(n-1), M) \to F(SO(n-1), M)$ by $T(x) = \phi(g_0, x)$. Then T is an involution on $F(SO(n-1), M)$ and the fixed point set agrees with $F(SO(n), M)$. Then orbit space $T\backslash F(SO(n-1), M)$ is naturally homeomorphic to a connected space M^*. Let Y be a connected component of $F(SO(n-1), M)$ such that $Y \cap F(SO(n), M)$ is non-empty. Then $TY = Y$ and the orbit space $T\backslash Y$ is a connected component of $T\backslash F(SO(n-1), M)$. Hence $Y = F(SO(n-1), M)$ is connected. Hence (b) follows. By the assumption, Lemma 2.3 and Proposition 2.5, we have the following:

$$F(SO(n-1), M) = F(L(n), M) \cup F(L^*(n), M)\ ,$$
$$F(SO(n), M) = F(L(n), M) \cap F(L^*(n), M) = F(SL(n, R), M)\ .$$

It follows from the above argument that X has at most two connected components. If X is connected, then it is easy to see that $F(SO(n-1), M)$ coincides with either $F(L(n), M)$ or $F(L^*(n), M)$. Suppose that X has two connected components X_1 and X_2. Then $TX_1 = X_2$. Since $g_0 L(n) g_0^{-1} = L(n)$ and $g_0 L^*(n) g_0^{-1} = L^*(n)$, we see that if X_1 is contained in $F(L(n), M)$ (resp. $F(L^*(n), M)$), then X_2 is also contained in $F(L(n), M)$ (resp. $F(L^*(n), M)$). Hence (c) follows.

Suppose now that $F(SO(n-1), M) = F(L(n), M)$. Consider the following commutative diagram:



Here $X = F(SO(n-1), M) - F(SO(n), M) = F(L(n), M) - F(SL(n, R), M)$; π, π' are the natural projections; ϕ_1, ϕ_2 are the restrictions of the map ϕ; ϕ_1', ϕ_2' are the induced maps. Then ϕ_1' is an $SO(n)$ equivariant real analytic isomorphism. Since $SL(n, R) = SO(n) \cdot N(n)$, it is easy to see that the map j is a surjection. Here the group $N(n)$ is defined in Section 2. It follows that ϕ_2' is an $SL(n, R)$ equivariant real analytic isomorphism. q.e.d.

We require the following result due to Guillemin and Sternberg [4]:

LEMMA 3.4. *Let* $\mathfrak{g}$ *be a real semi-simple Lie algebra and let* $\rho: \mathfrak{g} \to L(M)$ *be a Lie algebra homomorphism of* $\mathfrak{g}$ *into a Lie algebra of real analytic vector fields on a real analytic m-manifold M. Let p be a point at which the vector fields in the image* $\rho(\mathfrak{g})$ *have common zero. Then there exists an analytic system of coordinates* $(U; u_1, \cdots, u_m)$, *with origin at p, in which all of the vector fields in* $\rho(\mathfrak{g})$ *are linear. Namely, there exists* $a_{ij} \in \mathfrak{g}^* = \mathrm{Hom}_R(\mathfrak{g}, R)$ *such that*

$$\rho(X)_q = -\sum_{i,j} a_{ij}(X) u_j(q)(\partial/\partial u_i) \quad for \quad X \in \mathfrak{g}\,, \quad q \in U\,.$$

REMARK 3.5. The correspondence $X \to (a_{ij}(X))$ defines a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(m, R)$. Let $P = (p_{ij}) \in GL(m, R)$. Define an analytic system of coordinates $(U; v_1, \cdots, v_m)$ by $v_i(q) = \sum_{j=1}^m p_{ij} u_j(q)$, $q \in U$. Then $\rho(X)_q = -\sum_{i,j} b_{ij}(X) v_j(q)(\partial/\partial v_i)$ for $X \in \mathfrak{g}$, $q \in U$. Here $(b_{ij}(X)) = P(a_{ij}(X))P^{-1}$.

LEMMA 3.6. *Suppose* $n \geqq 3$. *Let* $\phi: SL(n, R) \times M \to M$ *be a real analytic action on m-manifold. Suppose that the restricted* $SO(n)$ *action of* ϕ *has just two orbit types* $SO(n)/SO(n-1)$ *and* $SO(n)/SO(n)$. *Suppose* $F(SO(n-1), M) = F(L(n), M)$. *Then for each* $p \in F(SL(n, R), M)$ *there exists an analytic system of coordinates* $(U; u_1, \cdots, u_m)$, *with origin at p, such that*

$$\phi^+((x_{ij}))_q = -\sum_{i,j=1}^{n} x_{ij} u_j(q)(\partial/\partial u_i) \quad for \quad (x_{ij}) \in \mathfrak{sl}(n, R)\,, \quad q \in U\,.$$

PROOF. By Lemma 3.4, there exists an analytic system of coordinates $(U; v_1, \cdots, v_m)$ with origin at p and there exists $a_{ij} \in \mathfrak{sl}(n, R)^*$ such that $\phi^+(X)_q = -\sum_{i,j=1}^m a_{ij}(X) v_j(q)(\partial/\partial v_i)$ for $X \in \mathfrak{sl}(n, R)$, $q \in U$. Then $F(SO(n), M) \cap U = \{q \in U: \phi^+(X)_q = 0 \text{ for } X \in \mathfrak{so}(n)\} = \{q \in U: \sum_{j=1}^m a_{ij}(X) v_j(q) = 0 \text{ for } X \in \mathfrak{so}(n), 1 \leqq i \leqq m\}$. Since $\dim F(SO(n), M) = m - n$ by Lemma 3.3 (a), we can assume $F(SO(n), M) \cap U = \{q \in U: v_1(q) = \cdots = v_n(q) = 0\}$ by Remark 3.5. Then $a_{ij}(X) = 0$ for $n+1 \leqq j \leqq m$, $1 \leqq i \leqq m$ for each $X \in \mathfrak{sl}(n, R)$, because $F(SO(n), M) = F(SL(n, R), M)$ by Lemma 3.3. Therefore the representation $X \to (a_{ij}(X))$ of $\mathfrak{sl}(n, \boldsymbol{R})$ has $(m-n)$-dimensional trivial subspace. It is well known that any real representation of $\mathfrak{sl}(n, \boldsymbol{R})$ is completely reducible (cf. Humphreys [6, Section 6]). Hence the representation $X \to (a_{ij}(X))$ is a direct sum of an n-dimensional representation and $(m-n)$-dimensional trivial representation. It is known that an n-dimensional real representation of $\mathfrak{sl}(n, \boldsymbol{R})$ is equivalent to the canonical representation $X \to X$ or the contragredient representation $X \to -{}^tX$. By Remark 3.5, there exists an analytic system of coordinates $(U; u_1, \cdots, u_m)$, with origin at p, such that

(a) $$\phi^+((x_{ij}))_q = -\sum_{i,j=1}^{n} x_{ij}u_j(q)(\partial/\partial u_i)$$

or

(b) $$\phi^+((x_{ij}))_q = \sum_{i,j=1}^{n} x_{ji}u_j(q)(\partial/\partial u_i)$$

for $(x_{ij}) \in \mathfrak{sl}(n, \boldsymbol{R})$, $q \in U$. The case (b) contradicts the assumption $F(\boldsymbol{SO}(n-1), M) = F(L(n), M)$. q.e.d.

THEOREM 3.7. *Suppose $n \geqq 3$. Let $\phi: \boldsymbol{SL}(n, \boldsymbol{R}) \times M \to M$ be a real analytic action on a connected paracompact m-manifold. Suppose that the restricted $\boldsymbol{SO}(n)$ action of ϕ has just two orbit types $\boldsymbol{SO}(n)/\boldsymbol{SO}(n-1)$ and $\boldsymbol{SO}(n)/\boldsymbol{SO}(n)$. Suppose $F(\boldsymbol{SO}(n-1), M) = F(L(n), M)$. Put $F = F(\boldsymbol{SL}(n, \boldsymbol{R}), M)$. Then* (i) *there exists a real analytic left principal $\boldsymbol{R}^\times$ bundle $p: E \to F$, and there exists a real analytic isomorphism h of $\boldsymbol{R}^n \times_{\boldsymbol{R}^\times} E$ onto an open set of M such that*

(a) $h(0, u) = p(u)$ *for* $u \in E$,

(b) $h(gx, u) = \phi(g, h(x, u))$ *for* $g \in \boldsymbol{SL}(n, \boldsymbol{R})$, $x \in \boldsymbol{R}^n$, $u \in E$.

Moreover, (ii) *if there exists a real analytic left principal $\boldsymbol{R}^\times$ bundle $p': E' \to F$ and if there exists a real analytic isomorphism h' of $\boldsymbol{R}^n \times_{\boldsymbol{R}^\times} E'$ onto an open set of M such that*

(a′) $h'(0, u') = p'(u')$ *for* $u' \in E'$,

(b′) $h'(gx, u') = \phi(g, h'(x, u'))$ *for* $g \in \boldsymbol{SL}(n, \boldsymbol{R})$, $x \in \boldsymbol{R}^n$, $u' \in E'$,

then there exists a real analytic $\boldsymbol{R}^\times$ bundle isomorphism $f: E \to E'$ such that $h(x, u) = h'(x, f(u))$ for $x \in \boldsymbol{R}^n$, $u \in E$.

PROOF. From Lemma 3.1 and Lemma 3.6, there exists an open covering $\{V_\alpha, \alpha \in A\}$ of F and there exists a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ equivariant isomorphism h_α of $\boldsymbol{R}^n \times V_\alpha$ onto an open set of M for each $\alpha \in A$, such that $h_\alpha(0, v) = v$ for $v \in V_\alpha$. Put $U = \bigcup_{\alpha \in A} h_\alpha(\boldsymbol{R}^n \times V_\alpha)$. Then U is an $\boldsymbol{SL}(n, \boldsymbol{R})$ invariant open neighborhood of F in M. Put $E = F(L(n),$

$U-F$), and define k_α: $\boldsymbol{R}^\times \times V_\alpha \to E$ by $k_\alpha(t, v) = h_\alpha(te, v)$ for $t \in \boldsymbol{R}^\times$, $v \in V_\alpha$. Here $\boldsymbol{e} = (1, 0, \cdots, 0) \in \boldsymbol{R}^n$. The group $NL(n)/L(n) = \boldsymbol{R}^\times$ acts naturally on E, and the map k_α is $\boldsymbol{R}^\times$ equivariant. It follows from Lemma 3.1 that $E = \bigcup_{\alpha \in A} k_\alpha(\boldsymbol{R}^\times \times V_\alpha)$ and $k_\alpha(\boldsymbol{R}^\times \times V_\alpha) \cap k_\beta(\boldsymbol{R}^\times \times V_\beta) = k_\alpha(\boldsymbol{R}^\times \times (V_\alpha \cap V_\beta))$ for $\alpha, \beta \in A$, and there exists a unique real analytic function $g_{\alpha\beta}$: $V_\alpha \cap V_\beta \to \boldsymbol{R}^\times$ such that $k_\beta(t, v) = k_\alpha(g_{\alpha\beta}(v)t, v)$ for $t \in \boldsymbol{R}^\times$, $v \in V_\alpha \cap V_\beta$.

