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TORUS EMBEDDINGS AND TANGENT COMPLEXES

Dedicated to Professor Yoshikazu Nakai on his sixtieth birthday

MASA-NORI ISHIDA AND TADAO ODA

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Introduction. Let Δ be a combinatorial triangulation of the ordinary $(d-1)$-sphere, or, more generally, a $(d-1)$-dimensional finite simplicial *k-homology sphere* with a field k (cf. (1.10) below). Number the vertices (= 0-simplices) of Δ from 1 through r. Consider the collection Ξ_Δ of subsets of $\{1, \cdots, r\}$ defined as follows: a subset ξ belongs to Ξ_Δ if and only if either ξ is the empty set or there exists a simplex σ of Δ such that ξ is the set of vertices of σ.

In the r-dimensional affine space A_r over k, let Y be the d-dimensional closed reduced subscheme obtained as the union

$$Y = \bigcup_{\xi \in \Xi_\Delta} V(\xi)$$

of the affine subspaces $V(\xi)$ defined by

$$V(\xi) = \{t = (t_1, \cdots, t_r) \in A_r;\ t_j = 0 \quad \text{if} \quad j \notin \xi\} .$$

Since $V(\xi) \supset V(\eta)$ if $\xi \supset \eta$, it suffices to take $V(\xi)$'s with ξ in Ξ_Δ of cardinality d to cover Y. Thus the normalization of Y is the union of these d-dimensional affine subspaces, hence is nonsingular. Here, let us tentatively call such Y a d-dimensional *k-spherical scheme with the nonsingular normalization.* Hochster and Ishida showed that such Y is Gorenstein (cf. (1.8) and (1.10) below).

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In terms of rings, we have $Y = \mathrm{Spec}(S)$, where S is the residue ring

$$S = k[t_1, \cdots, t_r]/J$$

of the polynomial ring by the ideal J generated by the monomials

$$t_{i_1} t_{i_2} \cdots t_{i_s}$$

for all subsets $\{i_1, i_2, \cdots, i_s\}$ of $\{1, \cdots, r\}$ not belonging to Ξ_Δ. Thus we need, in general, too many equations to define Y, hence Y is far from being a complete intersection.

Nevertheless, we can use the rich combinatorial information on Ξ_Δ to compute the hyperextension sheaves, the "tangent complex",

$$\mathscr{E}xt^i_{\mathscr{O}_Y}(L^Y_\cdot, \mathscr{O}_Y) \qquad i = 0, 1, 2 ,$$

where $L^Y_\cdot$ is the cotangent complex of Y introduced and studied by Lichtenbaum-Schlessinger [LS], Grothendieck [G_1], Rim [R] and Illusie [I_1] (cf. (1.14) below).

We carried out the computation for the following reasons. For one thing, we intend eventually to study the deformations of those varieties locally and formally isomorphic to such Y. As far as we know, no explicit computation of the second hyperextension sheaf was ever systematically carried out for nonnormal higher dimensional varieties which are not local complete intersections. For another, we wanted to test the reasonableness of the notion of k-sphericity. We hope we succeeded in doing so.

As a consequence of our slightly more general main theorems announced in Section 2 and proved in later sections, we get the following results:

(0) (cf. Remark after Corollary 2.2). The zeroth hyperextension sheaf

$$\mathscr{E}xt^0_{\mathscr{O}_Y}(L^Y_\cdot, \mathscr{O}_Y) ,$$

which classifies local infinitesimal automorphisms of Y and is nothing but the sheaf $\mathscr{D}er_k(\mathscr{O}_Y)$ of germs of k-derivations of $\mathscr{O}_Y$, is canonically isomorphic to the kernel of the homomorphism induced by the restriction maps

$$\bigoplus_\alpha \Theta_{V(\alpha)}(-\log D(\alpha)) \to \bigoplus_\beta \Theta_{V(\beta)}(-\log D(\beta))$$

with α and β running through the sets belonging to Ξ_Δ with $|\alpha| = d$ and $|\beta| = d - 1$, respectively, where for $\xi \in \Xi_\Delta$ in general, we let

$$\Theta_{V(\xi)}(-\log D(\xi)) = \bigoplus_{j \in \xi} \mathscr{O}_{V(\xi)} t_j \partial/\partial t_j ,$$

which is the *sheaf of germs of k-derivations on* $V(\xi)$ *with logarithmic zeros along the divisor* $D(\xi)$ on the affine subspace $V(\xi)$ defined by

$$D(\xi) = \sum_{\eta} V(\eta)$$

with η running through the subsets of ξ satisfying $|\xi| - |\eta| = 1$. This is a generalization of a result obtained by Nakamura [N_1, Proposition 2.5].

(1) (cf. Theorem 2.3). The first hyperextension sheaf

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(L^Y_{\cdot}, \mathcal{O}_Y) ,$$

which coincides with the ordinary extension sheaf

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y)$$

for the sheaf Ω^1_Y of 1-forms, is canonically isomorphic to the kernel of a homomorphism

$$\varepsilon : \bigoplus_{\beta} \mathcal{G}(\beta) \to \bigoplus_{\gamma} \mathcal{G}(\gamma)$$

with β and γ running through the sets belonging to Ξ_Δ with $|\beta| = d - 1$ and $|\gamma| = d - 2$, respectively, where the $\mathcal{O}_Y$-modules $\mathcal{G}(\beta)$ and $\mathcal{G}(\gamma)$ defined in Section 2 have the following properties: For $|\beta| = d - 1$, $\mathcal{G}(\beta)$ is an invertible $\mathcal{O}_{V(\beta)}$-module on the $(d - 1)$-dimensional affine subspace $V(\beta)$, while for $|\gamma| = d - 2$, $\mathcal{G}(\gamma)$ is either zero or a locally free $\mathcal{O}_{V(\gamma)}$-module of rank two on the $(d - 2)$-dimensional affine subspace $V(\gamma)$. This is a generalization of another result obtained by Nakamura [N_1, Section 5].

(2) (cf. Theorem 2.4 and Corollary 2.5). The most difficult to compute is the second hyperextension sheaf

$$\mathcal{E}xt^2_{\mathcal{O}_Y}(L^Y_{\cdot}, \mathcal{O}_Y) ,$$

which measures local obstructions for deformations of Y. We reduce its vanishing to that of more computable $H_{d-1}(\Phi(s', s''), k)$ for various s' and s'', which coincide with the $(d - 1)$-dimensional reduced k-homology group of certain "local" subcomplexes of Δ. In particular for $d = \dim Y \leqq 3$, we can completely classify those Y's with the vanishing second hyperextension sheaf as follow:

Case $d \leqq 1$. Always. Here Y is either a point or a transversal intersection of two affine lines (thus with the ordinary double singularity at the origin).

Case $d = 2$. Y, in general, is the *elliptic polygonal r-cone* in the sense of Mumford [M] for $r \geqq 3$, i.e., Y is the union in A_r of the (t_1, t_2)-, (t_2, t_3)-, $\cdots$, (t_{r-1}, t_r)- and (t_r, t_1)-planes. The second hyperextension sheaf for Y vanishes if and only if $r \leqq 5$. Here the cases $r = 3$ and $r = 4$, i.e., $Y = \mathrm{Spec}(k[t_1, \cdots, t_3]/(t_1t_2t_3))$ and $Y = \mathrm{Spec}(k[t_1, \cdots, t_4]/(t_1t_3, t_2t_4))$ are complete intersections, while the case $r = 5$ is not.

Case $d = 3$. There are many combinatorially different triangulations Δ of the 2-sphere even if the number r of the vertices is fixed. Among them, there are only ten different Δ's listed in Corollary 2.5 for which the second hyperextension sheaf vanishes. Only three of them are complete intersections.

The proof of these results is accomplished after lengthy combinatorial study in Sections 3, 4, 5 and 6.

Recently Kagami [K_1] showed that all these Y's of dimension three with the vanishing second hyperextension sheaf have *stable singularity* at the origin, generalizing Mumford's result in the case of elliptic polygonal cones in [M]. Note that there are some other Y's with the nonvanishing second hyperextension sheaf, which have stable or semistable singularity at the origin.

Here is the motivation for our study of k-spherical schemes Y with the nonsingular normalization, as we have already announced in [IO].

In connection with the compactification problem of various moduli spaces, we encounter many examples of "degenerate varieties", for instance, (i) stable curves of Deligne-Mumford [DM], (ii) degenerate jacobian varieties of Oda-Seshadri [OS] and Ishida [I_2] or, more generally, degenerate abelian varieties of Namikawa [N_3] and Nakamura [N_1], (iii) degenerate hyperelliptic surfaces of Tsuchihashi [T], (iv) degenerate forms of Hopf surfaces and other class VII_0 surfaces by Kodaira [K_2], Miyake-Oda [MO] and Nakamura [N_2] and (v) degenerate K3 surfaces by Kulikov [K_3], Persson [P] and Persson-Pinkham [PP].

These degenerate varieties are usually reduced and connected. But, in general, they are reducible with the irreducible components *not* crossing normally. Their singularities are very often formally isomorphic to our k-spherical schemes Y with the nonsingular normalization. As we saw above, however, it is rather hard to deal with them through their too many defining equations. Fortunately, we have a way of dealing systematically with monomials by means of the theory of torus embeddings, or Demazure varieties, introduced and studied by Demazure [D], Mumford et al. [TE], Satake [S] and Miyake-Oda [MO].

Ishida [I_3] already began to study, more generally, closed invariant reduced subschemes Y of normal torus embeddings. He gave a good description of the dualizing complex $K_Y^{\cdot}$ of Y and described when Y is Cohen-Macaulay or Gorenstein. He could single out a very nice class of

k-spherical Y's. We recall in Section 1 some of his results in a dual formulation convenient for our purpose.

The results in Ishida [I_3] as well as here grew out of our effort to understand and generalize those of Nakamura [N_1, Lemma 2.2, Lemma 5.2 and Proposition 5.1] in the case of degenerate abelian varieties. A generalization in another direction is being carried out by Ishida [I_4].

Now that the local theory of "degenerate varieties" is reasonably well established, we hope to formulate a good global theory of "degenerate varieties", which, in a sense, is a generalization of the theory of toroidal embeddings by Mumford et al. [TE].

1. The review of relevant results and amplifications. Partly to fix notations, we recall and supplement some of the relevant results obtained by Ishida [I_3] and Miyake-Oda [MO], in the dual formulation which is more convenient for our purpose. We also recall the hyperextension sheaves of the contagent complex necessary for the formal deformation theory developed by Lichtenbaum-Schlessinger [LS], Grothendieck [G_1], Rim [R] and Illusie [I_1].

Throughout, we fix a field k and a free $\boldsymbol{Z}$-module M of finite rank r. Let $N = \mathrm{Hom}_{\boldsymbol{Z}}(M, \boldsymbol{Z})$ be the dual $\boldsymbol{Z}$-module with the canonical pairing $\langle\ ,\ \rangle: M \times N \to \boldsymbol{Z}$.

(1.1) Let ϖ be a convex rational polyhedral cone in $M_{\boldsymbol{R}} = \boldsymbol{R} \otimes_{\boldsymbol{Z}} M$ which generates $M_{\boldsymbol{R}}$ as an $\boldsymbol{R}$-vector space, i.e., there exist elements $m_1, \cdots, m_s \in M$ which span $M_{\boldsymbol{R}}$ over $\boldsymbol{R}$ such that $\varpi = \boldsymbol{R}_{\geqq 0}m_1 + \cdots + \boldsymbol{R}_{\geqq 0}m_s$. Then $\varpi \cap (-\varpi)$ is the largest $\boldsymbol{R}$-subspace contained in ϖ. Here $\boldsymbol{R}_{\geqq 0}$ is the set of nonnegative real numbers.

(1.2) We consider the group ring $k[M]$ of M over k by introducing the multiplicative base $e(m)$ satisfying $e(0) = 1$ and $e(m + m') = e(m)e(m')$ for $m, m' \in M$ so that $k[M] = \bigoplus_{m \in M} ke(m)$. We consider $k[M]$ as an M-graded ring by letting $e(m)$ to be homogeneous of degree m.

(1.3) For ϖ as above, let $P = k[M \cap \varpi]$ be the semigroup ring of the subsemigroup $M \cap \varpi$ of M. Then Spec(P) is exactly a normal affine embedding of the torus $T = \mathrm{Spec}(k[M])$. For details, we refer the reader to [MO]. For instance, P is smooth over k if and only if ϖ is *nonsingular*, i.e., there exists a $\boldsymbol{Z}$-basis $\{m_1, \cdots, m_r\}$ of M and $s \leqq r$ such that $\varpi = \boldsymbol{R}_{\geqq 0}m_1 + \cdots + \boldsymbol{R}_{\geqq 0}m_s + \boldsymbol{R}m_{s+1} + \cdots + \boldsymbol{R}m_r$ by [MO, (5.6), p. 21].

(1.4) Let $\Gamma(\varpi)$ be the set of the *faces* ξ of ϖ, i.e., those subsets ξ for which there exists a $\boldsymbol{Z}$-linear functional $n \in N$ on M such that n has nonnegative values on ϖ and that ξ is exactly the set of points of ϖ on which n vanishes. Those ξ's are again convex rational polyhedral cones. $\Gamma(\varpi)$ is a finite partially ordered set with the largest element ϖ and the smallest element $\varpi \cap (-\varpi)$ via the order $\xi > \eta$ meaning η is a face of ξ. For $\xi, \eta \in \Gamma(\varpi)$, we denote by $\xi \cup \eta$ the smallest face of ϖ containing ξ and η as faces, while the intersection $\xi \cap \eta$ is the largest face of ϖ contained in ξ and in η. We regard $\Gamma(\varpi)$ as an abstract complex. A subset $\Xi \subset \Gamma(\varpi)$ is a *subcomplex* (resp. *star closed subset*, resp. *local subcomplex*) if $\xi \in \Xi$ and $\xi > \eta$ imply $\eta \in \Xi$ (resp. if $\eta \in \Xi$ and $\xi > \eta$ imply $\xi \in \Xi$, resp. if Ξ is the intersection of a subcomplex and a star closed subset). More generally, subcomplexes, star closed subsets and local subcomplexes of a local subcomplex $\Xi \subset \Gamma(\varpi)$ are defined in a similar manner.

(1.5) For ξ's in $\Gamma(\varpi)$, $\mathfrak{p}(\xi) = k[M \cap \varpi \setminus M \cap \xi] = \bigoplus_{m \in M \cap \varpi \setminus M \cap \xi} ke(m)$ are exactly the M-homogeneous prime ideals of $P = k[M \cap \varpi]$ by [MO, pp. 16–18].

DEFINITION. For a subcomplex $\Xi \subset \Gamma(\varpi)$, we define the M-homogeneous semiprime ideal $J = J(\Xi)$ and the M-graded quotient ring $S = S(\Xi)$ by $J(\Xi) = \bigcap_{\xi \in \Xi} \mathfrak{p}(\xi)$ and $S(\Xi) = P/J(\Xi)$.

$S(\Xi)$, as a k-vector space, coincides with $k[\bigcup_{\xi \in \Xi}(M \cap \xi)]$, with, however, the multiplication defined for $m, m' \in \bigcup_{\xi \in \Xi}(M \cap \xi)$ by $e(m)e(m') = e(m + m')$ if there exists $\xi \in \Xi$ containing both m and m' while $e(m)e(m') = 0$ otherwise. For $\xi \in \Gamma(\varpi)$, $P/\mathfrak{p}(\xi)$ is isomorphic to the semigroup ring $k[M \cap \xi]$, which we always regard as an M-graded quotient ring of P. $Y(\Xi) = \mathrm{Spec}(S(\Xi))$ is a reduced T-invariant closed subscheme of the torus embedding $Z = \mathrm{Spec}(P) \supset T = \mathrm{Spec}(k[M])$. For $\xi \in \Gamma(\varpi)$, let $V(\xi) = \mathrm{Spec}(P/\mathfrak{p}(\xi))$, which is an irreducible reduced T-invariant closed subscheme of Z with $\dim V(\xi) = \dim \xi$ and with $V(\xi) \supset V(\eta)$ if and only if $\xi > \eta$. We have $V(\xi) \subset Y(\Xi)$ if and only if $\xi \in \Xi$ so that $Y(\Xi) = \bigcup_{\xi \in \Xi} V(\xi) = \bigcup_{\alpha} V(\alpha)$, where α runs through the maximal elements of Ξ.

(1.6) For a local subcomplex $\Xi \subset \Gamma(\varpi)$ and $j \geqq 0$, let

$$\Xi_j = \{\xi \in \Xi;\ \dim \xi = j\}$$

and let

$$d = \dim \Xi = \max\{\dim \xi;\ \xi \in \Xi\}\ .$$

For $\xi \in \Xi_j$, consider the $\boldsymbol{Z}$-submodule $\boldsymbol{Z}(M \cap \xi)$ of M of rank j generated by $M \cap \xi$, and its highest exterior power

$$\det \boldsymbol{Z}(M \cap \xi) = \bigwedge^{j} \boldsymbol{Z}(M \cap \xi)\ .$$

For $\eta \in \Xi_{j-1}$ with $\xi > \eta$, there exists a primitive element $n \in N$, unique mod $N \cap \xi^{\perp}$, such that n has nonnegative values on ξ and that η is exactly the set of points of ξ at which n vanishes. Thus we have an exact sequence

$$0 \to \boldsymbol{Z}(M \cap \eta) \to \boldsymbol{Z}(M \cap \xi) \xrightarrow{n} \boldsymbol{Z} \to 0\ ,$$

hence a canonical isomorphism $\det \boldsymbol{Z}(M \cap \xi) \xrightarrow{\sim} \det \boldsymbol{Z}(M \cap \eta)$ sending $m_1 \wedge \cdots \wedge m_j$, with $m_1 \in \boldsymbol{Z}(M \cap \xi)$ and $m_2, \cdots, m_j \in \boldsymbol{Z}(M \cap \eta)$, to $\langle m_1, n\rangle m_2 \wedge \cdots \wedge m_j$. For $j \geqq 0$, let

$$C_j(\Xi) = \bigoplus_{\xi \in \Xi_j} \det \boldsymbol{Z}(M \cap \xi)$$

with the map

$$\partial\colon C_j(\Xi) \to C_{j-1}(\Xi)$$

defined so that its (ξ, η)-component for $\xi \in \Xi_j$ and $\eta \in \Xi_{j-1}$ is zero if η is not a face of ξ, and is the above isomorphism $\det \boldsymbol{Z}(M \cap \xi) \xrightarrow{\sim} \det \boldsymbol{Z}(M \cap \eta)$ if $\xi > \eta$. Then by Ishida [I_3, Proposition 1.6], we see that $C_.(\Xi)$ together with ∂ is a finite complex of free $\boldsymbol{Z}$-modules.

If we fix a $\boldsymbol{Z}$-basis u_ξ of $\det \boldsymbol{Z}(M \cap \xi)$ for each $\xi \in \Xi$, we see that for $\xi \in \Xi_j$

$$\partial(u_\xi) = \sum_{\eta \in \Xi_{j-1}} [\xi\colon \eta] u_\eta$$

for numbers $[\xi\colon \eta] = 0, 1$ or -1, which we call the *incidence numbers*.

For a subcomplex $\Xi' \subset \Xi$, we can canonically regard $C_.(\Xi')$ as a subcomplex of $\boldsymbol{Z}$-modules of $C_.(\Xi)$. Then the quotient complex $C_.(\Xi)/C_.(\Xi')$ coincides with $C_.(\Xi \setminus \Xi')$ defined for the local subcomplex $\Xi \setminus \Xi'$. We denote by $H_.(\Xi, k)$ and $H^{\cdot}(\Xi, k)$ the homology group of $C_.(\Xi, k) = C_.(\Xi) \otimes_{\boldsymbol{Z}} k$ and the cohomology group of $C^{\cdot}(\Xi, k) = \mathrm{Hom}_{\boldsymbol{Z}}(C_.(\Xi), k)$. We call Ξ *homologically trivial* (resp. *k-homologically trivial*) if $H_.(\Xi) = 0$ (resp. $H_.(\Xi, k) = 0$).

(1.7) For $\zeta \in \Xi$, let

$$\mathrm{Star}_\zeta(\Xi) = \{\xi \in \Xi;\ \xi > \zeta\} = \Xi \setminus \{\eta \in \Xi;\ \eta \not> \zeta\}$$

be the *star* of ζ in Ξ, which is a star closed subset of Ξ with the smallest element ζ. Thus we can think of $C_.(\mathrm{Star}_\zeta(\Xi))$ as a quotient complex of $C_.(\Xi)$.

Recall that for $\zeta \in \Gamma(\varpi)$, we have bijections

$$\mathrm{Star}_\zeta(\Gamma(\varpi)) \xrightarrow{\sim} \Gamma(\varpi + \boldsymbol{R}\zeta) \xrightarrow{\sim} \Gamma((\varpi + \boldsymbol{R}\zeta)/\boldsymbol{R}\zeta)$$

by sending ξ to $\xi + \boldsymbol{R}\zeta$ and then to $(\xi + \boldsymbol{R}\zeta)/\boldsymbol{R}\zeta$ (cf. [MO, Proposition 3.1] and [I_3, Proposition 1.3]).

(1.8) A local subcomplex $\Xi \subset \Gamma(\varpi)$ with the smallest element ϕ is called *k-spherical* if for any $\zeta \in \Xi$ we have

$$H_j(\mathrm{Star}_\zeta(\Xi), k) = \begin{cases} k & j = \dim \Xi \\ 0 & \text{otherwise} . \end{cases}$$

A local subcomplex $\Xi \subset \Gamma(\varpi)$ is called *k-semispherical* if there exists $\rho \in \Xi$ such that $\mathrm{Star}_\rho(\Xi)$ contains all the maximal elements of Ξ and that $\mathrm{Star}_\rho(\Xi)$ is k-spherical.

Note that a k-spherical Ξ is k-semispherical by taking ρ to be the smallest element ϕ. Note also that by Ishida [I_3, Proposition 5.8], ρ in the definition of the k-semisphericity of Ξ is uniquely determined, and for $\zeta \in \Xi$ we have

$$\mathrm{Star}_\zeta(\Xi) \quad \text{is} \quad \begin{cases} k\text{-spherical if} \quad \zeta > \rho \\ k\text{-homologically trivial if} \quad \zeta \not> \rho . \end{cases}$$

Moreover, by [ibid. Corollary 5.6], we see that for $\zeta > \rho$ and $d = \dim \Xi$, the canonical map

$$H_d(\mathrm{Star}_\rho(\Xi), k) \xrightarrow{\sim} H_d(\mathrm{Star}_\zeta(\Xi), k)$$

is an isomorphism.

The importance of the k-sphericity and k-semisphericity lies in the following basic:

THEOREM (Ishida [I_3, Theorem 5.10 and Proposition 5.13]. See also Hochster [H, Added in proof].). *If a subcomplex $\Xi \subset \Gamma(\varpi)$ with the smallest element ϕ is k-spherical, then $S(\Xi)$ is a Gorenstein ring. If ϖ is nonsingular, then $S(\Xi)$ is a Gorenstein ring if and only if Ξ is k-semispherical. In this case, $S(\Xi)$ is noncanonically isomorphic to the tensor product over k of $S(\Xi')$, for a k-spherical $\Xi' \subset \Gamma(\varpi')$ for a nonsingular rational polyhedral cone ϖ', and the k-smooth ring $k[M \cap \rho]$, where ρ is the one appearing in the definition of the k-semisphericity of Ξ.*

(1.9) Let $\Xi \subset \Gamma(\varpi)$ be a subcomplex. If $Y = Y(\Xi)$ has the nonsingular normalization $\tilde{Y}$, then we have

$$\tilde{Y} = \coprod_{\xi} V(\xi)$$

with ξ running through the maximal elements of Ξ. In this case, replacing ϖ by a possibly higher dimensional cone if necessary, we may assume ϖ itself is nonsingular.

If ϖ is nonsingular, with

$$\varpi = \boldsymbol{R}_{\geq 0} m_1 + \cdots + \boldsymbol{R}_{\geq 0} m_s + \boldsymbol{R} m_{s+1} + \cdots + \boldsymbol{R} m_r$$

for a $\boldsymbol{Z}$-basis $\{m_1, \cdots, m_r\}$ of M and $s \leqq r$, then the complex $\Gamma(\varpi)$ is isomorphic to that of the family of subsets of $\{1, \cdots, r\}$ containing $\{s+1, \cdots, r\}$ via the map sending $\xi \in \Gamma(\varpi)$ to $\{i; 1 \leqq i \leqq r, m_i \in \xi\}$.

If $Y = Y(\Xi)$ with a k-semispherical $\Xi \subset \Gamma(\varpi)$ has a nonsingular normalization $\tilde{Y}$, then Y is noncanonically isomorphic to the product

$$Y = Y(\Xi') \times \boldsymbol{A}_s \times T'$$

of an affine space $\boldsymbol{A}_s$, an algebraic torus T' and $Y(\Xi')$ for a k-spherical subcomplex $\Xi' \subset \Gamma(\varpi')$ for a nonsingular convex rational polyhedral cone ϖ' with $\omega' \cap (-\varpi') = \{0\}$. Thus for many purposes it is enough to study $Y(\Xi)$ for a k-spherical subcomplex $\Xi \subset \Gamma(\varpi)$ for a nonsingular convex rational polyhedral cone ϖ with $\varpi \cap (-\varpi) = \{0\}$. Thus if

$$\varpi = \boldsymbol{R}_{\geqq 0} m_1 + \cdots + \boldsymbol{R}_{\geqq 0} m_r$$

for a $\boldsymbol{Z}$-basis $\{m_1, \cdots, m_r\}$ of M, then Ξ can be identified with a subcomplex of the complex of subsets of $\{1, \cdots, r\}$ via the map sending $\xi \in \Xi$ to $\{i; 1 \leqq i \leqq r, m_i \in \xi\}$.