Define p: $E \to F$ by $pk_\alpha(t, v) = v$ for $t \in \boldsymbol{R}^\times$, $v \in V_\alpha$. This is a desired real analytic left principal $\boldsymbol{R}^\times$ bundle. We can define a map h: $\boldsymbol{R}^n \times_{\boldsymbol{R}^\times} E \to M$ by $h(x, k_\alpha(t, v)) = h_\alpha(tx, v)$ for $x \in \boldsymbol{R}^n$, $t \in \boldsymbol{R}^\times$, $v \in V_\alpha$. The map h is a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ equivariant isomorphism onto U. This is a desired map. Suppose finally that there exists a real analytic left principal $\boldsymbol{R}^\times$ bundle p': $E' \to F$ and there exists a real analytic isomorphism h' of $\boldsymbol{R}^n \times_{\boldsymbol{R}^\times} E'$ onto an open set of M, satisfying the conditions (a′), (b′). It is easy to see from Lemma 3.1 (ii) that image $h = U =$ image h'. It follows that there exists a unique $\boldsymbol{SL}(n, \boldsymbol{R})$ equivariant real analytic isomorphism

$$\bar{f}\colon \boldsymbol{R}^n \underset{\boldsymbol{R}^\times}{\times} E \to \boldsymbol{R}^n \underset{\boldsymbol{R}^\times}{\times} E'$$

such that $h(x, u) = h'(\bar{f}(x, u))$ for $x \in \boldsymbol{R}^n$, $u \in E$. Considering the fixed point sets of the restricted $L(n)$ action, we have a real analytic $\boldsymbol{R}^\times$ equivariant isomorphism f: $E \to E'$ such that $\bar{f}(te, u) = (te, f(u))$ for $t \in \boldsymbol{R}$, $u \in E$. Then f: $E \to E'$ is a bundle isomorphism of principal $\boldsymbol{R}^\times$ bundles, because $p(u) = h(0, u) = h'(\bar{f}(0, u)) = h'(0, f(u)) = p'(f(u))$ for $u \in E$.

q.e.d.

4. Smooth $\boldsymbol{SO}(n)$ actions on homotopy spheres. First we state the following two lemmas of which proofs are given in Section 7.

LEMMA 4.1. *Suppose $n \geqq 5$. Let G be a closed connected proper subgroup of $\boldsymbol{O}(n)$ such that* $\dim \boldsymbol{O}(n)/G \leqq 2n-2$. *Then it is one of the following listed in Table* 1 *up to an inner automorphism of $\boldsymbol{O}(n)$. Here*

$$\rho_k\colon \boldsymbol{SO}(k) \to \boldsymbol{O}(k)\ , \quad \mu_k\colon \boldsymbol{U}(k) \to \boldsymbol{O}(2k)\ , \quad \mu_k^0\colon \boldsymbol{SU}(k) \to \boldsymbol{O}(2k)$$

are the canonical inclusions, θ^k is the trivial representation of degree k, and Δ_7, ω, β are irreducible representations, respectively.

LEMMA 4.2. *Suppose $5 \leqq n \leqq k \leqq 2n-2$. Then an orthogonal non-trivial representation of $\boldsymbol{SO}(n)$ of degree k is equivalent to $\rho_n \oplus \theta^{k-n}$ by an inner automorphism of $\boldsymbol{O}(k)$.*

Now we shall prove the following result.

LEMMA 4.3. *Suppose $5 \leqq n \leqq k \leqq 2n-2$. Let Σ^k be a homotopy k-sphere with a non-trivial smooth $\boldsymbol{SO}(n)$ action. Then the principal*

TABLE 1

n	G	$i\colon G\to \boldsymbol{O}(n)$	$\dim \boldsymbol{O}(n)/G$
n	$\boldsymbol{SO}(n-1)$	$\rho_{n-1}\oplus\theta^1$	$n-1$
n	$\boldsymbol{SO}(n-2)$	$\rho_{n-2}\oplus\theta^2$	$2n-3$
n	$\boldsymbol{SO}(n-2)\times\boldsymbol{SO}(2)$	$\rho_{n-2}\oplus\rho_2$	$2n-4$
9	$\boldsymbol{Spin}(7)$	$\Delta_7\oplus\theta^1$	$15=2n-3$
8	$\boldsymbol{Spin}(7)$	Δ_7	$7=n-1$
8	$\boldsymbol{G}_2$	$\omega\oplus\theta^1$	$14=2n-2$
8	$\boldsymbol{U}(4)$	μ_4	$12=2n-4$
8	$\boldsymbol{SU}(4)$	$\mu_4{}^0$	$13=2n-3$
7	$\boldsymbol{G}_2$	ω	$7=n$
7	$\boldsymbol{U}(3)$	$\mu_3\oplus\theta^1$	$12=2n-2$
7	$\boldsymbol{SO}(3)\times\boldsymbol{SO}(4)$	$\rho_3\oplus\rho_4$	$12=2n-2$
6	$\boldsymbol{SO}(3)\times\boldsymbol{SO}(3)$	$\rho_3\oplus\rho_3$	$9=2n-3$
6	$\boldsymbol{U}(3)$	μ_3	$6=n$
6	$\boldsymbol{SU}(3)$	$\mu_3{}^0$	$7=2n-5$
6	$\boldsymbol{U}(2)\times\boldsymbol{U}(1)$	$\mu_2\oplus\mu_1$	$10=2n-2$
5	$\boldsymbol{U}(2)$	$\mu_2\oplus\theta^1$	$6=2n-4$
5	$\boldsymbol{SU}(2)$	$\mu_2{}^0\oplus\theta^1$	$7=2n-3$
5	$\boldsymbol{U}(1)\times\boldsymbol{U}(1)$	$\mu_1\oplus\mu_1\oplus\theta^1$	$8=2n-2$
5	$\boldsymbol{SO}(3)$	β	$7=2n-3$

isotropy type is $(\boldsymbol{SO}(n-1))$ *and the fixed point set* $F(\boldsymbol{SO}(n), \Sigma^k)$ *is non-empty.*

Let us start with some observations. In the following, let M be a closed connected k-dimensional manifold with a non-trivial smooth $\boldsymbol{SO}(n)$ action, let (H) be the principal isotropy type, and suppose $5 \leqq n \leqq k \leqq 2n-2$. Denote by H^0 the identity component of H.

OBSERVATION 4.4. *If* $F(\boldsymbol{SO}(n), M)$ *is non-empty, then* $(H) = (\boldsymbol{SO}(n-1))$.

This is a direct consequence of Lemma 4.2, by considering the isotropy representation at a fixed point.

OBSERVATION 4.5. *Suppose that* M *is* 2*-connected and the* $\boldsymbol{SO}(n)$ *action is transitive. Then* $M = \boldsymbol{SO}(n)/\boldsymbol{SO}(n-2)$ *or* $M = \boldsymbol{SO}(5)/\beta\boldsymbol{SO}(3)$.

This is a direct consequence of Lemma 4.1.

OBSERVATION 4.6. *Suppose that the principal isotropy type* (H) *is one of the following listed in Table* 2. *Then* M *is not* 3*-connected.*

PROOF. Since $F(\boldsymbol{SO}(n), M)$ is empty by Observation 4.4 and H^0 is a proper maximal connected subgroup of $\boldsymbol{SO}(n)$ by Lemma 4.1, there is an equivariant decomposition: $M = \boldsymbol{SO}(n)/H^0 \times_W F(H^0, M)$, where $W = N(H^0)/H^0$ is a finite group. If M is simply connected, then $M = \boldsymbol{SO}(n)/$

$H^0 \times F$ and it is not 3-connected, where F is a connected component of $F(H^0, M)$.

TABLE 2

n	H^0	$\pi_i(SO(n)/H^0)$
n	$SO(n-2)\times SO(2)$	$\pi_2=Z$
8	$Spin(7)$	$\pi_1=Z_2$
8	$U(4)$	$\pi_2=Z$
7	G_2	$\pi_1=Z_2$
7	$SO(3)\times SO(4)$	$\pi_2=Z_2$
6	$SO(3)\times SO(3)$	$\pi_2=Z_2$
6	$U(3)$	$\pi_2=Z$
5	$\beta SO(3)$	$\pi_3\neq 0$

OBSERVATION 4.7. *Suppose that* (H) *is one of the following*:

$$H^0 = SO(n-2)\times SO(2);\ \ U(4),\ n = 8;\ \ U(3),\ n = 6;\ \ U(2),\ n = 5\ .$$

Then M *is not stably parallelizable.*

PROOF. If M is stably parallelizable, then the principal orbit $SO(n)/H$ is stably parallelizable; hence $SO(n)/H^0$ is also stable parallelizable.

OBSERVATION 4.8. *Suppose that* $\dim M=2n-2$, $\pi_1(M)=\{1\}$, $\chi(M)\neq 0$, *and* H^0 *is conjugate to* $SO(n-2)$. *Then* $\chi(M) \geqq 4$. *Here* $\chi(M)$ *is the Euler characteristic of* M.