(1.10) (cf. Hochster [H]). As above, let $\varpi = \boldsymbol{R}_{\geqq 0} m_1 + \cdots + \boldsymbol{R}_{\geqq 0} m_r$ for a $\boldsymbol{Z}$-basis $\{m_1, \cdots, m_r\}$ of M and let $\Xi \subset \Gamma(\varpi)$ be a d-dimensional subcomplex. Then Ξ is k-spherical if and only if $\Xi = \Xi_\Delta$ for a $(d-1)$-dimensional *simplicial k-homology sphere* Δ with r vertices $\{1, \cdots, r\}$, where $\Xi_\Delta \subset \Gamma(\varpi)$ is the collection of all $\xi \in \Gamma(\varpi)$ such that $\{i; 1 \leqq i \leqq r, m_i \in \xi\}$ is the set of vertices of a simplex of Δ or the empty set. The reason for this is that for $\{0\} \neq \zeta \in \Xi_\Delta$, $H_{j+1}(\mathrm{Star}_\zeta(\Xi_\Delta), k)$ coincides with the $(j - \dim \zeta)$-dimensional reduced k-homology group of the *link* in Δ of the simplex corresponding to ζ, while for $\zeta = \{0\}$, it is the j-dimensional reduced k-homology group of Δ itself.

Combinatorial triangulations Δ of the ordinary $(d-1)$-sphere are typical examples. Conversely for $d = 1, 2, 3$, $(d-1)$-dimensional simplicial k-homology spheres are known necessarily to be combinatorial triangulations of the ordinary sphere. Thus when $d = 1$, we have

$$S(\Xi_\Delta) = k[t_1, t_2]/(t_1 t_2) ,$$

hence $Y(\Xi_\Delta)$ is a curve with an ordinary double point at the origin. When $d = 2$, then Δ is necessarily a decomposition of a circle into r arcs. $Y(\Xi_\Delta)$ is realized in the r-dimensional affine space with the coordinates $(t_1, \cdots, t_r)$ as the union of (t_1, t_2)-, (t_2, t_3)-, $\cdots$, (t_{r-1}, t_r)- and (t_r, t_1)-planes. Such $Y(\Xi_\Delta)$ is called the *elliptic polygonal r-cone* by Mumford [M]. The ring $S(\Xi_\Delta)$ is of the following form:

$$S(\Xi_\Delta) = \begin{cases} k[t_1, t_2, t_3]/(t_1t_2t_3) & r = 3 \\ k[t_1, t_2, t_3, t_4]/(t_1t_3, t_2t_4) & r = 4 \\ k[t_1, \cdots, t_r]/(t_it_j;\ j \not\equiv i-1, i, i+1 \pmod r) & r \geqq 5\,. \end{cases}$$

When $d = 3$, Δ is again necessarily a triangulation of the ordinary 2-sphere. There are, however, many combinatorially different ones even when the number r of the vertices is fixed. By Steinitz's theorem (cf Grünbrum [G_2, Chapter 13]), the combinatorial classification of Δ's coincides with that of 3-dimensional simplicial convex polytopes, i.e., bounded convex polyhedra in $\boldsymbol{R}^3$ whose 2-dimensional faces are triangles. We encountered this classification problem in another context in [MO, Section 9].

(1.11) For simplicity, we adopt the following notation: If $\Xi \subset \Gamma(\varpi)$ is a d-dimensional local subcomplex, then we denote d-dimensional cones in Ξ by $\alpha, \alpha', \cdots$, $(d-1)$-dimensional cones in Ξ by $\beta, \beta', \cdots$, and $(d-2)$-dimensional cones in Ξ by $\gamma, \gamma', \cdots$. Suppose $\Xi \subset \Gamma(\varpi)$ is a d-dimensional k-spherical subcomplex with ϖ nonsingular. For $\beta \in \Xi_{d-1}$ we see by (1.7), (1.8) and (1.9) that $\mathrm{Star}_\beta(\Xi)$ is isomorphic to a one-dimensional k-spherical subcomplex of the complex of subsets of $\{1, \cdots, r\}\setminus\beta$. Hence by (1.10), there exist exactly two distinct $\alpha, \alpha' \in \Xi_d$ satisfying $\alpha, \alpha' > \beta$. On the other hand for $\gamma \in \Xi_{d-2}$, we see again by (1.7), (1.8) and (1.9) that $\mathrm{Star}_\gamma(\Xi)$ is isomorphic to a 2-dimensional k-spherical subcomplex of the complex of subsets of $\{1, \cdots, r\}\setminus\gamma$. Again by (1.10), the latter is necessarily isomorphic to a subdivision of a circle into v arcs with $v \geqq 3$. Thus $\mathrm{Star}_\gamma(\Xi)$ consists exactly of $\gamma \in \Xi_{d-2}$, $\beta_1, \cdots, \beta_v \in \Xi_{d-1}$ and $\alpha_1, \cdots, \alpha_v \in \Xi_d$ such that $\alpha_{i-1} > \beta_i < \alpha_i$ for $1 \leqq i \leqq v$, where we let $\alpha_0 = \alpha_v$. In this case, we call γ *v-valent* and denote $v(\gamma) = v$.

(1.12) For a k-scheme V, we denote, as usual, by $\Theta_V = \mathscr{D}er_k(\mathscr{O}_V)$ the sheaf of germs of k-derivations of the structure sheaf $\mathscr{O}_V$. For a closed subscheme $W \subset V$, we denote by $\mathscr{N}_{W/V}$ the *normal sheaf* of W in V, i.e., the $\mathscr{O}_W$-dual of the conormal sheaf I/I^2, where I is the ideal sheaf of W in V. Recall, furthermore, the following:

DEFINITION. For a k-scheme V and its effective divisor D with the ideal sheaf J, we define the sheaf $\Theta_V(-\log D)$ of germs of *k-derivations with logarithmic zeros along D* to be that of k-derivations δ of $\mathscr{O}_V$ satisfying $\delta(J) \subset J$. Its $\mathscr{O}_V$-dual is denoted by $\Omega_V^1(\log D)$ and is called the sheaf of germs of *differential 1-forms with logarithmic poles along D.*

DEFINITION. For a subcomplex $\Xi \subset \Gamma(\varpi)$ and $\xi \in \Xi$, let $D(\xi)$ be the reduced effective divisor of $V(\xi) = \mathrm{Spec}(k[M \cap \xi])$ defined by $D(\xi) =$

$\sum_\eta V(\eta)$, where η runs through the codimension one faces of ξ (cf. (1.5)).

In connection with this, we will have occasions later to need the decomposition

$$D(\beta) = \sum_{v \geqq 3} D_v(\beta)$$

when Ξ is k-spherical and $\beta \in \Xi_{d-1}$ as follows (cf. (1.11)):

DEFINITION. Suppose Ξ is d-dimensional and k-spherical with ϖ nonsingular. For $\beta \in \Xi_{d-1}$ and $v \geqq 3$, we denote by $D_v(\beta)$ the divisor of $V(\beta)$ defined by $D_v(\beta) = \sum_\gamma V(\gamma)$, where γ runs through the v-valent $(d-2)$-dimensional faces of β.

To an element n of the $\boldsymbol{Z}$-module N dual to M, we have a k-derivation δ_n of $k[M]$ defined by

$$\delta_n(e(m)) = \langle m, n\rangle e(m)$$

for all $m \in M$. We see easily that the map $n \mapsto \delta_n$ induces an isomorphism from $k \otimes_{\boldsymbol{Z}} N$ to the Lie algebra Lie(T) of T. More concretely, let $\{m_1, \cdots, m_r\}$ be a $\boldsymbol{Z}$-basis of M and let $\{n_1, \cdots, n_r\}$ be the dual basis of N. Then in terms of the coordinates $t_i = e(m_i)$ of T, we have $\delta_{n_i} = t_i\partial/\partial t_i$.

The k-derivation δ_n obviously preserves the subring $k[M \cap \varpi]$ and its M-homogeneous prime ideals $\mathfrak{p}(\xi) = k[M \cap \varpi \setminus M \cap \xi]$ defining $V(\xi)$ for $\xi \in \Gamma(\varpi)$ (cf. (1.5)). Thus δ_n induces a global section of $\Theta_{V(\xi)}$. The divisor $D(\xi)$ on $V(\xi)$ is defined by the ideal $k[M \cap$ (the relative interior of $\xi)]$ of $k[M \cap \xi]$, which is preserved by δ_n. Thus we have a canonical homomorphism

$$\mathcal{O}_{V(\xi)} \otimes_{\boldsymbol{Z}} N \to \Theta_{V(\xi)}(-\log D(\xi))$$

which kills the elements of $\mathcal{O}_{V(\xi)} \otimes_{\boldsymbol{Z}} (N \cap \xi^\perp)$. We have:

PROPOSITION. *For $\xi \in \Gamma(\varpi)$, there are canonical isomorphisms*

$$\mathcal{O}_{V(\xi)} \otimes_{\boldsymbol{Z}} (N/N \cap \xi^\perp) \xrightarrow{\sim} \Theta_{V(\xi)}(-\log D(\xi))$$

$$\mathcal{O}_{V(\xi)} \otimes_{\boldsymbol{Z}} \boldsymbol{Z}(M \cap \xi) \xrightarrow{\sim} \Omega^1_{V(\xi)}(\log D(\xi)) .$$

PROOF. It is enough to prove the isomorphy of the first homomorphism on the ring level for $A = k[M \cap \xi]$. Let $I \subset A$ be the ideal defining $D(\xi)$. Then we have $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$, where $\eta_1, \cdots, \eta_s$ are the codimension one faces of ξ and $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ are the corresponding M-homogeneous height one prime ideals of A defined by $\mathfrak{p}_i = k[M \cap \xi \setminus M \cap \eta_i]$. There exist primitive elements $n_1, \cdots, n_s$ in N, having nonnegative values on ξ and being uniquely determined mod $N \cap \xi^{\perp}$, such that $\eta_i = \xi \cap \{n_i\}^{\perp}$. By [MO, p. 18, Remark], we see that for each $m \in M \cap \xi$, we have a primary decomposition

$$Ae(m) = \mathfrak{p}_1^{(\mu_1)} \cap \cdots \cap \mathfrak{p}_s^{(\mu_s)} ,$$

where $\mathfrak{p}^{(\mu)}$ is the μ-th symbolic power of $\mathfrak{p}$ and $\mu_i = \langle m, n_i \rangle$. Clearly a k-derivation δ of A with $\delta(I) \subset I$ preseves the symbolic powers. Thus we have $\delta(e(m)) = u(m)e(m)$ for an element $u(m) \in A$. Obviously u is an additive semigroup homomorphism from $M \cap \xi$ to A. Thus u can be uniquely extended to a group homomorphism from $\boldsymbol{Z}(M \cap \xi)$ to A, hence to an element of $A \otimes_{\boldsymbol{Z}} (N/N \cap \xi^{\perp})$.