PROOF. The principal orbit $SO(n)/H$ is of codimension one. Since $\pi_1(M) = \{1\}$, there are just two singular orbits (cf. Uchida [11, Lemma 1.2.1]). By Observation 4.4, $F(SO(n), M)$ is empty. Hence the following are the only possibilities of the singular orbit types:

$$SO(n)/SO(n-1) = S^{n-1}\ ,\quad SO(n)/S(O(n-1)\times O(1)) = P_{n-1}(R)\ ,$$
$$SO(n)/SO(n-2)\times SO(2) = Q_{n-2}\ ,\quad SO(n)/S(O(n-2)\times O(2)) = Q_{n-2}/Z_2\ .$$

By the general position theorem and the assumption $\pi_1(M) = \{1\}$, it is easy to see that the pair of singular orbits is none of the following: $(S^{n-1}, P_{n-1}(R))$, $(S^{n-1}, Q_{n-2}/Z_2)$, $(P_{n-1}(R), P_{n-1}(R))$, $(P_{n-1}(R), Q_{n-2}/Z_2)$. Since $\chi(M) = \chi$ (singular orbits), we have the desired result.

OBSERVATION 4.9. *Suppose that* $\dim M = 2n-2$ *and* (H) *is one of the following*:

$$H^0 = Spin(7),\ n = 9;\ \ SU(4),\ n = 8;\ \ SU(2),\ n = 5\ .$$

Then $\pi_1(M) \neq \{1\}$ *or* $\chi(M) \geqq 4$.

This is similarly proved as Observation 4.8.

OBSERVATION 4.10. *Suppose that* $n = 6$ *and* H^0 *is conjugate to* $\boldsymbol{SU}(3)$. *Then* M *is not* 2-*connected*.

PROOF. By Observation 4.4, $F(\boldsymbol{SO}(6), M)$ is empty. Hence the identity component of an isotropy group is conjugate to $\boldsymbol{SU}(3)$ or $\boldsymbol{U}(3)$ for each point of M. It follows that there is an equivariant decomposition: $M = \boldsymbol{SO}(6)/\boldsymbol{SU}(3) \times_W F(\boldsymbol{SU}(3), M)$, where $W = N\boldsymbol{SU}(3)/\boldsymbol{SU}(3) = \boldsymbol{U}(1)$. Then it is seen that M is not 2-connected by the following homotopy exact sequence:

$$\pi_2(M) \to \pi_1(W) \to \pi_1(\boldsymbol{SO}(6)/\boldsymbol{SU}(3)) \times \pi_1(F(\boldsymbol{SU}(3), M)) \to \pi_1(M) .$$

PROOF OF LEMMA 4.3. It is sufficient to prove that the set $F(\boldsymbol{SO}(n), \Sigma^k)$ is non-empty by Observation 4.4. It is well known that every homotopy sphere is stably parallelizable (cf. Kervaire and Milnor [7, Theorem 3.1]). Let (H) be the principal isotropy type of a non-trivial smooth $\boldsymbol{SO}(n)$ action on a homotopy k-sphere Σ^k. Then it follows that H^0 is conjugate to $\boldsymbol{SO}(n-1)$ by Lemma 4.1 and the above Observations. Suppose that $F(\boldsymbol{SO}(n), \Sigma^k)$ is empty. Then there is an equivariant decomposition: $\Sigma^k = \boldsymbol{SO}(n)/\boldsymbol{SO}(n-1) \times_W F(\boldsymbol{SO}(n-1), \Sigma^k)$, where $W = N\boldsymbol{SO}(n-1)/\boldsymbol{SO}(n-1) = \boldsymbol{Z}_2$. But this is impossible for $k \geqq n$. q.e.d.

THEOREM 4.11. *Suppose* $5 \leqq n \leqq k \leqq 2n-2$. *Let* Σ^k *be a homotopy* k-*sphere with a non-trivial smooth* $\boldsymbol{SO}(n)$ *action. Then there is an equivariant decomposition*: $\Sigma^k = \partial(\boldsymbol{D}^n \times Y)$ *as a smooth* $\boldsymbol{SO}(n)$ *manifold. Here* Y *is a compact contractible* $(k-n+1)$-*manifold with trivial* $\boldsymbol{SO}(n)$ *action, and* $\boldsymbol{D}^n$ *is the standard* n-*disk with the canonical* $\boldsymbol{SO}(n)$ *action*.

PROOF. Put $F = F(\boldsymbol{SO}(n), \Sigma^k)$. By Lemma 4.3, F is non-empty. It follows from Lemma 4.2 that each connected component of F is of $(k-n)$-dimension. Let U be a closed $\boldsymbol{SO}(n)$ invariant tubular neighborhood of F in Σ^k. Then U is regarded as an n-disk bundle over F with a smooth $\boldsymbol{SO}(n)$ action as bundle isomorphisms. It follows that there is an equivariant decomposition: $U = \boldsymbol{D}^n \times_W F(\boldsymbol{SO}(n-1), \partial U)$, where $W = N\boldsymbol{SO}(n-1)/\boldsymbol{SO}(n-1) = \boldsymbol{Z}_2$. Put $E = \Sigma^k - \operatorname{int} U$. Then there is an equivariant decomposition: $E = \boldsymbol{SO}(n)/\boldsymbol{SO}(n-1) \times_W F(\boldsymbol{SO}(n-1), E)$. Notice that $F(\boldsymbol{SO}(n-1), \partial U) = \partial F(\boldsymbol{SO}(n-1), E)$. It is easy to see that $\pi_1(E) = \{1\}$ by the general position theorem. Hence $F(\boldsymbol{SO}(n-1), E)$ has just two connected components. Let Y be a connected component of $F(\boldsymbol{SO}(n-1), E)$. Then Y is a compact simply connected $(k-n+1)$-manifold with non-empty boundary, and there is an equivariant decomposition: $\Sigma^k = U \cup E = \partial(\boldsymbol{D}^n \times Y)$.

It remains to prove that Y is contractible. By the Poincaré Lefschetz duality, $H_i(\boldsymbol{D}^n \times Y, \Sigma^k; \boldsymbol{Z}) = H^{k+1-i}(\boldsymbol{D}^n \times Y; \boldsymbol{Z}) = \{0\}$ for each $i < n$. Consider the homology exact sequence: $H_{i+1}(\boldsymbol{D}^n \times Y, \Sigma^k; \boldsymbol{Z}) \to H_i(\Sigma^k; \boldsymbol{Z}) \to H_i(\boldsymbol{D}^n \times Y; \boldsymbol{Z}) \to H_i(\boldsymbol{D}^n \times Y, \Sigma^k; \boldsymbol{Z})$. Then $H_i(Y; \boldsymbol{Z}) = \{0\}$ for $0 < i \leqq n-2$. On the other hand, Y is a compact simply connected manifold with non-empty boundary, and $\dim Y \leqq n-1$ by the assumption $k \leqq 2n-2$. It follows that Y is contractible. q.e.d.

REMARK. Theorem 4.11 for $n \geqq 9$ has been proved already by Hsiang [5, Theorem III].

5. **Decomposition and classification.** Suppose $5 \leqq n \leqq m \leqq 2n-2$. Let ϕ be a non-trivial real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ action on S^m. Consider the restricted $\boldsymbol{SO}(n)$ action of ϕ. By Theorem 4.11, there exists an equivariant decomposition: $S^m = \partial(\boldsymbol{D}^n \times Y)$ as a smooth $\boldsymbol{SO}(n)$ manifold. In particular, the $\boldsymbol{SO}(n)$ action has just two orbit types $\boldsymbol{SO}(n)/\boldsymbol{SO}(n-1)$ and $\boldsymbol{SO}(n)/\boldsymbol{SO}(n)$. Then, by Lemma 3.3, $F(\boldsymbol{SO}(n-1), S^m)$ coincides with either $F(L(n), S^m)$ or $F(L^*(n), S^m)$. We shall show first the following decomposition theorem.

THEOREM 5.1. *Suppose* $5 \leqq n \leqq m \leqq 2n-2$. *Let* ϕ *be a non-trivial real analytic* $\boldsymbol{SL}(n, \boldsymbol{R})$ *action on* S^m. *Suppose*

$$F(\boldsymbol{SO}(n-1), S^m) = F(L(n), S^m) .$$

Then, (i) $\Sigma = F(L(n), S^m)$ *is a real analytic* $(m-n+1)$*-dimensional closed submanifold of* S^m *which is homotopy equivalent to a sphere, and* $\boldsymbol{R}^\times = NL(n)/L(n)$ *acts naturally on* Σ, (ii) $F = F(\boldsymbol{SL}(n, \boldsymbol{R}), S^m)$ *is a real analytic* $(m-n)$*-dimensional closed submanifold of* Σ, *and there exists a real analytic* $\boldsymbol{R}^\times$ *equivariant isomorphism* j *of* $\boldsymbol{R} \times F$ *onto an open set of* Σ *such that* $j(0, x) = x$ *for* $x \in F$, (iii) *there exists an equivariant decomposition*:

$$S^m = \boldsymbol{R}^n \times F \bigcup_f (\boldsymbol{R}^n - 0) \times_{\boldsymbol{R}^\times} (\Sigma - F)$$

as a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ *manifold, where* $\boldsymbol{SL}(n, \boldsymbol{R})$ *acts naturally on* $\boldsymbol{R}^n$, $\boldsymbol{R}^\times$ *acts on* $\boldsymbol{R}^n - 0$ *by the scalar multiplication, and* f *is an equivariant isomorphism of* $(\boldsymbol{R}^n - 0) \times F$ *onto an open set of* $(\boldsymbol{R}^n - 0) \times_{\boldsymbol{R}^\times} (\Sigma - F)$ *defined by* $f(u, x) = (u, j(1, x))$ *for* $u \in \boldsymbol{R}^n - 0$, $x \in F$.

PROOF. Consider the restricted $\boldsymbol{SO}(n)$ action of ϕ. By Theorem 4.11, there exists an equivariant decomposition: $S^m = \partial(\boldsymbol{D}^n \times Y)$ as a smooth $\boldsymbol{SO}(n)$ manifold. Here Y is a compact contractible smooth $(m-n+1)$-manifold. Then $\Sigma = F(\boldsymbol{SO}(n-1), S^m)$ is a real analytic $(m-n+1)$-dimensional closed submanifold of S^m which is C^∞ diffeomorphic to a double of Y; hence Σ is a homotopy sphere. By Lemma 3.3, $F = F(SO(n), S^m)$ is a real analytic $(m-n)$-dimensional closed submanifold of S^m which is C^∞ diffeomorphic to ∂Y; hence F is homology equivalent to a sphere. Moreover, there exists an equivariant decomposition:

$$S^m - F = \boldsymbol{SL}(n, \boldsymbol{R})/L(n) \underset{NL(n)/L(n)}{\times} (\Sigma - F) = (\boldsymbol{R}^n - 0) \underset{\boldsymbol{R}^\times}{\times} (\Sigma - F)$$

as a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ manifold. By Theorem 3.7, there exists a real analytic left principal $\boldsymbol{R}^\times$ bundle $p: E \to F$ and there exists a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ equivariant isomorphism h of $\boldsymbol{R}^n \times_{\boldsymbol{R}^\times} E$ onto an open set of S^m such that $h(0, u) = p(u)$ for $u \in E$. It is easy to see that the bundle $p: E \to F$ is trivial as a C^∞ bundle by the decomposition $S^m = \partial(\boldsymbol{D}^n \times Y)$.

To show that E is trivial as a real analytic $\boldsymbol{R}^\times$ bundle, we need the following.

LEMMA 5.2. *Let $p: V \to X$ be a real analytic vector bundle over a paracompact real analytic manifold X. Then the bundle V admits a real analytic Riemannian metric.*

PROOF. Let $i: X \to V$ be the zero section. Then it follows from a calculation of transition functions that there is an isomorphism $i^*\tau(V) = V \oplus \tau(X)$ as real analytic vector bundles. Here $\tau(\ )$ denotes the tangent bundle. Since V is a paracompact real analytic manifold, there exists a real analytic embedding $f: V \to \boldsymbol{R}^N$ such that $f(V)$ is a closed real analytic submanifold of $\boldsymbol{R}^N$ (cf. Grauert [3]). It follows that there is an isomorphism $\tau(V) \oplus \nu = \boldsymbol{R}^N \times V$ as real analytic vector bundles. Here ν denotes the normal bundle. Therefore there is an isomorphism $V \oplus \tau(X) \oplus i^*\nu = \boldsymbol{R}^N \times X$ as real analytic vector bundles. The product bundle $\boldsymbol{R}^N \times X$ admits canonically a real analytic Riemannian metric; hence its real analytic subbundle V admits a real analytic Riemannian metric.

q.e.d.

We now return to the proof of Theorem 5.1. Let $\boldsymbol{R} \times_{\boldsymbol{R}^\times} E \to F$ be the line bundle associated to the principal bundle $p: E \to F$. Then it has a real analytic Riemannian metric; hence the associated sphere bundle is a real analytic double covering over F. Since $p: E \to F$ is trivial as a C^∞ bundle, the sphere bundle is trivial as a real analytic bundle, and hence the principal bundle $p: E \to F$ has a real analytic cross-section. Therefore E is trivial as a real analytic $\boldsymbol{R}^\times$ bundle. It follows that there exists a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ equivariant isomorphism $h: \boldsymbol{R}^n \times F \to S^m$ onto an open set of S^m such that $h(0, x) = x$ for $x \in F$.

Consider the fixed point sets of restricted $L(n)$ actions. We have a real analytic $\boldsymbol{R}^{\times}$ equivariant isomorphism $j: \boldsymbol{R}\times F \to \Sigma$ onto an open set of $\Sigma = F(L(n), S^m)$, defined by $j(t, x) = h(te, x)$ for $t \in \boldsymbol{R}, x \in F$. Here $\boldsymbol{e} = (1, 0, \cdots, 0) \in \boldsymbol{R}^n$, and $\boldsymbol{R}^{\times}$ acts canonically on Σ through the identification $\boldsymbol{R}^{\times} = NL(n)/L(n)$. It is easy to see that there exists an equivariant decomposition:

$$S^m \mathrel{\dot{=}} \boldsymbol{R}^n \times F \underset{f}{\cup} (\boldsymbol{R}^n - 0) \underset{\boldsymbol{R}^{\times}}{\times} (\Sigma - F)$$

as a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ manifold. Here f is an equivariant isomorphism of $(\boldsymbol{R}^n - 0)\times F$ onto an open set of $(\boldsymbol{R}^n - 0)\times_{\boldsymbol{R}^{\times}} (\Sigma - F)$ defined by $f(u, x) = (u, j(1, x))$ for $u \in \boldsymbol{R}^n - 0, x \in F$. This completes the proof of Theorem 5.1.

REMARK. By this theorem, the action ϕ on S^m is completely determined up to an equivariant isomorphism by $\Sigma = F(L(n), S^m)$ with $\boldsymbol{R}^{\times}$ action and an equivariant map $j: \boldsymbol{R}\times F \to \Sigma$.

To state a classification theorem, we introduce the following notions. Let G be a Lie group, and let $\phi_i: G\times M_i \to M_i$ be a real analytic G action for $i = 1, 2$. We say that ϕ_1 is weakly C^r equivariant to ϕ_2 if there exists an automorphism h of G and there exists a C^r diffeomorphism $f: M_1 \to M_2$ such that the following diagram is commutative:

$$\text{(5-a)} \qquad \begin{array}{ccc} G\times M_1 & \xrightarrow{\phi_1} & M_1 \\ \big\downarrow h\times f & & \big\downarrow f \\ G\times M_2 & \xrightarrow{\phi_2} & M_2 \, . \end{array}$$

In particular, ϕ_1 is said to be C^r equivariant to ϕ_2 if the identity map of G can be chosen as the automorphism h.

Let h be an automorphism of G, and let $\phi: G\times M \to M$ be a real analytic G action. Define a new real analytic G action $h^{\#}\phi$ on M as follows: $(h^{\#}\phi)(g, x) = \phi(h(g), x)$ for $g \in G, x \in M$. Then the action $h^{\#}\phi$ is weakly C^{ω} equivariant to ϕ, because the following diagram is commutative:

$$\text{(5-b)} \qquad \begin{array}{ccc} G\times M & \xrightarrow{h^{\#}\phi} & M \\ \big\downarrow h\times \mathrm{id} & & \big\downarrow \mathrm{id} \\ G\times M & \xrightarrow{\phi} & M \, . \end{array}$$

Let I_g denote the inner automorphism of G defined by $I_g(g') = gg'g^{-1}$ for $g, g' \in G$. Then, for any real analytic G action ϕ on M, ϕ is C^{ω} equivariant to $I_g^{\#}\phi$, because the following diagram is commutative:

(5-c)
$$\begin{array}{ccc} G\times M & \xrightarrow{\phi} & M \\ \Big\downarrow{\scriptstyle \mathrm{id}\times f} & & \Big\downarrow{\scriptstyle f} \\ G\times M & \xrightarrow{I_g^{\#}\phi} & M \end{array},$$

where $f(x) = \phi(g, x)$ for $x \in M$.

THEOREM 5.3. *Suppose $5 \leqq n \leqq m \leqq 2n - 2$. Then there is a natural one-to-one correspondence between the weak C^r equivariance classes of non-trivial real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on the standard m-sphere and the C^r equivariance classes of real analytic $\boldsymbol{R}^\times$ actions on homotopy $(m - n + 1)$-spheres satisfying the condition* (P), *for each $r = 0, 1, \cdots, \infty, \omega$. The correspondence is given by the construction in Section* 1.

PROOF. Let $A_r(n, m)$ denote the weak C^r equivariance classes of non-trivial real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on the standard m-sphere, let $A'_r(n, m)$ denote the C^r equivariance classes of non-trivial real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on the standard m-sphere such that $F(\boldsymbol{SO}(n-1), S^m) = F(L(n), S^m)$, and let $B_r(k)$ denote the C^r equivariance classes of real analytic $\boldsymbol{R}^\times$ actions on homotopy k-spheres satisfying the condition (P) in Section 1.

Let $\psi: \boldsymbol{R}^\times \times \Sigma \to \Sigma$ be a real analytic $\boldsymbol{R}^\times$ action on a homotopy k-sphere Σ satisfying the condition (P). We constructed, in Section 1, a compact real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ manifold $M(\psi, j)$ such that the C^ω equivariance class of $M(\psi, j)$ does not depend on the choice of j, $F(\boldsymbol{SO}(n-1), M(\psi, j)) = F(L(n), M(\psi, j))$, and $M(\psi, j)$ is real analytically isomorphic to the standard $(n + k - 1)$-sphere for $n + k \geqq 6$. The correspondence $\psi \to M(\psi, j)$ defines a mapping $c_r: B_r(k) \to A'_r(n, n + k - 1)$ for $r = 0, 1, \cdots, \infty, \omega$ and each $n + k \geqq 6$. It follows from Theorem 5.1 that c_r is a bijection $(r = 0, 1, \cdots, \infty, \omega)$ if $n \geqq 5$ and $1 \leqq k \leqq n - 1$.

It remains to show that there is a natural one-to-one correspondence between $A'_r(n, m)$ and $A_r(n, m)$. Let ϕ be a real analytic non-trivial $\boldsymbol{SL}(n, \boldsymbol{R})$ action on S^m such that $F(\boldsymbol{SO}(n-1), S^m) = F(L(n), S^m)$. Then ϕ represents a class of $A'_r(n, m)$ and a class of $A_r(n, m)$. Hence there is a natural mapping $i_r: A'_r(n, m) \to A_r(n, m)$.

We shall show that i_r is a bijection $(r = 0, 1, \cdots, \infty, \omega)$ if $5 \leqq n \leqq m \leqq 2n - 2$. Let σ be the automorphism of $\boldsymbol{SL}(n, \boldsymbol{R})$ defined by $\sigma(X) = {}^tX^{-1}$ for $X \in \boldsymbol{SL}(n, \boldsymbol{R})$. Then it is seen that σ is an involution and $\sigma(L(n)) = L^*(n)$. Let ϕ be a real analytic non-trivial $\boldsymbol{SL}(n, \boldsymbol{R})$ action on S^m. Then, by Lemma 3.3 (c) we have that $F(\boldsymbol{SO}(n-1), S^m)$ coincides with $F(L(n), S^m)$ or $F(L^*(n), S^m)$. Since $\sigma(L(n)) = L^*(n)$, we see that if

$F(SO(n-1), S^m) = F(L^*(n), S^m)$ for ϕ, then $F(SO(n-1), S^m) = F(L(n), S^m)$ for the induced action $\sigma^\sharp\phi$. By the diagram (5-b), $\sigma^\sharp\phi$ is weakly C^ω equivariant to ϕ; hence the natural mapping i_r is surjective.