(1.13) Let $\Xi \subset \Gamma(\varpi)$ be a d-dimensional subcomplex for a convex rational polyhedral cone ϖ. Ξ, considered as an ordered set is almost never a directed set. Nevertheless, we can consider the projective system $\{\mathcal{O}_{V(\xi)}\}_{\xi \in \Xi}$ of $\mathcal{O}_{Y(\Xi)}$-algebras with the restriction maps $\mathcal{O}_{V(\xi)} \to \mathcal{O}_{V(\eta)}$ for $\xi > \eta$ as the transition homomorphisms. Sometimes convenient is the following fact, whose proof can be easily carried out on the ring level:

PROPOSITION. *For a subcomplex $\Xi \subset \Gamma(\varpi)$ for a convex rational polyhedral cone ϖ, we have a canonical isomorphism*

$$\mathcal{O}_{Y(\Xi)} \xrightarrow{\sim} \operatorname*{proj\,lim}_{\xi \in \Xi} \mathcal{O}_{V(\xi)} .$$

Ishida [I_3, Corollary 3.5] showed, in the dual formulation, that $Y(\Xi)$ is Cohen-Macaulay if and only if

$$H_j(\mathrm{Star}_\zeta(\Xi), k) = 0$$

for all $\zeta \in \Xi$ and all $j \neq d = \dim \Xi$. If this condition is satisfied, we call Ξ *k-Cohen-Macaulay*. Thus k-semispherical Ξ is k-Cohen-Macaulay (cf. (1.8)). Note that if Ξ is k-Cohen-Macaulay, then it is equidimensional, i.e., for each $\xi \in \Xi$, there exists $\alpha \in \Xi_d$ with $\alpha > \xi$. In the k-Cohen-Macaulay situation, the projective limits taken over Ξ is determined by the information on codimensions zero and one as follows:

LEMMA. *Let $\{X_\xi\}_{\xi \in \Xi}$ be a projective system of sets with the transition maps $f_{\eta\xi}\colon X_\xi \to X_\eta$ for $\xi > \eta$. If Ξ is d-dimensional k-Cohen-Macaulay, then there exists a canonical bijection*

$$\operatorname*{proj\,lim}_{\xi \in \Xi} X_\xi \xrightarrow{\sim} \operatorname*{proj\,lim}_{\eta \in \Xi_d \cup \Xi_{d-1}} X_\eta .$$

PROOF. We have canonical maps $f_\eta\colon \operatorname{proj\,lim}_{\xi \in \Xi} X_\xi \to X_\eta$, hence

$$\operatorname*{proj\,lim}_{\xi \in \Xi} X_\xi \to \operatorname*{proj\,lim}_{\mu \in \Xi_d \cup \Xi_{d-1}} X_\eta .$$

We first claim f to be injective. Indeed, if $x = (x_\xi)$ and $x' = (x'_\xi)$ are in $\operatorname{proj\,lim}_{\xi \in \Xi} X_\xi$ with $x_\eta = x'_\eta$ for each $\eta \in \Xi_d \cup \Xi_{d-1}$, then $x_\xi = x'_\xi$ for each $\xi \in \Xi$ since by the equidimensionality, there exists $\alpha \in \Xi_d$ with $\alpha > \xi$. Let us show the surjectivity of f. Let $x = (x_\eta)$ be an element of the projective limit for $\Xi_d \cup \Xi_{d-1}$. Each $\zeta \in \Xi$ is a face of some $\eta' \in \Xi_d \cup \Xi_{d-1}$ by the equidimensionality. Let us fix one such η' for each ζ and define x_ζ to be $f_{\zeta\eta'}(x_{\eta'})$. We claim that x_ζ is independent of the choice of η', hence (x_ζ) is an element of the projective limit for Ξ. Indeed, consider the subsets of $\operatorname{Star}_\zeta(\Xi)$ defined by

$$\Xi' = \{\eta \in \Xi_d \cup \Xi_{d-1};\ \eta > \zeta,\ f_{\zeta\eta}(x_\eta) = x_\zeta\}\ ,$$
$$\Xi'' = \{\eta \in \Xi_d \cup \Xi_{d-1};\ \eta > \zeta,\ f_{\zeta\eta}(x_\eta) \neq x_\zeta\}\ .$$

Then we obviously have $\Xi' \cap \Xi'' = \emptyset$ and $\Xi' \cup \Xi'' = (\Xi_d \cup \Xi_{d-1}) \cap \operatorname{Star}_\zeta(\Xi)$. Ξ' is nonempty, since it contains η'. Thus by Ishida [I_3, Proposition 5.3], we conclude that $\Xi'' = \emptyset$ and $\Xi' = (\Xi_d \cup \Xi_{d-1}) \cap \operatorname{Star}_\zeta(\Xi)$.

Combining our Proposition and Lemma, we have:

COROLLARY. *If $Y(\Xi)$ is d-dimensional and is k-Cohen-Macaulay, then we have a canonical isomorphism*

$$\mathcal{O}_{Y(\Xi)} \xrightarrow{\sim} \operatorname*{proj\,lim}_{\eta \in \Xi_d \cup \Xi_{d-1}} \mathcal{O}_{V(\eta)}\ .$$

(1.14) Let X be a k-scheme. Illusie [I_1] defined the cotangent complex $L^X_\cdot$ of $\mathcal{O}_X$-modules using the homotopical algebraic technique. This complex plays the following role in the theory of infinitesimal deformations of X.

Let R be an Artin local k-algebra with the residue field k. Then a deformation of X over R is an R-scheme X_R, flat over R, such that $X_R \otimes k = X$. Let R' be another Artin local k-algebra with a surjective local k-homomorphism $R' \to R$ whose kernel has length one. Given a deformation X_R of X over R, the obstruction for lifting X_R to a deformation $X_{R'}$ over R' lies in the second hyperextension group

$$\operatorname{Ext}^2_{\mathcal{O}_X}(L^X_\cdot, \mathcal{O}_X)\ .$$

When the obstruction vanishes, the set of liftings $X_{R'}$ of X_R to R' is a principal homogeneous space under the first hyperextension group

$$\operatorname{Ext}^1_{\mathcal{O}_X}(L^X_\cdot, \mathcal{O}_X)\ .$$

Incidentally, the set of R'-automorphisms of $X_{R'}$ inducing the identity on X_R is the group

$$\operatorname{Ext}^0_{\mathcal{O}_Y}(L^X_\cdot, \mathcal{O}_X)\ .$$

The computation of these hyperextension groups is reduced to that of the local hyperextension sheaves

$$\mathcal{E}xt^{j}_{X}(L^{X}_{\cdot}, \mathcal{O}_X) \quad j = 0, 1, 2$$

via the local-global spectral sequence

$$H^{i}(X, \mathcal{E}xt^{j}_{\mathcal{O}_X}(L^{X}_{\cdot}, \mathcal{O}_X)) \Rightarrow \mathrm{Ext}^{i+j}_{X}(L^{X}_{\cdot}, \mathcal{O}_X) .$$

Fortunately, by Rim [R], we can compute these local hyperextension sheaves as follows:

(0) $\mathcal{E}xt^{0}_{X}(L^{X}_{\cdot}, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = \mathcal{D}er_k(\mathcal{O}_X)$.

(1) If X is reduced, then $\mathcal{E}xt^{1}_{X}(L^{X}_{\cdot}, \mathcal{O}_X) = \mathcal{E}xt^{1}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ (cf. Artin [A, Proposition 6.1]).

(1′) Suppose $X = \mathrm{Spec}(S)$ is affine. If $S = P/J$ for a polynomial k-algebra P and an ideal J of P, then the $\mathcal{O}_X$-module corresponding to the S-module

$$\mathrm{Ext}^1_S(L^S_{\cdot}, S) = \mathrm{coker}\,[\mathrm{Der}_k(P, S) \to \mathrm{Hom}_S(J/J^2, S)]$$

is exactly $\mathcal{E}xt^{1}_{X}(L^{X}_{\cdot}, \mathcal{O}_X)$.

(2) Let $X = \mathrm{Spec}(S)$ with $S = P/J$ as above. Suppose there exists a surjective P-homomorphism $F \to J \to 0$ from a free P-module F of finite rank. Consider the Koszul complex constructed out of this surjection

$$\mathrm{Kosz} = [\cdots \to \wedge^3 F \to \wedge^2 F \to F \to P] .$$

Then $\mathcal{E}xt^{2}_{\mathcal{O}_X}(L^{X}_{\cdot}, \mathcal{O}_X)$ is exactly the $\mathcal{O}_X$-module corresponding to the S-module

$$\begin{aligned}\mathrm{Ext}^2_S(L^S_{\cdot}, S) &= \mathrm{coker}\,[H^1(\mathrm{Kosz}, S) \to \mathrm{Hom}_S(H_1(\mathrm{Kosz}), S)] \\ &= \mathrm{coker}\,[\mathrm{Hom}_P(F, S) \to \mathrm{Hom}_P(H_1(\mathrm{Kosz}), S)] .\end{aligned}$$

2. The announcement of the main results. We are ready to state our main theorems and their consequences, whose proofs will be completed at the end of the paper.

We need much less condition on Y for our description of the zeroth hyperextension sheaf $\mathcal{D}er_k(\mathcal{O}_Y)$ than for that of the higher hyperextension sheaves.

THEOREM 2.1. *Let $\Xi \subset \Gamma(\varpi)$ be a subcomplex for a convex rational polyhedral cone ϖ. Then we have a canonical injective homomorphism*

$$\operatorname*{proj\,lim}_{\xi \in \Xi} \Theta_{V(\xi)}(-\log D(\xi)) \hookrightarrow \mathcal{D}er_k(\mathcal{O}_{Y(\Xi)})$$

which is an isomorphism if and only if one of the following equivalent conditions are satisfied:

(1) *Each $\eta \in \Xi$ is the intersection of all the maximal elements $\xi \in \Xi$*

satisfying $\xi > \eta$.

(2) *The cardinality of* $\mathrm{Star}_\eta(\Xi)$ *is not two for each* $\eta \in \Xi$.

COROLLARY 2.2. *If* $\Xi \subset \Gamma(\varpi)$ *is a d-dimensional k-Cohen-Macaulay subcomplex* (cf. (1.13)) *for a convex rational polyhedral cone* ϖ *and if* $H_d(\mathrm{Star}_\beta(\Xi), k) \neq 0$ *for any* $\beta \in \Xi_{d-1}$, *then we have canonical isomorphisms*

$$\mathscr{D}er_k(\mathscr{O}_{Y(\Xi)}) \xleftarrow{\sim} \operatorname{proj\,lim}_{\xi \in \Xi} \Theta_{V(\xi)}(-\log D(\xi))$$

$$\xrightarrow{\sim} \operatorname{proj\,lim}_{\eta \in \Xi_d \cup \Xi_{d-1}} \Theta_{V(\eta)}(-\log D(\eta)) .$$

REMARK. This is, in a sense, a generalization of a result obtained by Nakamura [N_1, Proposition 2.5]. Note that by (1.8), the conditions in Corollary 2.2 are satisfied if Ξ is k-spherical. See Ishida [I_4] for a long exact sequence extending the isomorphism in Corollary 2.2.

EXAMPLE. Let X be an algebraic surface over k with an ordinary double curve C as the only singularity. Let $\nu: \tilde{X} \to X$ be the normalization. If C is nonsingular, then $\nu^{-1}(C)$ is the disjoint union of isomorphic nonsingular curves C' and C'' on $\tilde{X}$ isomorphic to C via ν. We have an exact sequence

$$0 \to \Theta_{\tilde{X}}(-\log(C' + C'')) \to \Theta_{\tilde{X}} \to \mathscr{N}_{C'/X} \oplus \mathscr{N}_{C''/X} \to 0 .$$

By an easy local calculation, we have a canonical isomorphism

$$\Theta_X \cong \ker[\nu_* \Theta_{\tilde{X}}(-\log(C' + C'')) \to \Theta_C] ,$$

where the homomorphism is the difference of the two obvious restriction maps. Our result generalizes this to special but combinatorially more complicated varieties of the form $Y(\Xi)$. Obviously, X above is locally isomorphic to $Y(\Xi)$ for a very simple Ξ.

Recall that for a closed subscheme W of a scheme V, we denoted by $\mathscr{N}_{W/V}$ the normal sheaf of W in V (cf. (1.12)). In stating our description of $\mathscr{E}xt^1_X(L^X_\cdot, \mathscr{O}_Y) = \mathscr{E}xt^1_{\mathscr{O}_Y}(\Omega^1_Y, \mathscr{O}_Y)$, we need the following:

DEFINITION. For a convex rational polyhedral cone ϖ, let $\Xi \subset \Gamma(\varpi)$ be a d-dimensional k-spherical subcomplex with $Y(\Xi)$ having the nonsingular normalization.