To show that i_r is injective, we consider the automorphism group of $SL(n, R)$. Let $Aut\, SL(n, R)$, $Inn\, SL(n, R)$ denote the automorphism group and the inner automorphism group of $SL(n, R)$, respectively. Define an automorphism γ of $SL(n, R)$ by $\gamma(X) = YXY^{-1}$ for $X \in SL(n, R)$, where Y is the diagonal matrix with diagonal elements $-1, 1, \cdots, 1$. Then it is known that σ and γ generate the quotient group $Out\, SL(n, R) = Aut\, SL(n, R)/Inn\, SL(n, R)$. In fact

$$Out\, SL(n, R) = \begin{cases} Z_2 & \text{for} \quad n\colon \text{odd} \geqq 3 \\ Z_2 \oplus Z_2 & \text{for} \quad n\colon \text{even} \geqq 4 \,, \end{cases}$$

and γ is an inner automorphism for n odd (cf. Murakami [9]).

Let ϕ, ϕ' be real analytic non-trivial $SL(n, R)$ actions on S^m. Suppose that ϕ' is weakly C^r equivariant to ϕ. Then by the diagrams (5-a), (5-b), (5-c) ϕ' is C^r equivariant to one of the following: ϕ, $\sigma^\sharp\phi$, $\gamma^\sharp\phi$, $\sigma^\sharp\gamma^\sharp\phi$. Notice that if $F(SO(n-1), S^m) = F(L(n), S^m)$ for ϕ, then $F(SO(n-1), S^m) = F(L(n), S^m)$ for $\gamma^\sharp\phi$, and $F(SO(n-1), S^m) = F(L^*(n), S^m)$ for $\sigma^\sharp\phi$, $\sigma^\sharp\gamma^\sharp\phi$. Therefore, if ϕ and ϕ' represent classes of $A'_r(n, m)$, respectively, and if ϕ' is weakly C^r equivariant to ϕ, then ϕ' is C^r equivariant to ϕ or $\gamma^\sharp\phi$. To show that i_r is injective, it suffices to prove $\gamma^\sharp\phi$ is C^ω equivariant to ϕ. Consider the real analytic $SL(n, R)$ manifold

$$M(\psi, j) = R^n \times F \bigcup_f (R^n - 0) \times_{R^\times} (\Sigma - F)$$

constructed in Section 1. Define a real analytic isomorphism $g\colon M(\psi, j) \to M(\psi, j)$ by

$$\begin{aligned} g(u, x) &= (Y\cdot u, x) \quad \text{for} \quad (u, x) \in R^n \times F \,, \\ g(v, y) &= (Y\cdot v, y) \quad \text{for} \quad (v, y) \in (R^n - 0) \times_{R^\times} (\Sigma - F) \,. \end{aligned}$$

Here the matrix Y is as before. Then the following diagram is commutative:

$$\begin{array}{ccc} SL(n, R) \times M(\psi, j) & \xrightarrow{\ \phi\ } & M(\psi, j) \\ \downarrow{\scriptstyle \gamma\times g} & & \downarrow{\scriptstyle g} \\ SL(n, R) \times M(\psi, j) & \xrightarrow{\ \phi\ } & M(\psi, j) \,, \end{array}$$

where ϕ is the natural $SL(n, R)$ action on $M(\psi, j)$. By the diagram (5-b), we have the following commutative diagram:

$$\begin{array}{ccc} SL(n, \boldsymbol{R})\times M(\psi, j) & \xrightarrow{\gamma^{\#}\phi} & M(\psi, j) \\ \Big\downarrow{\scriptstyle \gamma\times \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ SL(n, \boldsymbol{R})\times M(\psi, j) & \xrightarrow{\phi} & M(\psi, j) \end{array} .$$

Since $\gamma^2 = 1$, it follows that $\gamma^{\#}\phi$ is C^ω equivariant to ϕ; hence the mapping i_r is bijective. q.e.d.

6. $\boldsymbol{R}^\times$ actions on spheres. In the previous section, we showed that the classification of real analytic $SL(n, \boldsymbol{R})$ actions on the m-sphere can be reduced to that of real analytic $\boldsymbol{R}^\times$ actions on homotopy $(m-n+1)$-spheres satisfying the condition (P). So we study now $\boldsymbol{R}^\times$ actions on spheres.

Let S^k be the standard k-sphere in $\boldsymbol{R}^{k+1}$, $k \geqq 1$. Let T be an involution of S^k defined by $T(x_0, x_1, \cdots, x_k) = (-x_0, x_1, \cdots, x_k)$. Put

$$\xi^a = x_0(1 - x_0^2)a(x_0^2)(\partial/\partial x_0) - x_0^2 a(x_0^2) \sum_{i=1}^{k} x_i(\partial/\partial x_i) ,$$

where $a(t)$ is a real analytic function defined on an open neighborhood of the closed interval $[0, 1]$. It is easy to see that ξ^a is a real analytic tangent vector field on S^k such that $T_*\xi^a = \xi^a$. Let $\{\theta_t;\ t \in \boldsymbol{R}\}$ be the one-parameter group of real analytic transformations of S^k associated with the vector field ξ^a. It follows from $T_*\xi^a = \xi^a$ that $T\cdot\theta_t = \theta_t\cdot T$ for $t \in \boldsymbol{R}$. Now we can define a real analytic $\boldsymbol{R}^\times$ action ψ^a on S^k by

$$\psi^a((-1)^n e^t, x) = T^n(\theta_t(x)) \quad \text{for} \quad x \in S^k , \quad t \in \boldsymbol{R} , \quad n \in \boldsymbol{Z} .$$

It is easy to see that the $\boldsymbol{R}^\times$ action ψ^a satisfies the condition (P)-(i). We shall give a sufficient condition for ψ^a to satisfy the condition (P)-(ii).

PROPOSITION 6.1. *If $a(0) = 1$, then the $\boldsymbol{R}^\times$ action ψ^a satisfies the condition* (P).

PROOF. It is sufficient to construct a real analytic into isomorphism $j: \boldsymbol{R}\times F \to S^k$ satisfying the following conditions:

(1) $$j(0, x) = x ,$$

(2) $$T(j(t, x)) = j(-t, x) ,$$

(3) $$j(e^s t, x) = \psi^a(e^s, j(t, x))$$

for $x \in F$; $t, s \in \boldsymbol{R}$. Here F is the fixed point set of T. It is easy to see that the condition (3) is equivalent to the following condition:

(3′) $$j_*(t(\partial/\partial t)) = \xi^a .$$

By the assumption $a(0) = 1$, there is a real analytic function $b(t)$ such that $a(t) = 1 + t\cdot b(t)$. Put $F(t, u) = -tu^3 + tu^3b(t^2u^2) - t^3u^5b(t^2u^2)$. Then there is a unique real analytic function $c(t)$ defined on an interval $(-\varepsilon, \varepsilon)$ for a positive real ε such that $(d/dt)c(t) = F(t, c(t))$, $c(0) = 1$, $-1 < t\cdot c(t) < 1$.

Define a real analytic mapping $j_1: (-\varepsilon, \varepsilon)\times F \to S^k$ by $j_1(t, x) = (t\cdot c(t), (1 - t^2c(t)^2)^{1/2}x)$. Then it is easy to see that $j_{1*}(t(\partial/\partial t)) = \xi^a$ at $j_1(t, x)$. Since $F(-t, u) = -F(t, u)$, we have $c(t) = c(-t)$. Therefore the map j_1 satisfies the following conditions: (1) $j_1(0, x) = (0, x)$, (2) $T(j_1(t, x)) = j_1(-t, x)$, (3′) $j_{1*}(t(\partial/\partial t)) = \xi^a$ at $j_1(t, x)$, for $x \in F$, $-\varepsilon < t < \varepsilon$. By the definition of the action ψ^a, the curve $s \to \psi^a(e^s, j_1(t, x))$ is an integral curve of the vector field ξ^a. By the condition (3′), the curve $s \to j_1(e^st, x)$ is also an integral curve of ξ^a. It follows that

$$(*) \qquad \psi^a(e^s, j_1(t, x)) = j_1(e^st, x)$$

for $x \in F$, $-\varepsilon < t < \varepsilon$, $-\varepsilon < e^st < \varepsilon$. Define a mapping $j: \boldsymbol{R}\times F \to S^k$ by

$$j(t, x) = \begin{cases} \psi^a(2t/\varepsilon, j_1(\varepsilon/2, x)) & \text{for} \quad t \neq 0 \\ (0, x) & \text{for} \quad t = 0 . \end{cases}$$

Then j is an extension of j_1 by $(*)$; hence j is real analytic. By definition, the map j satisfies the conditions (1), (2) and (3).

Finally, we shall show that j is an into isomorphism. Let $\boldsymbol{O}(k)$ be the orthogonal transformation group of the Euclidean space $\boldsymbol{R}^{k+1}$ leaving fixed the x_0-coordinate. Then the vector field ξ^a and the map j_1 are $\boldsymbol{O}(k)$ invariant by definition. Hence we have

$$(**) \qquad A(j(t, x)) = j(t, Ax) \quad \text{for} \quad A \in \boldsymbol{O}(k)\ , \quad (t, x) \in \boldsymbol{R}\times F\ .$$

Since $c(0) = 1$, the map j is non-singular at each point of $0\times F$. It remains to show that j is injective. Assume $j(t_1, x_1) = j(t_2, x_2)$ for some $(t_i, x_i) \in \boldsymbol{R}\times F$. Then $j(st_1, x_1) = j(st_2, x_2)$ for any $s \neq 0$ by the definition of j. Let $s \to 0$. Then $j(0, x_1) = j(0, x_2)$. Hence we have $x_1 = x_2$ and $j(t_1, x_1) = j(t_2, x_1)$. It follows from $(**)$ that $j(t_1, x) = j(t_2, x)$ for any $x \in F$. Assume $t_1 \neq t_2$. Then j induces a real analytic isomorphism of $S^1\times F$ onto an open set of S^k. This is a contradiction. Therefore the map j is injective. q.e.d.

By Proposition 6.1, we can construct many examples of real analytic $\boldsymbol{R}^\times$ actions on the standard k-sphere satisfying the condition (P). Let

$$\boldsymbol{a} = (a_1, a_2, \cdots, a_N) \in \boldsymbol{R}^N \quad \text{for} \quad N = 1, 2, \cdots ,$$

and define a real analytic tangent vector field ξ^a on S^k as follows:

$$\xi^a = \left(\prod_{i=1}^{N} (1 - a_i x_0^2)\right)\cdot(x_0(1 - x_0^2)(\partial/\partial x_0) - x_0^2 \sum_{i=1}^{k} x_i(\partial/\partial x_i)) .$$

Let ψ^a be the real analytic $\boldsymbol{R}^\times$ action on S^k determined by the vector field ξ^a and the involution T. Then the action ψ^a satisfies the condition (P).

PROPOSITION 6.2. *Let* $\boldsymbol{a} = (a_1, \cdots, a_N)$ *and* $\boldsymbol{a}' = (a'_1, \cdots, a'_N)$.

(i) *If* ψ^a *is* C^0 *equivariant to* $\psi^{a'}$, *then the cardinality of the set* $\{a_j\colon a_j > 1\}$ *is equal to that of the set* $\{a'_j\colon a'_j > 1\}$.

(ii) *If* ψ^a *is* C^2 *equivariant to* $\psi^{a'}$, *then* $\prod_{j=1}^N (1 - a_j) = \prod_{j=1}^N (1 - a'_j)$.

PROOF. The points $x_0 = \pm 1$ are isolated zeros of the vector field ξ^a, and the other zeros of ξ^a are the hypersurfaces

$$x_0 = 0 \quad \text{and} \quad x_0 = \pm 1/a_j^{1/2} \quad \text{for} \quad a_j > 1 .$$

If there is an equivariant homeomorphism of S^k with the $\boldsymbol{R}^\times$ action ψ^a to S^k with the $\boldsymbol{R}^\times$ action $\psi^{a'}$, then the zeros of the vector field ξ^a is homeomorphic to the zeros of the vector field $\xi^{a'}$. Hence the cardinality of the set $\{a_j\colon a_j > 1\}$ is equal to that of the set $\{a'_j\colon a'_j > 1\}$.

Suppose next that there is an equivariant C^2 diffeomorphism f of S^k with the $\boldsymbol{R}^\times$ action ψ^a to S^k with the $\boldsymbol{R}^\times$ action $\psi^{a'}$. We shall show that there is an equivariant C^2 diffeomorphism g of S^1 with the $\boldsymbol{R}^\times$ action ψ^a to S^1 with the $\boldsymbol{R}^\times$ action $\psi^{a'}$. Put

$$A(x) = \{(t, (1 - t^2)^{1/2}x) \in S^k\colon -1 < t < 1\} ,$$
$$C(x) = \{(\sin\theta, \cos\theta\cdot x) \in S^k\colon \theta \in \boldsymbol{R}\} ,$$

for $x \in F$. Then $C(x)$ is the closure of the union $A(x) \cup A(-x)$. Since the map f is equivariant, we have $f(A(x)) = A(f(x))$ for $x \in F$. Then we have $f(-x) = -f(x)$ for $x \in F$, by the differentiability of f at $x_0 = 1$. Hence $f(C(x)) = C(f(x))$ for $x \in F$. Since the $\boldsymbol{R}^\times$ action ψ^a is compatible with the $\boldsymbol{O}(k)$ action (see the proof of Proposition 6.1), we can assume $f(y) = y$ for some $y \in F$. Then the restriction $f\colon C(y) \to C(y)$ can be regarded as an equivariant C^2 diffeomorphism g of S^1 with the $\boldsymbol{R}^\times$ action ψ^a to S^1 with the $\boldsymbol{R}^\times$ action $\psi^{a'}$.

Finally we shall show that the existence of g implies $\prod_{j=1}^N (1 - a_j) = \prod_{j=1}^N (1 - a'_j)$. Since g is equivariant, we have $g_*(\xi^a) = \xi^{a'}$. Let $\pi\colon S^1 \to \boldsymbol{R}$ be a map defined by $\pi(x_0, x_1) = x_1$. Then π is a local diffeomorphism at $x_0 = \pm 1$, and

$$\pi_*(\xi^a) = -x_1(1 - x_1^2) \prod_{j=1}^{N} (1 - a_j(1 - x_1^2))(d/dx_1) .$$

There is a local C^2 diffeomorphism h of $\boldsymbol{R}$ such that $h(0) = 0$, $\pi\cdot g =$

$h\cdot\pi$. Then it follows from $h_*(\pi_*(\xi^a)) = \pi_*(\xi^{a'})$ that $-x_1(1-x_1^2)\prod_{j=1}^{N}(1-a_j(1-x_1^2))(dh/dx_1)(x_1) = -y_1(1-y_1^2)\prod_{j=1}^{N}(1-a_j'(1-y_1^2))$ for $y_1 = h(x_1)$. Differentiate by x_1, and put $x_1 = 0$. Then we have the desired equation, because $dh/dx_1(0) \neq 0$. q.e.d.

7. Closed subgroups of $O(n)$. In this section, we shall prove Lemmas 4.1 and 4.2. The method used here is essentially due to Dynkin[2].

PROOF OF LEMMA 4.1. Let G be a connected closed subgroup of $O(n)$. Suppose that

$$(*) \qquad n \geqq 5\,, \quad 0 < \dim O(n)/G \leqq 2n-2\,.$$

The inclusion map $i: G \to O(n)$ gives an orthogonal faithful representation of G.

(A) Suppose first that the representation i is irreducible.

(A-1) Suppose that G is not semi-simple. Let T be a one-dimensional closed central subgroup of G. Since i is irreducible, the centralizer of T in $O(n)$ agrees with $U(n/2)$ by an inner automorphism of $O(n)$ (cf. Uchida [12, Lemma 5.1]). Put $n = 2k$. Then it can be assumed that G is a subgroup of $U(k)$ and the inclusion $G \to U(k)$ is irreducible. It follows that the center of G is one-dimensional by Schur's lemma. Moreover the condition $(*)$ implies $k(k-1) = \dim O(2k)/U(k) \leqq 4k-2$. Hence $k = 3, 4$. It is easy to see that $SU(3)$ has no semi-simple proper subgroup of codimension $\leqq 4$, and $SU(4)$ has no semi-simple proper subgroup of codimension $\leqq 2$. Therefore the case (A-1) occurs only when $n = 6, 8$; G agrees with $U(n/2)$ up to an inner automorphism of $O(n)$.

(A-2) Suppose that G is semi-simple and the complexification i^c of the representation i is reducible. Then $n = 2k$, G is isomorphic to a subgroup G' of $U(k)$, and the inclusion $G' \to U(k)$ is irreducible. Hence $k = 3, 4$ and $G' = SU(k)$. Calculating the centralizer of the center of G in $O(n)$, we can show that G agrees with $SU(n/2)$ up to an inner automorphism of $O(n)$.

(A-3) Suppose that G is semi-simple, non-simple, and i^c is irreducible. Let G^* be the universal covering group of G, and let $p: G^* \to G$ be the projection. Since G is not simple, there are closed semi-simple normal subgroups H_1, H_2 of G^* such that $G^* = H_1 \times H_2$. Consider the representation $i^c p: G^* \to U(n)$. Then there are irreducible complex representations $r_t: H_t \to U(n_t)$ for $t = 1, 2$ such that the tensor product $r_1 \otimes r_2$ is equivalent to $i^c p$. Since $i^c p$ has a real form, the representations r_1, r_2 are self-conjugate; hence r_1 (resp. r_2) has a real form or a quaternionic structure, but not both (cf. Adams [1, Proposition 3.56]).

Moreover, if r_1 has a real form (resp. quaternionic structure), then r_2 has also a real form (resp. quaternionic structure).

Suppose first that r_1, r_2 have quaternionic structures. Then it follows that n_1, n_2 are even, and $\dim H_t \leqq \dim \boldsymbol{Sp}(n_t/2) = n_t(n_t+1)/2$ for $t = 1, 2$. The condition $(*)$ implies $\dim \boldsymbol{O}(n) - \dim \boldsymbol{Sp}(n_1/2) - \dim \boldsymbol{Sp}(n_2/2) \leqq 2n-2$, $n = n_1 n_2$. Therefore $n^2 - 3n + 4 \leqq (n_1+n_2)(n_1+n_2+1) \leqq (2+n/2) \times (3+n/2)$. Hence $n \leqq 7$. But n is a multiple of 4 and $n \geqq 5$. Therefor r_1, r_2 cannot have quaternionic structures simultaneously.

Suppose next that r_1, r_2 have real forms. Then, since H_t is semisimple, it follows that $n_t \geqq 3$ for $t = 1, 2$. Moreover, $\dim H_t \leqq \dim \boldsymbol{O}(n_t) = n_t(n_t-1)/2$ for $t = 1, 2$. The condition $(*)$ implies $\dim \boldsymbol{O}(n) - \dim \boldsymbol{O}(n_1) - \dim \boldsymbol{O}(n_2) \leqq 2n-2$, $n = n_1 n_2$. Therefore $n^2 - 3n + 4 \leqq (n_1+n_2)(n_1+n_2-1) \leqq (3+n/3)(2+n/3)$. Hence $n \leqq 5$. But $n = n_1 n_2 \geqq 9$. Therefore r_1, r_2 cannot have real forms simultaneously. Therefore the case (A-3) does not happen.

(A-4) Suppose finally that G is simple and i^C is irreducible. Put $r = \operatorname{rank} G$, and denote by G^* the universal covering group of G. Denote by $L_1, L_2, \cdots, L_r$ the fundamental weights of G^*. Then there is a one-to-one correspondence between complex irreducible representations of G^* and sequences $(a_1, \cdots, a_r)$ of non-negative integers such that $a_1L_1 + \cdots + a_rL_r$ is the highest weight of a corresponding representation (cf. Dynkin [2, Theorems 0.8 and 0.9]; Humphreys [6, Section 21.2]). Denote by $d(a_1L + {}^1\cdots + a_rL_r)$ the degree of the complex irreducible representation of G^* with the highest weight $a_1L_1 + \cdots + a_rL_r$. The degree can be computed by Weyl's dimension formula (cf. Dynkin [2, Theorem 0.24, (0.148)-(0.155)]; Humphreys [6, Section 24.3]). Notice that if $a_i \geqq a'_i$ for $i = 1, 2, \cdots, r$, then $d(a_1L_1 + \cdots + a_rL_r) \geqq d(a'_1L_1 + \cdots + a'_rL_r)$ and the equality holds only if $a_i = a'_i$ for $i = 1, 2, \cdots, r$.