(1) For $\beta \in \Xi_{d-1}$, let α, α' be the two distinct elements of Ξ_d satisfying $\alpha > \beta$ and $\alpha' > \beta$. Then we define an invertible $\mathscr{O}_{V(\beta)}$-module $\mathscr{G}(\beta)$ by

$$\mathscr{G}(\beta)=\mathscr{N}_{V(\beta)/V(\alpha)}\otimes\mathscr{N}_{V(\beta)/V(\alpha')}\otimes\mathscr{O}_{V(\beta)}(D_3(\beta)-\sum_{v\geqq 5}D_v(\beta)) .$$

(2) For $\gamma\in\Xi_{d-2}$, we define a locally free $\mathscr{O}_{V(\gamma)}$-module $\mathscr{G}(\gamma)$ as follows: If $v(\gamma)\geqq 5$, then we let $\mathscr{G}(\gamma)=0$. If $v(\gamma)=4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ are the distinct elements of Ξ_{d-1} satisfying $\beta_i>\gamma$ and $\beta_1\cup\beta_3, \beta_2\cup\beta_4\notin\Xi$, we let

$$\mathscr{G}(\gamma, \beta_1)=\mathscr{G}(\gamma, \beta_3)=\mathscr{N}_{V(\gamma)/V(\beta_2)}\otimes\mathscr{N}_{V(\gamma)/V(\beta_4)}\otimes\mathscr{O}_{V(\gamma)}(D(\gamma))$$
$$\mathscr{G}(\gamma, \beta_2)=\mathscr{G}(\gamma, \beta_4)=\mathscr{N}_{V(\gamma)/V(\beta_1)}\otimes\mathscr{N}_{V(\gamma)/V(\beta_3)}\otimes\mathscr{O}_{V(\gamma)}(D(\gamma)) .$$

Then

$$\mathscr{G}(\gamma)=\{\mathscr{G}(\gamma, \beta_1)\oplus\mathscr{G}(\gamma, \beta_3)\}/\text{diagonal} \oplus\{\mathscr{G}(\gamma, \beta_2)\oplus\mathscr{G}(\gamma, \beta_4)\}/\text{diagonal}.$$

Finally, for $v(\gamma)=3$, let $\beta_1, \beta_2, \beta_3$ be the distinct elements in Ξ_{d-1} with $\beta_i>\gamma$. Let

$$\mathscr{G}(\gamma, \beta_1)=\mathscr{G}(\gamma, \beta_2)=\mathscr{G}(\gamma, \beta_3) =\mathscr{N}_{V(\gamma)/V(\beta_1)}\otimes\mathscr{N}_{V(\gamma)/V(\beta_2)}\otimes\mathscr{N}_{V(\gamma)/V(\beta_3)}\otimes\mathscr{O}_{V(\gamma)}(D(\gamma)) .$$

Then

$$\mathscr{G}(\gamma)=\{\mathscr{G}(\gamma, \beta_1)\oplus\mathscr{G}(\gamma, \beta_2)\oplus\mathscr{G}(\gamma, \beta_3)\}/\text{diagonal} .$$

(3) We define an $\mathscr{O}_{Y(\Xi)}$-homomorphism

$$\varepsilon:\bigoplus_{\beta\in\Xi_{d-1}}\mathscr{G}(\beta)\to\bigoplus_{\gamma\in\Xi_{d-2}}\mathscr{G}(\gamma)$$

as follows: Let $\beta\in\Xi_{d-1}$ and $\gamma\in\Xi_{d-2}$. The (β, γ)-component of ε is zero if either $\beta\not>\gamma$ or $v(\gamma)\geqq 5$. If $v(\gamma)=4$ and $\beta>\gamma$, then $\beta=\beta_i$ for some $i=1, 2, 3, 4$ as above. In this case, the (β, γ)-component of ε is the composite $\mathscr{G}(\beta)\to\mathscr{O}_{V(\gamma)}\otimes\mathscr{G}(\beta)\to\mathscr{G}(\gamma, \beta)\to\mathscr{G}(\gamma)$ of the restriction map, the obvious injection and the map induced by the inclusion into the (γ, β_i)-factor. Finally if $v(\gamma)=3$ and $\beta>\gamma$, then $\beta=\beta_i$ for some $i=1, 2, 3$. In this case, the (β, γ)-component of ε is the composite $\mathscr{G}(\beta)\to\mathscr{O}_{V(\gamma)}\otimes\mathscr{G}(\beta)\to\mathscr{G}(\gamma, \beta)\to\mathscr{G}(\gamma)$ exactly as above.

THEOREM 2.3. *Let ϖ be a convex rational polyhedral cone. For a d-dimensional k-spherical subcomplex $\Xi\subset\Gamma(\varpi)$ with $Y(\Xi)$ having the nonsingular normalization, we have a canonical isomorphism*

$$\mathscr{E}xt^1_{\mathscr{O}_{Y(\Xi)}}(L^{Y(\Xi)}_{\cdot}, \mathscr{O}_{Y(\Xi)})\xrightarrow{\sim}\ker\left[\bigoplus_{\beta\in\Xi_{d-1}}\mathscr{G}(\beta)\xrightarrow{\varepsilon}\bigoplus_{\gamma\in\Xi_{d-2}}\mathscr{G}(\gamma)\right].$$

REMARK. This is a generalization of a result obtained by Nakamura [N_1, Section 5].

THEOREM 2.4. *For a $\boldsymbol{Z}$-basis $\{m_1, \cdots, m_r\}$ of M, let $\varpi = \boldsymbol{R}_{\geqq 0}m_1 + \cdots + \boldsymbol{R}_{\geqq 0}m_r$ and let $\Xi \subset \Gamma(\varpi)$ be a d-dimensional k-spherical subcomplex. Then we have*

$$\mathscr{E}xt^2_{Y(\Xi)}(L_\bullet^{Y(\Xi)}, \mathscr{O}_{Y(\Xi)}) = 0$$

if and only if $H_{d-1}(\Phi(s', s''), k) = 0$ for all $s' \in \Xi$ and $s'' \in \Gamma(\varpi)$ satisfying $s' \cap s'' = \{0\}$ and $|s''| \geqq 2$, where by freely identifying the elements of $\Gamma(\varpi)$ with the subsets of $\{1, \cdots, r\}$ as in (1.9), *we define $|s''|$ to be the cardinality of s'' and*

$$\Phi(s', s'') = \{\xi \in \Xi;\ \xi > s',\ \xi \cup s'' \setminus \{l\} \in \Xi \quad \text{for all} \quad l \in s''\}\ .$$

When $d = \dim \Xi \leqq 3$, we can simplify this condition further in the following way, where by (1.10), we identify Ξ with Ξ_Δ for a $(d-1)$-dimensional finite simplicial k-homology sphere Δ, which is nothing but a triangulation of the ordinary $(d-1)$-sphere now:

COROLLARY 2.5. *For $d \leqq 3$, let Δ be a combinatorial triangulation of the $(d-1)$-sphere. Then for $Y = Y(\Xi_\Delta)$, we have*

$$\mathscr{E}xt^2_Y(L_\bullet^Y, \mathscr{O}_Y) = 0$$

if and only if one of the following is satisfied:

(1) $d \leqq 1$.

(2) *$d = 2$ and $v(\{0\}) \leqq 5$, i.e., Y is an elliptic polygonal r-cone with $r \leqq 5$.*

(3) *$d = 3$ and the stereographic projection, from one of the vertices, of Δ to the plane looks like one of the diagrams* except [8–14] *in Figure* 1, *where the names* [4–1] *through* [8–14] *are those used in* [MO, p. 77].

REMARK. Among those listed in Corollary 2.5, only the following are complete intersections (cf. (1.10) and below), hence the vanishing of the $\mathscr{E}xt^2$ in these cases are already known:

(1) always.

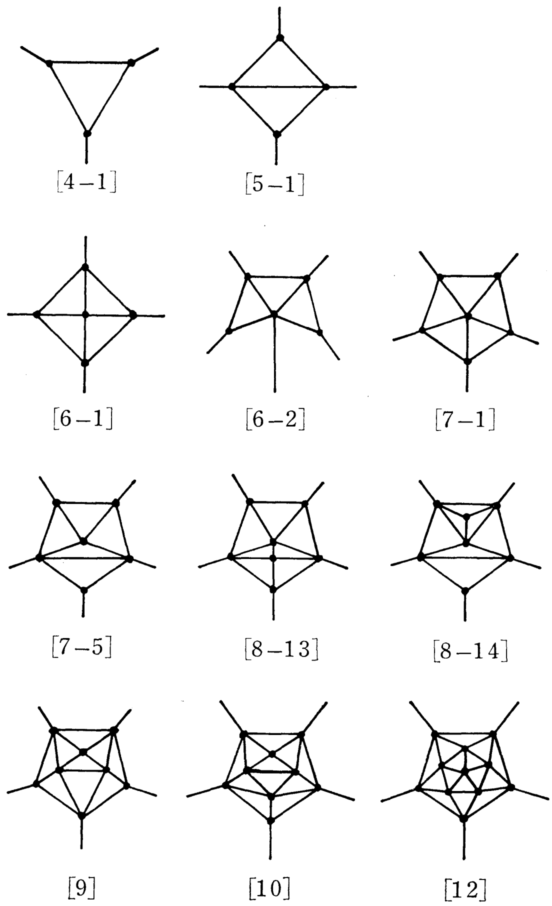
(2) with $v(\{0\}) = 3$ or 4.

(3) [4–1] $S = k[x_1, \cdots, x_4]/(x_1x_2x_3x_4)$

[5–1] $S = k[x_1, \cdots, x_5]/(x_1x_2x_3, x_4x_5)$

[6–1] $S = k[x_1, \cdots, x_6]/(x_1x_2, x_3x_4, x_5x_6)$.

REMARK. As we see in Proposition 6.3, the eleven triangulations [4–1] through [12] in Figure 1 are exactly those for which each vertex is incident to five or less edges.



The triangulations of the 2-sphere with all the vertices having the valency $\leqq 5$.

FIGURE 1

3. Barycentric subdivisions. In computing the higher hyperextension sheaves in question, we will later need the results in this section on the barycentric subdivision, which is an adaptation of the usual one to our situation.

Let Φ be a partially ordered set. Later, we will be mainly concerned with the case where Φ is a local subcomplex of $\Gamma(\varpi)$ for a convex rational polyhedral cone ϖ.

DEFINITION. For $i \geqq 0$, define the set $\mathrm{Sd}_i(\Phi)$ as follows: $\mathrm{Sd}_0(\Phi)$ is the one-element set consisting of the symbol ( ). For $i \geqq 1$, $\mathrm{Sd}_i(\Phi)$ is the set of strictly increasing sequences $(\xi_1 \lneqq \xi_2 \lneqq \cdots \lneqq \xi_i)$ of length i in Φ. $\mathrm{Sd}(\Phi) = \coprod_{i \geqq 0} \mathrm{Sd}_i(\Phi)$ is called the *barycentric subdivision* of Φ. We regard it as an abstract complex by taking faces to be subsequences. The chain complex of $\boldsymbol{Z}$-modules $C_.(\mathrm{Sd}(\Phi))$ is defined by letting $C_i(\mathrm{Sd}(\Phi))$ to be the free $\boldsymbol{Z}$-module with the basis $\mathrm{Sd}_i(\Phi)$ and by letting the boundary map $\partial: C_{i+1} \to C_i$ to be

$$\partial(\xi_1 \lneqq \cdots \lneqq \xi_{i+1}) = \sum_{1 \leqq l \leqq i+1} (-1)^{l+1}(\xi_1 \lneqq \cdots \overset{l}{\vee} \cdots \lneqq \xi_{i+1}) .$$

We define the homology group $H_.(\mathrm{Sd}(\Phi))$, the cohomology group $H^{\cdot}(\mathrm{Sd}(\Phi))$ and $H_.(\mathrm{Sd}(\Phi), k)$, $H^{\cdot}(\mathrm{Sd}(\Phi), k)$ for a field k in the usual manner. We call $\mathrm{Sd}(\Phi)$ homologically trivial (resp. k-homologically trivial) if $H_.(\mathrm{Sd}(\Phi)) = 0$ (resp. $H_.(\mathrm{Sd}(\Phi), k) = 0$).

It is easy to see that if Φ is empty, then we have

$$H_i(\mathrm{Sd}(\Phi)) = \begin{cases} \boldsymbol{Z} & i = 0 \\ 0 & i \neq 0 , \end{cases}$$

while if Φ is nonempty, then we have $H_0(\mathrm{Sd}(\Phi)) = 0$.

REMARK. It is sometimes more convenient to consider, instead, another barycentric subdivision $\widetilde{\mathrm{Sd}}(\Phi)$ consisting of all the *nondecreasing* sequences $(\xi_1 < \xi_2 < \cdots < \xi_i)$ in Φ. Thus $\mathrm{Sd}(\Phi)$ is a subcomplex of $\widetilde{\mathrm{Sd}}(\Phi)$. It is standard to show that the induced map $C_.(\mathrm{Sd}(\Phi)) \to C_.(\widetilde{\mathrm{Sd}}(\Phi))$ is a quasi-isomorphism, i.e., the complement $D(\Phi) = \widetilde{\mathrm{Sd}}(\Phi) \setminus \mathrm{Sd}(\Phi)$ consisting of the degenerate sequences is homologically trivial. Just define the chain homotopy for $C_.(D(\Phi))$ by sending $(\xi_1 < \cdots < \xi_i)$ to $(-1)^{l+1}(\xi_1 < \cdots < \xi_l = \xi_l < \xi_{l+1} < \cdots < \xi_i)$, where l is the smallest j such that $\xi_j = \xi_{j+1}$.

The following lemmas are standard and useful below.

LEMMA 3.1. *Let Φ' be a subset of an ordered set Φ. Suppose there exists an order preserving retraction map $\rho: \Phi \to \Phi'$, i.e., $\rho \circ i = \mathrm{id}_{\Phi'}$ for the inclusion map $i: \Phi' \to \Phi$. If either* (1) *$\rho(\xi) > \xi$ for all $\xi \in \Phi$ or* (2) *$\rho(\xi) < \xi$ for all $\xi \in \Phi$, then i induces an isomorphism*

$$i_*: H_.(\mathrm{Sd}(\Phi')) \xrightarrow{\sim} H_.(\mathrm{Sd}(\Phi)) .$$

PROOF. By the above remark, we may replace Sd by $\widetilde{\mathrm{Sd}}$ allowing nondecreasing sequences. Then, as usual, we construct the chain homotopy s for $C_.(\widetilde{\mathrm{Sd}}(\Phi))$ connecting $i \circ \rho$ and id_Φ by sending $(\xi_1 < \cdots < \xi_j)$ to $s(\xi_1 < \cdots < \xi_j) = \sum_{1 \leqq l \leqq j} (-1)^l (\xi_1 < \cdots < \xi_l < \rho(\xi_l) < \cdots < \rho(\xi_j))$ in case (1), and to $s(\xi_1 < \cdots < \xi_j) = \sum_{1 \leqq l \leqq j} (-1)^l (\rho(\xi_1) < \cdots < \rho(\xi_l) < \xi_l < \cdots < \xi_j)$ in case (2).

LEMMA 3.2. *If Φ has either the smallest element or the largest element, then* $\mathrm{Sd}(\Phi)$ *is homologically trivial.*

PROOF. Apply Lemma 3.1 (1) (resp. (2)) to the subset Φ' consisting of the largest (resp. smallest) element only, with ρ sending every element of Φ to the unique element of Φ'. We are done, since $\mathrm{Sd}(\Phi')$ is obviously homologically trivial.

The proof of the following lemma is left to the reader.

LEMMA 3.3. *Let Ξ' be a subcomplex of a local subcomplex $\Xi \subset \Gamma(\varpi)$ for a convex rational polyhedral cone ϖ. Let $d' = \dim \Xi'$ and $d = \dim \Xi$. Then we have the following:*

(i) *There is a decreasing filtration*

$$\mathrm{Sd}(\Xi \setminus \Xi') = F^{d'+1} \subset F^{d'} \subset \cdots \subset F^{j+1} \subset F^{j} \subset \cdots \subset \mathrm{Sd}(\Xi)$$

by the subcomplexes defined by $F^j = \{(\xi_1 \lneqq \cdots \lneqq \xi_i) \in \mathrm{Sd}(\Xi);$ either $\xi_1 \notin \Xi'$, or $\xi_1 \in \Xi'$ and $\dim \xi_1 \geqq j\}$. In particular, $F^j = \mathrm{Sd}(\Xi)$ if $j \leqq \min\{l; \Xi'_l$ is nonempty$\}$. Moreover, the quotient complex $C_{\cdot}(F^j \setminus F^{j+1})$ is of the form

$$C_i(F^j \setminus F^{j+1}) = \begin{cases} 0 & i = 0 \\ \bigoplus_{\zeta \in \Xi', \dim \zeta = j} C_{i-1}(\mathrm{Sd}(\mathrm{Star}_\zeta(\Xi) \setminus \{\zeta\})) & i \geqq 1 . \end{cases}$$

(ii) *There is an increasing filtration*

$$\mathrm{Sd}(\Xi') \subset \cdots \subset F'_j \subset F'_{j+1} \subset \cdots \subset F'_d = \mathrm{Sd}(\Xi)$$

by the subcomplexes defined by $F'_j = \{(\xi_1 \lneqq \cdots \lneqq \xi_i) \in \mathrm{Sd}(\Xi);$ either $\xi_i \in \Xi'$, or $\xi_i \notin \Xi'$ and $\dim \xi_i \leqq j\}$. In particular, $F'_j = \mathrm{Sd}(\Xi')$ if $j < \min\{l; \Xi_l \setminus \Xi'_l$ is nonempty$\}$. Moreover, the quotient complex $C_{\cdot}(F'_j \setminus F'_{j-1})$ is of the form

$$C_i(F'_j \setminus F'_{j-1}) = \begin{cases} 0 & i = 0 \\ \bigoplus_{\zeta \in \Xi \setminus \Xi', \dim \zeta = j} C_{i-1}(\mathrm{Sd}(\Xi \cap \Gamma(\zeta) \setminus \{\zeta\})) & i \geqq 1 . \end{cases}$$

When $\Xi \subset \Gamma(\varpi)$ is a local subcomplex with the smallest element for a convex rational polyhedral cone ϖ, we can relate the homology group of the barycentric subdivision and the homology group in (1.6) as follows. It is just the usual comparison theorem in disguise between the homology group of a finite simplicial complex and that of its barycentric subdivision.

LEMMA 3.4. *Let $\Xi \subset \varpi$ be a local subcomplex for a convex rational polyhedral cone ϖ. If Ξ has the smallest element ϕ, then we have canonical isomorphisms for all i*

$$\mathrm{sd} : H_i(\Xi) \xrightarrow{\sim} H_{i-\dim\phi}(\mathrm{Sd}(\Xi \setminus \{\phi\})) ,$$

where the left hand side is the homology group we defined in (1.6).

PROOF. As we saw in (1.6), we have the incidence number $[\xi:\eta]=0, 1$ or -1 for $\xi\in\Xi_i$ and $\eta\in\Xi_{i-1}$ so that $\partial u_\xi=\sum_{\eta\in\Xi_{i-1}}[\xi:\eta]u_\eta$ for a $\boldsymbol{Z}$-basis u_ξ of $\det \boldsymbol{Z}(M\cap\xi)$. We define the map

$$\mathrm{sd}_i\colon C_i(\Xi)\to C_{i-\dim\phi}(\mathrm{Sd}(\Xi\setminus\{\phi\}))$$

by $\mathrm{sd}_0(u_\phi)=(\quad)$ and, for $\xi\in\Xi_i$ with $i\geqq 1$, by

$$\mathrm{sd}_i(u_\xi)=\sum\sigma(\xi_{\dim\phi+1}\lneqq\cdots\lneqq\xi_i)(\xi_{\dim\phi+1}\lneqq\cdots\lneqq\xi_i)\ ,$$

with the summation taken over all $(\xi_{\dim\phi+1}\lneqq\cdots\lneqq\xi_i)\in\mathrm{Sd}(\Xi\setminus\{\phi\})$ satisfying $\xi_i=\xi$, where $\sigma(\xi_{\dim\phi+1}\lneqq\cdots\lneqq\xi_i)\in\boldsymbol{Z}$ is defined as follows: Denoting $\xi_l=\phi$ if $l=\dim\phi$, we let

$$\sigma(\xi_{\dim\phi+1}\lneqq\cdots\lneqq\xi_i)=(-1)^{(i-\dim\phi)(i-\dim\phi-1)/2}\prod_{\dim\phi\leqq l\leqq i-1}[\xi_{l+1}:\xi_l]\ .$$

Note that $\dim\xi_l=l$ for all $\dim\phi\leqq l\leqq i$. It is easy to check that $\mathrm{sd}_{i-1}\circ\partial=\partial\circ\mathrm{sd}_i$, since for any $\zeta\in\Xi_{i-2}$ and $\xi\in\Xi_i$ with $\xi>\zeta$, there exist exactly two $\eta,\eta'\in\Xi_{i-1}$ such that $\xi>\eta>\zeta$ and $\xi>\eta'>\zeta$ and that $[\xi:\eta][\eta:\zeta]+[\xi:\eta'][\eta':\zeta]=0$ (cf. Ishida [I_3, Lemma 1.4]). We thus have the induced homomorphisms $\mathrm{sd}\colon H_i(\Xi)\to H_{i-\dim\phi}(\mathrm{Sd}(\Xi\setminus\{\phi\}))$. We now show these to be isomorphisms by induction on the cardinality of Ξ.

First of all, suppose Ξ has the largest element ξ. We claim that sd are isomorphic in this case. If $\xi\neq\phi$, then $\Xi\setminus\{\phi\}$ also has the largest element, and the right hand side is trivial by Lemma 3.2. If $\xi=\phi$, then $\Xi\setminus\{\phi\}$ is empty, hence the right hand side is $\boldsymbol{Z}$ for $i=\dim\phi$ and zero otherwise, as we saw at the beginning of this section. On the other hand, if $\xi\neq\phi$, then the left hand side is trivial, since Ξ consists of all the faces of ξ containing ϕ (cf. (1.7) and, for instance, Ishida [I_3, the comment immediately after Corollary 2.3]). If $\xi=\phi$, then $\Xi=\{\phi\}$ and the left hand side is $\boldsymbol{Z}$ for $i=\dim\phi$ and zero otherwise. Moreover, u_ϕ is sent by sd to $(\quad)$ by definition.

In the general case, let ξ be an element of the largest dimension in Ξ. We may assume $\xi\neq\phi$, by what we saw above. The $\Xi\setminus\{\phi\}$ is a subcomplex of Ξ with fewer elements. Apply Lemma 3.3 (ii) to the subcomplex $\Xi\setminus\{\xi,\phi\}$ of $\Xi\setminus\{\phi\}$. Since the complement consists of ξ only, we see that the increasing filtration is of the form $\mathrm{Sd}(\Xi\setminus\{\xi,\phi\})=F'_{j-1}\subset F'_j=\mathrm{Sd}(\Xi\setminus\{\phi\})$ with $j=\dim\phi$. Hence we have

$$C_i(F'_j\setminus F'_{j-1})=\begin{cases}0 & i=0\\ C_{i-1}(\mathrm{Sd}(\Xi\cap\Gamma(\xi)\setminus\{\xi,\phi\})) & i\geqq 1\ .\end{cases}$$

Similarly, Lemma 3.3 (ii) applied to $\Xi\cap\Gamma(\xi)\setminus\{\xi,\phi\}\subset\Xi\cap\Gamma(\xi)\setminus\{\phi\}$ yields an increasing filtration $\mathrm{Sd}(\Xi \cap \Gamma(\xi) \setminus \{\xi, \phi\}) = F''_{j-1} \subset F''_j = \mathrm{Sd}(\Xi \cap \Gamma(\xi) \setminus \{\phi\})$ with $j = \dim \xi$. Hence

$$C_i(F''_j \setminus F''_{i-1}) = \begin{cases} 0 & i = 0 \\ C_{i-1}(\mathrm{Sd}(\Xi \cap \Gamma(\xi) \setminus \{\xi, \phi\})) & i \geqq 1 . \end{cases}$$

Thus the homomorphism $C_.(F''_j \setminus F''_{j-1}) \to C_.(F'_j \setminus F'_{j-1})$ induced by the inclusion is an isomorphism. We are done in view of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C.(\Xi \setminus \{\xi\}) & \longrightarrow & C.(\Xi) & \longrightarrow & C.(\{\xi\}) & \longrightarrow & 0 \\ & & \downarrow \mathrm{sd} & & \downarrow \mathrm{sd} & & \downarrow \mathrm{sd} & & \\ 0 & \longrightarrow & C.(F'_{j-1}) & \longrightarrow & C.(F'_j) & \longrightarrow & C.(F'_j \setminus F'_{j-1}) & \longrightarrow & 0 \end{array}$$

whose first column is a quasi-isomorphism by the induction hypothesis, and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C.(\Xi \cap \Gamma(\xi) \setminus \{\xi\}) & \longrightarrow & C.(\Xi \cap \Gamma(\xi)) & \longrightarrow & C.(\{\xi\}) & \longrightarrow & 0 \\ & & \downarrow \mathrm{sd} & & \downarrow \mathrm{sd} & & \downarrow \mathrm{sd} & & \\ 0 & \longrightarrow & C.(F''_{j-1}) & \longrightarrow & C.(F''_j) & \longrightarrow & C.(F''_j \setminus F''_{j-1}) & \longrightarrow & 0 \end{array}$$

the first and second columns, hence the third column, of which are quasi-isomorphisms by the induction hypothesis and our result at the beginning of this proof applied to the complex $\Xi \cap \Gamma(\xi)$ having the largest element ξ.

The following is a modification in our context of the usual *Alexander duality theorem.*

PROPOSITION 3.5. *For a convex rational polyhedral cone* ϖ, *let* $\Xi \subset \Gamma(\Xi)$ *be a d-dimensional k-spherical local subcomplex with the smallest element* ϕ. *Then for subcomplexes* $\Xi \supset \Phi \supset \Phi'$, *we have a canonical isomorphism*

$$H^j(\Phi \setminus \Phi', k) \xrightarrow{\sim} H_{d+1-j}(\mathrm{Sd}(\Xi \setminus \Phi') \setminus \mathrm{Sd}(\Xi \setminus \Phi), k)$$

for all $0 \leqq j \leqq d$. *In particular, we have a canonical isomorphism*

$$H^j(\Phi, k) \xrightarrow{\sim} H_{d-j}(\mathrm{Sd}(\Xi \setminus \Phi), k) \qquad \text{for all } j .$$

PROOF. The second part is the special case of the first with Φ' empty, since the long exact sequence arising from the inclusion $\mathrm{Sd}(\Xi \setminus \Phi) \subset \mathrm{Sd}(\Xi)$ induces a homomorphism $H_{d+1-j}(\mathrm{Sd}(\Xi) \setminus \mathrm{Sd}(\Xi \setminus \Phi), k) \to H_{d-j}(\mathrm{Sd}(\Xi \setminus \Phi), k)$, which is an isomorphism by the k-homological triviality of $\mathrm{Sd}(\Xi)$ in Lemma 3.2.