(A-4-1) Suppose that G is an exceptional Lie group. Then we have Table 3. Here $m(G)$ is the least degree of non-trivial complex irreducible representations of G^* (cf. Dynkin [2, p. 378, Table 30]). The condition $(*)$ implies that $\dim G \geqq \dim \boldsymbol{O}(n) - (2n-2) = (n-1)(n-4)/2$. Hence $(m-1)(m-4) \leqq 2k$. The possibility remains only when $G^* = \boldsymbol{G}_2$ and

TABLE 3

G^*	$k = \dim G$	$m = m(G)$
$\boldsymbol{G}_2$	14	7
$\boldsymbol{F}_4$	52	26
$\boldsymbol{E}_6$	78	27
$\boldsymbol{E}_7$	133	56
$\boldsymbol{E}_8$	248	248

$n \leqq 8$. Since $d(L_1) = 7$, $d(L_2) = 14$, $d(2L_1) = 27$ for $G^* = G_2$, there is no complex irreducible representation of G_2 of degree 8. The complex irreducible representation of G_2 of degree 7 has a real form. Therefore the case (A-4-1) occures only when $n = 7$ and $G = G_2$, where the inclusion $G_2 \to O(7)$ is uniquely determined up to an inner automorphism of $O(7)$.

(A-4-2) Suppose that G^* is isomorphic to $SU(r+1)$ for $r \geqq 1$. Since rank $G \leqq \text{rank} SO(n)$, it follows that

(a) $$2r \leqq n .$$

The condition $(*)$ implies that

(b) $$(n-1)(n-4)/2 \leqq r(r+2) \leqq n(n-1)/2 , \quad n \geqq 5 .$$

It is easy to see from (a), (b) that $n \leqq 13$. If the pair (n, r) satisfies the conditions (a), (b), then it is one of the following: (12, 6), (11, 5), (10, 5), (9, 4), (8, 4), (8, 3), (7, 3), (6, 2), (5, 2), (5, 1). Notice that $d(L_i) = {}_{r+1}C_i$, $d(L_1 + L_r) = r(r+2)$, $d(2L_1) = d(2L_r) = (r+1)(r+2)/2$. Hence there is no complex irreducible representation of $SU(r+1)$ of degrees $2r$ and $2r+1$ for $r \geqq 4$. If $r = 3$, then $d(L_1) = d(L_3) = 4$, $d(L_2) = 6$, $d(2L_1) = d(2L_3) = 10$, $d(2L_2) = d(L_1 + L_2) = d(L_2 + L_3) = 20$, $d(L_1 + L_3) = 15$. Hence there is no complex irreducible representation of $SU(4)$ of degrees 7 and 8. If $r = 2$, then $d(L_1) = d(L_2) = 3$, $d(2L_1) = d(2L_2) = 6$, $d(L_1 + L_2) = 8$. Hence there is no complex irreducible representation of $SU(3)$ of degree 5. There are just two complex irreducible representations of $SU(3)$ of degree 6 which are not self-conjugate. Therefore there is no possibility for $r \geqq 2$. Finally there is only one complex irreducible representation of $SU(2)$ of degree 5 which has a real form. Therefore the case (A-4-2) occurs only when $n = 5$ and $G = SO(3)$, where the inclusion $SO(3) \to O(5)$ is an irreducible representation uniquely determined up to an inner automorphism of $O(5)$.

(A-4-3) Suppose that G^* is isomorphic to $Sp(r)$ for $r \geqq 2$. The condition $(*)$ implies that $(n-1)(n-4)/2 \leqq r(2r+1) < n(n-1)/2$. Hence $n = 2r+2$ or $n = 2r+3$. Notice that $d(L_i) = {}_{2r+1}C_i - {}_{2r+1}C_{i-1}$, $d(2L_1) = r(2r+1)$. If $r \geqq 3$, then $d(L_1) < d(L_2) < \cdots < d(L_s) \geqq d(L_{s+1}) > \cdots > d(L_r)$ for some s. It is easy to see that there is no complex irreducible representation of $Sp(r)$ of degrees $2r+2$ and $2r+3$ for $r \geqq 3$. If $r = 2$, then $d(L_1) = 4$, $d(L_2) = 5$, $d(2L_1) = 10$, $d(2L_2) = 14$, $d(L_1 + L_2) = 16$. Hence there is no complex irreducible representation of $Sp(r)$ of degrees $2r+2$ and $2r+3$ for $r \geqq 2$. Therefore the case (A-4-3) does not happen.

(A-4-4) Suppose that G^* is isomorphic to $Spin(k)$ for $k \geqq 5$. The condition $(*)$ implies that $(n-1)(n-4) \leqq k(k-1) < n(n-1)$. Hence

$n = k + 1$ or $n = k + 2$. If $k = 2r$, then $d(L_i) = {}_{2r}C_i$ for $1 \leqq i \leqq r - 2$, $d(L_{r-1}) = d(L_r) = 2^{r-1}$, $d(2L_1) = (r + 1)(2r - 1)$, $d(2L_{r-1}) = d(2L_r) = {}_{2r-1}C_r$, $d(L_1 + L_{r-1}) = d(L_1 + L_r) = 2^{r-1}(2r - 1)$, $d(L_{r-1} + L_r) = {}_{2r}C_{r-1}$. Hence there is no complex irreducible representation of $\boldsymbol{Spin}(2r)$ of degrees $2r + 1$ and $2r + 2$. If $k = 2r + 1$, then $d(L_i) = {}_{2r+1}C_i$ for $1 \leqq i \leqq r - 1$, $d(L_r) = 2^r$, $d(2L_1) = r(2r + 3)$, $d(L_1 + L_r) = 2^{r+1}r$, $d(2L_r) = 2^{2r}$. Hence there is no complex irreducible representation of $\boldsymbol{Spin}(2r + 1)$ of degrees $2r + 2$ and $2r + 3$ for $r \neq 3$, there is no complex irreducible representation of $\boldsymbol{Spin}(7)$ of degree 9, but there is only one complex irreducible representation of $\boldsymbol{Spin}(7)$ of degree 8 which has a real form. Therefore the case (A-4-4) occurs only when $n = 8$ and $G = \boldsymbol{Spin}(7)$, the inclusion $\boldsymbol{Spin}(7) \to \boldsymbol{O}(8)$ is a real spin representation uniquely determined up to an inner automorphism of $\boldsymbol{O}(8)$.

Consequently, the case (A) occurs only when G is one of the following listed in Table 4 up to an inner automorphism of $\boldsymbol{O}(n)$. Here

TABLE 4

n	G	$i: G \to \boldsymbol{O}(n)$	$\dim \boldsymbol{O}(n)/G$
8	$\boldsymbol{Spin}(7)$	Δ_7	$7=n-1$
8	$\boldsymbol{U}(4)$	μ_4	$12=2n-4$
8	$\boldsymbol{SU}(4)$	$\mu_4{}^0$	$13=2n-3$
7	$\boldsymbol{G}_2$	ω	$7=n$
6	$\boldsymbol{U}(3)$	μ_3	$6=n$
6	$\boldsymbol{SU}(3)$	$\mu_3{}^0$	$7=2n-5$
5	$\boldsymbol{SO}(3)$	β	$7=2n-3$

μ_k: $\boldsymbol{U}(k) \to \boldsymbol{O}(2k)$, μ_k^0: $\boldsymbol{SU}(k) \to \boldsymbol{O}(2k)$ are the canonical inclusions, and Δ_7, ω, β are irreducible representations uniquely determined up to an inner automorphism of $\boldsymbol{O}(n)$, respectively.

(B) Suppose next that the representation $i: G \to \boldsymbol{O}(n)$ is reducible. Then, by an inner automorphism of $\boldsymbol{O}(n)$, G is isomorphic to a subgroup G' of $\boldsymbol{O}(k) \times \boldsymbol{O}(n - k)$ for some k such that $0 < k \leqq n/2$. The condition $(*)$ implies that

(c) $$k(n - k) = \dim \boldsymbol{O}(n)/\boldsymbol{O}(k) \times \boldsymbol{O}(n - k) \leqq 2n - 2 .$$

Hence $k = 1, 2$ or $k = 3$ and $n = 6, 7$. If $k = 3$ and $n = 6, 7$, then it is easy to see that $G' = \boldsymbol{SO}(3) \times \boldsymbol{SO}(3)$, $G' = \boldsymbol{SO}(3) \times \boldsymbol{SO}(4)$, respectively. Suppose $k = 2$. Then the inequality (c) implies $2 + \dim G' \geqq \dim \boldsymbol{O}(2) \times \boldsymbol{O}(n-2)$. Since $\boldsymbol{SO}(n - 2)$ is semi-simple for $n \geqq 5$, $\boldsymbol{SO}(n - 2)$ has no closed subgroup of codimension one. Therefore $G' = \boldsymbol{SO}(n - 2)$, $\boldsymbol{SO}(2) \times \boldsymbol{SO}(n - 2)$ or $G' = \boldsymbol{SO}(2) \times G''$, where G'' is a closed subgroup of $\boldsymbol{O}(n - 2)$ of codimension 2. If the inclusion $G'' \to \boldsymbol{O}(n - 2)$ is irreducible, then $n = 5, 6$

by the case (A). Hence $n=6$ and $G''=U(2)$. If the inclusion $G''\to O(n-2)$ is reducible, then $n=5$ and G'' is a maximal torus of $SO(3)$. Suppose $k=1$. Then G' is a closed subgroup of $O(n-1)$, and the inequality (c) implies $\dim O(n-1)/G'\leqq n-1$. It can be assumed that the inclusion $G'\to O(n-1)$ is irreducible. By the case (A), G' is one of the following listed in Table 5. Consequently, the case (B) occurs only when G is one of the following listed in Table 6 up to an inner automorphism of $O(n)$. Here ρ_k: $SO(k)\to O(k)$ is the canonical inclusion, and θ^k is the trivial representation of degree k. This completes the proof of Lemma 4.1.