Let us now construct a canonical duality homomorphism

$$D\colon C^j(\Phi\setminus\Phi', k) \to C_{d+1-j}(\mathrm{Sd}(\Xi\setminus\Phi')\setminus\mathrm{Sd}(\Xi\setminus\Phi), k)\ .$$

Since Ξ is d-dimensional and k-spherical, we have $H_d(\Xi, k) \cong k$. Hence by Ishida's result quoted in (1.8), there exists a map $\varepsilon\colon \Xi_d \to k^*$, the orientation, such that for any $\zeta \in \Xi$, the element $\sum_\alpha \varepsilon(\alpha)u_\alpha$ with α running through the d-dimensional cones of $\mathrm{Star}_\zeta(\Xi)$ gives rise to the k-basis of $H_d(\mathrm{Star}_\zeta(\Xi), k)$. In particular for any $\beta \in \Xi_{d-1}$, we have $\varepsilon(\alpha)[\alpha\colon \beta] + \varepsilon(\alpha')[\alpha'\colon \beta] = 0$ for the two d-dimensional cones α, α' satisfying $\alpha, \alpha' > \beta$. Let $\{u_\xi^*\}$ be the k-basis of $C^\cdot(\Phi\setminus\Phi', k)$ dual to $\{u_\xi\}$. Then for $\xi \in \Phi\setminus\Phi'$ with $\dim \xi = j$, we let

$$D(u_\xi^*) = \sum \varepsilon(\xi_j \lneqq \cdots \lneqq \xi_d)(\xi_j \lneqq \cdots \lneqq \xi_d)$$

with the summation taken over all $(\xi_j \lneqq \cdots \lneqq \xi_d)$ in $\mathrm{Sd}(\Xi\setminus\Phi')\setminus\mathrm{Sd}(\Xi\setminus\Phi)$ satisfying $\xi_j = \xi$, where $\varepsilon(\xi_j \lneqq \cdots \lneqq \xi_d) \in k$ is defined as

$$\varepsilon(\xi_j \lneqq \cdots \lneqq \xi_d) = \varepsilon(\xi_d) \prod_{j \leqq l \leqq d-1} [\xi_{l+1}; \xi_l]\ .$$

Note that $\dim \xi_l = l$ for all $j \leqq l \leqq d$. It is easy to check as in Lemma 3.4 that $\partial \circ D = D \circ \delta$, where δ is the coboundary map of $C^\cdot(\Phi\setminus\Phi', k)$ defined by

$$\delta(u_\xi^*) = \sum_\eta [\eta\colon \xi] u_\eta^*$$

for j-dimensional $\xi \in \Phi\setminus\Phi'$, with the summation taken over all $(j+1)$-dimensional $\eta \in \Phi\setminus\Phi'$.

To prove that the above D induces the required isomorphism in the proposition, we may restrict ourselves to the case where Φ' is empty, in view of the long exact sequence arising from the inclusion $\mathrm{Sd}(\Xi)\setminus\mathrm{Sd}(\Xi\setminus\Phi') \subset \mathrm{Sd}(\Xi)\setminus\mathrm{Sd}(\Xi\setminus\Phi)$. Thus we now show that D induces an isomorphism

$$H^j(\Phi, k) \xrightarrow{\sim} H_{d+1-j}(\mathrm{Sd}(\Xi)\setminus\mathrm{Sd}(\Xi\setminus\Phi), k) \quad \text{for all} \quad 0 \leqq j \leqq d$$

by induction on the cardinality of Φ.

If Φ is empty, then both sides vanish, and we are done. If Φ is not empty, let ζ be an element in Φ of the largest dimension and let $\Phi' = \Phi\setminus\{\zeta\}$, which is a subcomplex of Φ. Since D is canonical, we have a homomorphism from the long exact sequence arising from $\Phi' \subset \Phi$ to that arising from $\mathrm{Sd}(\Xi\setminus\Phi')\setminus\mathrm{Sd}(\Xi\setminus\Phi) \subset \mathrm{Sd}(\Xi)\setminus\mathrm{Sd}(\Xi\setminus\Phi)$. Thus by the induction hypothesis applied to Φ', it is enough to show that D induces an isomorphism

$$(*)\qquad H^j(\{\zeta\}, k) \xrightarrow{\sim} H_{d+1-j}(\mathrm{Sd}(\Xi\setminus\Phi')\setminus\mathrm{Sd}(\Xi\setminus\Phi), k)$$

for all $0\leqq j\leqq d$. The left hand side of $(*)$ is zero for $j\neq\dim\zeta$ and is ku_ζ^* for $j=\dim\zeta$. Moreover, $D(u_\zeta^*)=\sum\varepsilon(\xi_j\lneqq\cdots\lneqq\xi_d)(\xi_j\lneqq\cdots\lneqq\xi_d)$ with the summation taken over all $(\xi_j\lneqq\cdots\lneqq\xi_d)$ satisfying $\xi_j=\zeta$.

As for the right hand side of $(*)$, we have two isomorphisms

$$H_{d-j+\dim\zeta}(\mathrm{Star}_\zeta(\Xi), k) \underset{\sim}{\overset{\mathrm{sd}}{\to}} H_{d-j}(\mathrm{Sd}(\mathrm{Star}_\zeta(\Xi)\setminus\{\zeta\}), k)$$
$$\underset{\sim}{\overset{f}{\to}} H_{d+1-j}(\mathrm{Sd}(\Xi\setminus\Phi')\setminus\mathrm{Sd}(\Xi\setminus\Phi), k)\ .$$

Indeed, the first isomorphism sd was obtained in Lemma 3.4. Hence, moreover by the k-sphericity of $\mathrm{Star}_\zeta(\Xi)$, we see that the right hand side of sd is nonzero only when $j=\dim\zeta$, and then has the k-basis

$$\mathrm{sd}\Big(\sum_\alpha\varepsilon(\alpha)u_\alpha\Big)=\sum\varepsilon(\xi_d)\sigma(\xi_{j+1}\lneqq\cdots\lneqq\xi_d)(\xi_{j+1}\lneqq\cdots\lneqq\xi_d)\ .$$

Here the summations are taken over all d-dimensional α in $\mathrm{Star}_\zeta(\Xi)$ and over all $(\xi_{j+1}\lneqq\cdots\lneqq\xi_d)$ in $\mathrm{Sd}(\mathrm{Star}_\zeta(\Xi)\setminus\{\zeta\})$, respectively. On the other hand, the second isomorphism f is induced, up to sign, by the natural bijection from $\mathrm{Sd}(\mathrm{Star}_\zeta(\Xi)\setminus\{\zeta\})$ to $\mathrm{Sd}(\Xi\setminus\Phi')\setminus\mathrm{Sd}(\Xi\setminus\Phi)$ sending $(\eta_1\lneqq\cdots\lneqq\eta_l)$ to $(\zeta\lneqq\eta_1\lneqq\cdots\lneqq\eta_l)$.

Finally, since $\varepsilon(\xi_j\lneqq\xi_{j+1}\lneqq\cdots\lneqq\xi_d)=(-1)^{(d-j)(d-j-1)/2}\varepsilon(\xi_d)\sigma(\xi_{j+1}\lneqq\cdots\lneqq\xi_d)$ for $j=\dim\zeta$, we have

$$D(u_\zeta^*)=(-1)^{(d-j)(d-j-1)/2}f\circ\mathrm{sd}\Big(\sum_\alpha\varepsilon(\alpha)u_\alpha\Big)\ ,$$

hence $(*)$ is an isomorphism.

The following Propositions 3.6 and 3.8 will play a crucial role in Sections 5 and 6, where we compute the hyperextension sheaves in question. At the preliminary stage of our formulation, the discussion with Hiroshi Sato was useful.

PROPOSITION 3.6. *For a convex rational polyhedral cone* ϖ, *let* $\Xi\subset\Gamma(\varpi)$ *be a d-dimensional k-spherical local subcomplex with the smallest element* ϕ. *For* $s', s''\in\Gamma(\varpi)$ *satisfying* $s'\cap s''=\phi$, $s'>\phi$ *and* $s''>\phi$, *we define a local subcomplex* $\Xi(s', s'')$ *of* Ξ *by*

$$\Xi(s', s'')=\{\xi\in\Xi;\ s'\cup\xi\in\Xi,\ \xi\cap s''=\phi,\ \xi\cup s''\notin\Xi\}\ .$$

If either $s'\notin\Xi$ *or* $s''=\phi$, *then* $\Xi(s', s'')$ *is empty. On the other hand, if* $s'\in\Xi$ *and* $s''\neq\phi$, *then the barycentric subdivision* $\mathrm{Sd}(\Xi(s', s''))$ *is k-homologically trivial, hence, in particular,* $\Xi(s', s'')$ *is nonempty.*

PROOF. Obviously, it suffices to prove the k-homological triviality of $\mathrm{Sd}(\Xi(s', s''))$ when $s' \in \Xi$ and $s'' \neq \phi$. If $s'' \notin \Xi$, then $\Xi(s', s'')$ contains the smallest element ϕ, hence we are done by Lemma 3.2. Henceforth, we thus assume $s', s'' \in \Xi$, $s' \cap s'' = \phi$ and $s'' \neq \phi$. Let $\Phi = \{\xi \in \Xi;\ s' \cup \xi \in \Xi,\ \xi \cap s'' = \phi\}$ and $\Phi' = \{\zeta \in \Xi;\ s' \cup \zeta \in \Xi,\ \zeta \cap s'' = \phi,\ \zeta \cup s'' \in \Xi\}$. Obviously, $\Phi' \subset \Phi$ are nonempty subcomplexes of Ξ and $\Xi(s', s'') = \Phi \setminus \Phi'$. We now apply Lemma 3.3 (i). Consider the decreasing filtration $F^{\cdot}$ of $\mathrm{Sd}(\Phi)$ by subcomplexes $F^i = \{(\xi_1 \lneqq \cdots \lneqq \xi_j) \in \mathrm{Sd}(\Phi);$ either $\xi_1 \notin \Phi'$, or $\xi_1 \in \Phi'$ and $\dim \xi_1 \geqq i\}$. Then $F^{i+1} \subset F^i$, $F^{d+1} = \mathrm{Sd}(\Phi \setminus \Phi')$ and $F^0 = \mathrm{Sd}(\Phi)$. Combining Lemma 3.3 (i) and Lemma 3.4, we see that

$$H_j(F^i \setminus F^{i+1}, k) = \begin{cases} 0 & j = 0 \\ \bigoplus_{\zeta \in \Phi'_i} H_{j+i-1}(\mathrm{Star}_\zeta(\Phi), k) & j \geqq 1\,. \end{cases}$$

$F^0 = \mathrm{Sd}(\Phi)$ is k-homologically trivial by Lemma 3.2, since Φ contains the smallest element ϕ. Hence in view of the long exact sequences arising from the inclusions $F^{i+1} \subset F^i$, we have thus reduced the k-homological triviality of $\mathrm{Sd}(\Phi \setminus \Phi') = \mathrm{Sd}(\Xi(s', s''))$ to that of $\mathrm{Star}_\zeta(\Phi)$ for each $\zeta \in \Phi'$.