TABLE 5

$n-1$	G'	$G'\to O(n-1)$	$\dim O(n-1)/G'$
$n-1$	$SO(n-1)$	ρ_{n-1}	0
8	$Spin(7)$	Δ_7	7
7	G_2	ω	7
6	$U(3)$	μ_3	6
4	$U(2)$	μ_2	2
4	$SU(2)$	$\mu_2{}^0$	3

TABLE 6

n	G	$i: G\to O(n)$	$\dim O(n)/G$
n	$SO(n-1)$	$\rho_{n-1}\oplus\theta^1$	$n-1$
n	$SO(n-2)$	$\rho_{n-2}\oplus\theta^2$	$2n-3$
n	$SO(n-2)\times SO(2)$	$\rho_{n-2}\oplus\rho_2$	$2n-4$
9	$Spin(7)$	$\Delta_7\oplus\theta^1$	$15=2n-3$
8	G_2	$\omega\oplus\theta^1$	$14=2n-2$
7	$U(3)$	$\mu_3\oplus\theta^1$	$12=2n-2$
7	$SO(3)\times SO(4)$	$\rho_3\oplus\rho_4$	$12=2n-2$
6	$SO(3)\times SO(3)$	$\rho_3\oplus\rho_3$	$9=2n-3$
6	$U(2)\times U(1)$	$\mu_2\oplus\mu_1$	$10=2n-2$
5	$U(2)$	$\mu_2\oplus\theta^1$	$6=2n-4$
5	$SU(2)$	$\mu_2{}^0\oplus\theta^1$	$7=2n-3$
5	$U(1)\times U(1)$	$\mu_1\oplus\mu_1\oplus\theta^1$	$8=2n-2$

PROOF OF LEMMA 4.2. It is sufficient to prove that there is no irreducible real representation of $SO(n)$ of degree m for $5\leqq n<m\leqq 2n-2$, and a non-trivial orthogonal representation of $SO(n)$ of degree n is equivalent to the canonical representation ρ_n up to an inner automorphism of $O(n)$. The second half is well known and a proof is given in our previous paper [12, Section 5]. To prove the first half, suppose that there is an irreducible real representation σ of $SO(n)$ of degree m for $5\leqq n<m\leqq 2n-2$. Then it is easy to see that the complexification σ^C of σ is irreducible. Let p: $Spin(n)\to SO(n)$ be the covering pro-

jection. Then the composition $\sigma^c p$ is an irreducible complex representation of $\boldsymbol{Spin}(n)$, which has a real form. Suppose $n = 2r$. Then $d(L_i) = {}_{2r}C_i$ for $1 \leqq i \leqq r-2$, $d(L_{r-1}) = d(L_r) = 2^{r-1}$, $d(2L_1) = (r+1)(2r-1)$, $d(2L_{r-1}) = d(2L_r) = {}_{2r-1}C_r$, $d(L_1+L_{r-1}) = d(L_1+L_r) = 2^{r-1}(2r-1)$, $d(L_{r-1}+L_r) = {}_{2r}C_{r-1}$. Therefore the following are the only possibilities for the irreducible complex representation of $\boldsymbol{Spin}(2r)$ of degree m $(2r < m \leqq 4r-2)$:

$$\Delta_{2r}^+,\ \Delta_{2r}^-\colon \boldsymbol{Spin}(2r) \to \boldsymbol{U}(2^{r-1}) \quad \text{for} \quad r = 5\ ,$$
$$\tau,\ \tau^*\colon \boldsymbol{Spin}(6) = \boldsymbol{SU}(4) \to \boldsymbol{U}(10)\ .$$

Here the representation space of τ is the second symmetric product of the canonical representation space $\boldsymbol{C}^4$ of $\boldsymbol{SU}(4)$, and τ^* is the dual representation. Hence τ, τ^* have no real form. It is known that the half spin representations $\Delta_{2r}^+, \Delta_{2r}^-$ are not induced from a representation of $\boldsymbol{SO}(2r)$. Suppose $n = 2r+1$. Then $d(L_i) = {}_{2r+1}C_i$ for $1 \leqq i \leqq r-1$, $d(L_r) = 2^r$, $d(2L_1) = r(2r+3)$, $d(L_1+L_r) = 2^{r+1}r$, $d(2L_r) = 2^{2r}$. Therefore the following is the only possibility for the irreducible complex representation of $\boldsymbol{Spin}(2r+1)$ of degree m $(2r+1 < m \leqq 4r)$:

$$\Delta_{2r+1}\colon \boldsymbol{Spin}(2r+1) \to \boldsymbol{U}(2^r) \quad \text{for} \quad r = 3, 4\ .$$

It is known that the spin representation Δ_{2r+1} is not induced from a representation of $\boldsymbol{SO}(2r+1)$. Consequently, we have the desired result.

q.e.d.

8. **Concluding remark.** If $5 \leqq n \leqq m \leqq 2n-2$, then there exists only one linear $\boldsymbol{SO}(n)$ action $\rho_n \oplus \theta^{m-n+1}$ on the standard m-sphere (see Theorem 4.11). This action is the restriction of a linear $\boldsymbol{SL}(n, \boldsymbol{R})$ action. We shall show a counterexample for $n = 4$.

Recall that there is a surjective homomorphism $\pi\colon \boldsymbol{SO}(4) \to \boldsymbol{SO}(3)$. Through this homomorphism, $\boldsymbol{SO}(4)$ acts on $\boldsymbol{R}^3$ and the action is transitive on the unit sphere S^2 with the isotropy group $\boldsymbol{U}(2)$. Also $\boldsymbol{SO}(4)$ acts naturally on $\boldsymbol{R}^4$ and the action is transitive on the unit sphere S^3 with the isotropy group $\boldsymbol{SO}(3)$. Thus we have the diagonal action of $\boldsymbol{SO}(4)$ on the unit sphere S^6 of $\boldsymbol{R}^3 \oplus \boldsymbol{R}^4$. This action is a linear $\boldsymbol{SO}(4)$ action on S^6, the principal orbit type is $\boldsymbol{SO}(4)/\boldsymbol{SO}(2)$ and there are just two singular orbit types $\boldsymbol{SO}(4)/\boldsymbol{SO}(3)$ and $\boldsymbol{SO}(4)/\boldsymbol{U}(2)$.

PROPOSITION 8.1. *The above $\boldsymbol{SO}(4)$ action on S^6 is not extendable to any continuous $\boldsymbol{SL}(4, \boldsymbol{R})$ action on S^6.*

PROOF. Suppose that there exists a continuous $\boldsymbol{SL}(4, \boldsymbol{R})$ action on S^6 which is an extension of the $\boldsymbol{SO}(4)$ action. Let $x \in S^6$ be a point such that $SO(4)_x = U(2)$. Then

(1) $U(2) \subset SL(4, R)_x \neq SL(4, R)$,

(2) $\dim SL(4, R)/SL(4, R)_x \leqq 6$.

Here we shall show first the following result.

LEMMA 8.2. *Let* $\mathfrak{u}(2)$ *be the Lie algebra of* $U(2)$. *Let* $\mathfrak{g}$ *be a proper Lie subalgebra of* $\mathfrak{sl}(4, R)$ *which contains* $\mathfrak{u}(2)$. *Then* $\dim \mathfrak{g} = 4, 6, 7$ *or* 10.

PROOF. Recall

$$U(2) = \left\{\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in M_4(R): A^tA + B^tB = I_2,\ A^tB = B^tA\right\}.$$

Put

$$\mathfrak{u}(2) = \left\{\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M_4(R): X + {}^tX = 0,\ Y = {}^tY\right\},$$

$$\mathfrak{h}(2) = \left\{\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in M_4(R): X = {}^tX,\ Y + {}^tY = 0,\ \operatorname{trace} X = 0\right\},$$

$$\mathfrak{a} = \left\{\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in M_4(R): X = {}^tX,\ Y = {}^tY\right\},$$

$$\mathfrak{b} = \left\{\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in M_4(R): X + {}^tX = Y + {}^tY = 0\right\}.$$

Then $\mathfrak{sl}(4, R) = \mathfrak{u}(2) \oplus \mathfrak{h}(2) \oplus \mathfrak{a} \oplus \mathfrak{b}$ as a direct sum of $Ad(U(2))$ invariant linear subspaces. Here $\mathfrak{h}(2)$, $\mathfrak{a}$ and $\mathfrak{b}$ are irreducible, respectively, and $\dim \mathfrak{h}(2) = 3$, $\dim \mathfrak{a} = 6$, $\dim \mathfrak{b} = 2$. Moreover, we have $[\mathfrak{h}(2), \mathfrak{a}] = \mathfrak{b}$, $[\mathfrak{h}(2), \mathfrak{b}] = \mathfrak{a}$, $[\mathfrak{a}, \mathfrak{b}] = \mathfrak{h}(2)$, $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{u}(2)$, $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{u}(2)$, $[\mathfrak{h}(2), \mathfrak{h}(2)] \subset \mathfrak{u}(2)$. Therefore $\mathfrak{g}$ is one of the following: $\mathfrak{u}(2)$, $\mathfrak{u}(2) \oplus \mathfrak{a}$, $\mathfrak{u}(2) \oplus \mathfrak{b}$, $\mathfrak{u}(2) \oplus \mathfrak{h}(2)$. Then $\dim \mathfrak{g} = 4, 10, 6$ or 7, respectively. q.e.d.

We now return to the proof of Proposition 8.1. By the condition (1), (2), it follows from Lemma 8.2 that $\dim SL(4, R)_x = 10$. Therefore the orbit $SL(4, R)\cdot x$ contains the orbit $SO(4)\cdot x$ as a proper subset. Since the orbit $SO(4)\cdot x$ is isolated, the orbit $SL(4, R)\cdot x$ must intersect a principal orbit of the $SO(4)$ action. Hence there is an element $g \in SL(4, R)$ such that $SO(4)_{gx} = SO(2)$. Put $y = gx$. Then there is an embedding $SO(4)\cdot y \subset SL(4, R)\cdot y = SL(4, R)\cdot x$. But $\dim SO(4)\cdot y = \dim SL(4, R)\cdot x = 5$. Hence $SO(4)\cdot y = SL(4, R)\cdot x$. Since $SO(4)\cdot y$ is a principal orbit, we have $x \notin SO(4)\cdot y$. This is a contradiction. Therefore there is no continuous $SL(4, R)$ action on S^6 which is an extension of the $SO(4)$ action. q.e.d.

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