For $\zeta \in \Phi'$, however, we have $s' \cup \zeta$, $\zeta \cup s'' \in \mathrm{Star}_\zeta(\Xi)$, $(s' \cup \zeta) \cap (\zeta \cup s'') = \zeta$, $\zeta \cup s'' \neq \zeta$ and $\mathrm{Star}_\zeta(\Phi) = \{\xi \in \mathrm{Star}_\zeta(\Xi);\ (s' \cup \zeta) \cup \xi \in \mathrm{Star}_\zeta(\Xi),\ \xi \cap (\zeta \cup s'') = \zeta\}$. Thus replacing $\mathrm{Star}_\zeta(\Xi)$ by Ξ, ζ by ϕ, $s' \cup \zeta$ by s' and $\zeta \cup s''$ by s'', we have reduced ourselves to (ii) of the following:

LEMMA 3.7. *For a convex rational polyhedral cone ϖ, let $\Xi \subset \Gamma(\varpi)$ be a d-dimensional k-spherical local subcomplex with the smallest element ϕ. For $s', s'' \in \Xi$ with $s' \cap s'' = \phi$, let*

$$\mathscr{U}_{s'}(\Xi) = \{\xi \in \Xi;\ s' \cup \xi \in \Xi\}$$
$$\mathscr{V}_{s',s''}(\Xi) = \{\xi \in \Xi;\ s' \cup \xi \in \Xi,\ \xi \cap s'' = \phi\}\,.$$

Then $\mathscr{V}_{s',s''}(\Xi) \subset \mathscr{U}_{s'}(\Xi)$ are subcomplexes of Ξ and we have:

(i) *$\mathscr{U}_{s'}(\Xi)$ is k-semispherical with respect to $\rho = s'$. In particular, if $s' \neq \phi$, then $\mathscr{U}_{s'}(\Xi)$ is k-homologically trivial.*

(ii) *$\mathscr{V}_{s',s''}(\Xi)$ is k-homologically trivial unless $s' = s'' = \phi$.*

PROOF. The k-semisphericity of $\mathscr{U}_{s'}(\Xi)$ is immediate by definition. The second part of (i) was shown by Ishida, as we pointed out in (1.8). We now prove (ii) by induction on the codimension of ϕ in Ξ. If it is zero, then $\Xi = \{\phi\}$ and there is nothing to prove.

Consider the increasing filtration $\mathscr{V}^{\cdot}$ of $\mathscr{U}_{s'}(\Xi)$ by the subcomplexes $\mathscr{V}^i = \{\xi \in \Xi;\ s' \cup \xi \in \Xi,\ \dim(\xi \cap s'') \leqq i\}$. Obviously, $\mathscr{V}^i \subset \mathscr{V}^{i+1}$, $\mathscr{V}^{\dim\phi} = \mathscr{V}_{s',s''}(\Xi)$ and $\mathscr{V}^{\dim s''} = \mathscr{U}_{s'}(\Xi)$. The complement $\mathscr{V}^i \setminus \mathscr{V}^{i-1}$ is the disjoint union of its subcomplexes $\{\xi \in \Xi;\ s' \cup \xi \in \Xi,\ \xi \cap s'' = \zeta\}$ with ζ running through the elements in Ξ_i satisfying $s''>\zeta$ and $s'\cup\zeta\in\Xi$. But these subcomplexes coincide with $\mathscr{Y}_{s'\cup\zeta,s''}(\mathrm{Star}_\zeta(\Xi))=\{\xi\in\mathrm{Star}_\zeta(\Xi);(s'\cup\zeta)\cup\xi\in\mathrm{Star}_\zeta(\Xi),\ \xi\cap s''=\zeta\}$ with $s'\cup\zeta$, $s''\in\mathrm{Star}_\zeta(\Xi)$. If $\zeta\neq\phi$, then the codimension of ζ in $\mathrm{Star}_\zeta(\Xi)$ is smaller. Hence by the induction assumption, we see that if $\zeta\neq\phi$, then $\mathscr{Y}_{s'\cup\zeta,s''}(\mathrm{Star}_\zeta(\Xi))$ is k-homologically trivial unless $s'\cup\zeta=s''=\zeta$, i.e., $s'=\phi$ and $\zeta=s''$. Note that $\mathrm{Star}_\zeta(\Xi)$ is k-spherical by (1.8).

Thus if $s'\neq\phi$, the long exact sequences arising from the inclusions $\mathscr{Y}^{i-1}\subset\mathscr{Y}^i$ yield the isomorphisms $H_.(\mathscr{Y}_{s',s''}(\Xi),k)=H_.(\mathscr{Y}^{\dim\phi},k)\xrightarrow{\sim}\cdots\xrightarrow{\sim}H_.(\mathscr{Y}^{\dim s''},k)=H_.(\mathscr{U}_{s'}(\Xi),k)$, which vanishes by (i).

It remains to consider the case $s'=\phi$. We have $\mathscr{Y}^{\dim s''}=\mathscr{U}_{s'}(\Xi)=\Xi$. Even in this case, the long exact sequences arising from the inclusions $\mathscr{Y}^{i-1}\subset\mathscr{Y}^i$ yield the isomorphisms $H_.(\mathscr{Y}_{\phi,s''}(\Xi),k)=H_.(\mathscr{Y}^{\dim\phi},k)\xrightarrow{\sim}\cdots\xrightarrow{\sim}H_.(\mathscr{Y}^{\dim s''-1},k)$ as well as the long exact sequence $\cdots\to H_j(\mathscr{Y}^{\dim s''-1},k)\to H_j(\Xi,k)\to H_j(\mathrm{Star}_{s''}(\Xi),k)\to\cdots$, since for $s'=\phi$ and $\zeta=s''$, we have $\mathscr{Y}_{s'\cup\zeta,s''}(\mathrm{Star}_\zeta(\Xi))=\mathrm{Star}_{s''}(\Xi)$. Since $\mathrm{Star}_{s''}(\Xi)$ is k-spherical, we see that

$$H_j(\Xi,k)=H_j(\mathrm{Star}_{s''}(\Xi),k)=\begin{cases}k & j=d=\dim\Xi\\ 0 & \text{otherwise}\,.\end{cases}$$

Moreover, the induced map $H_d(\Xi,k)\to H_d(\mathrm{Star}_{s''}(\Xi,k)$ is an isomorphism by a result of Ishida, as we pointed out in (1.8). Hence we are done.

PROPOSITION 3.8. *For a* $\boldsymbol{Z}$*-basis* $\{m_1,\cdots,m_r\}$ *of* M, *let* $\varpi=\boldsymbol{R}_{\geq0}m_1+\cdots+\boldsymbol{R}_{\geq0}m_r$ *and let* $\Xi\subset\Gamma(\varpi)$ *be a* d*-dimensional* k*-spherical subcomplex. By freely identifying the elements of* $\Gamma(\varpi)$ *with the subsets of* $\{1,\cdots,r\}$ *as in* (1.9), *let* $\Xi'(s',s'')=\{\xi\in\Xi;s'\cup\xi\in\Xi,\ \xi\cap s''=\phi,\ \xi\cup s''\setminus\{l\}\notin\Xi$ *for some* $l\in s''\}$ *for* $s',s''\in\Gamma(\varpi)$. *On the other hand, let*

$$\Phi(s',s'')=\{\xi\in\Xi;\xi>s',\ \xi\cup s''\setminus\{l\}\in\Xi\quad\textit{for all}\quad l\in s''\}\,.$$

Then we have a canonical isomorphism

$$H^j(\mathrm{Sd}(\Xi'(s',s'')),k)\xrightarrow{\sim}H_{d-j}(\Phi(s',s''),k)\,.$$

PROOF. For simplicity, let $\Psi=\{\zeta\in\Xi;\zeta>s',\ \zeta\cup s''\setminus\{l\}\notin\Xi$ for some $l\in s''\}$ and $\Psi'=\{\eta\in\Xi;\eta>s',\ \eta\cap s''=\phi,\ \eta\cup s''\setminus\{l\}\notin\Xi$ for some $l\in s''\}$. Then Ψ' is a subset of $\Xi'(s',s'')$ with the order preserving retraction ρ sending $\xi\in\Xi'(s',s'')$ to $\rho(\xi)=s'\cup\xi>\xi$. Hence by Lemma 3.1 (1), $H_.(\mathrm{Sd}(\Psi'),k)$ is isomorphic to $H_.(\mathrm{Sd}(\Xi'(s',s'')),k)$. On the other hand, Ψ' is a subset of Ψ with the order preserving retraction ρ sending $\zeta\in\Psi$ to $\rho(\zeta)=\zeta\setminus s''<\zeta$. Hence by Lemma 3.1 (2), $H_.(\mathrm{Sd}(\Psi'),k)$ is isomorphic to $H_.(\mathrm{Sd}(\Psi),k)$.

Thus $H_.(\mathrm{Sd}(\Xi'(s',s'')),k)$ is isomorphic to $H_.(\mathrm{Sd}(\Psi),k)$. But obviously,

$\Phi = \Phi(s', s'')$ is a subcomplex of d-dimensional k-spherical $\mathrm{Star}_{s'}(\Xi)$ with the smallest element s' and $\Psi = \mathrm{Star}_{s'}(\Xi)\setminus\Phi$. Hence by Proposition 3.5, we are done.

4. The proofs of Theorem 2.1 and Corollary 2.2. Compared with those of Theorems 2.3 and 2.4, the proofs of Theorem 2.1 and Corollary 2.2 are much simpler.

Le $\Xi\subset\Gamma(\varpi)$ be a subcomplex for a convex rational polyhedral cone ϖ. Let us simply denote $Y(\Xi)$ by Y. By Proposition in (1.13), we have a canonical isomorphism $\mathcal{O}_Y \xrightarrow{\sim} \mathrm{proj\,lim}_{\xi\in\Xi}\mathcal{O}_{V(\xi)}$, where, moreover, the transition homomorphisms on the right hand side are all surjective. On the other hand, in view of Proposition in (1.12), the restriction map for $\xi, \eta\in\Xi$ with $\xi>\eta$ induces a homomorphism

$$\Theta_{V(\xi)}(-\log D(\xi)) \to \Theta_{V(\eta)}(-\log D(\eta)) ,$$

hence we have the projective limit

$$\mathrm{proj}\lim_{\xi\in\Xi}\Theta_{V(\xi)}(-\log D(\xi)) .$$

Since an element in this limit gives rise to a compatible system of k-derivations on the projective system $\{\mathcal{O}_{V(\xi)}\}_{\xi\in\Xi}$, we obviously have a canonical homomorphism

$$\mathrm{proj}\lim_{\xi\in\Xi}\Theta_{V(\xi)}(-\log D(\xi)) \to \mathcal{D}er_k(\mathcal{O}_Y) ,$$

which is easily seen to be injective by the surjectivity of $\mathcal{O}_Y\to\mathcal{O}_{V(\xi)}$ for all $\xi\in\Xi$.

We now prove that (1) in Theorem 2.1 implies the isomorphy of the canonical homomorphism above. It suffices to prove the corresponding assertion on the ring level. Let us denote $P = k[M\cap\varpi]$, $J = J(\Xi)$ and $S = S(\Xi)$. Then $\mathfrak{q}(\xi) = \mathfrak{p}(\xi)/J$ is the prime ideal defining $V(\xi)$ in Y, where $\mathfrak{p}(\xi) = k[M\cap\varpi\setminus M\cap\xi]$. Obviously, $\mathfrak{q}(\xi)$'s with ξ running through the maximal elements of Ξ are exactly the minimal prime ideals of S. It is well known that a derivation automatically preserves each minimal prime ideal. On the other hand by (1), each $\eta\in\Xi$ is the intersection of the maximal ξ's in Ξ with $\xi>\eta$. Hence we have $\mathfrak{q}(\eta) = \sum_\xi\mathfrak{q}(\xi)$ with the summation taken over the maximal elements ξ in Ξ with $\xi>\eta$. Thus a k-derivation δ of S preserves $\mathfrak{q}(\eta)$ for each $\eta\in\Xi$, hence induces a k-derivation δ_η of $S/\mathfrak{q}(\eta) = k[M\cap\eta]$. By definition, the ideal defining $D(\eta)$ in $V(\eta)$ is $\bigcap_\zeta(\mathfrak{q}(\zeta)/\mathfrak{q}(\eta))$, where ζ runs through the elements of Ξ with $\eta>\zeta$ and $\dim\eta-\dim\zeta = 1$. Hence δ_η automatically is a k-derivation with logarithmic zeros along $D(\eta)$, and we are done.

We next show that (2) implies (1). Let η be an element of Ξ. We prove (1) is satisfied by induction on $\dim \eta$. If η is maximal in Ξ, then there is nothing to prove. If it is not, then by (2), either (i) there exists $\xi \in \Xi$ with $\xi > \eta$ and $\dim \xi - \dim \eta \geqq 2$ or (ii) there exist distinct $\eta', \eta'' \in \Xi$ with $\eta' > \eta, \eta'' > \eta$ and with $\dim \eta' - \dim \eta = \dim \eta'' - \dim \eta = 1$. But in case (i), there certainly exist faces η' and η'' of ξ satisfying the properties of (ii). Thus we are in case (ii) anyway. Then obviously, $\eta = \eta' \cap \eta''$. Applying the induction hypothesis to η' and η'', we are done.

It remains to show that the isomorphy of the canonical homomorphism implies (2). Suppose there exists $\eta \in \Xi$ such that $\mathrm{Star}_\eta(\Xi)$ consists of η and $\xi \gneqq \eta$. Then we necessarily have $\dim \xi - \dim \eta = 1$, and $U = V(\xi) \setminus \bigcup_\zeta V(\zeta)$, with ζ running through the faces of ξ different from ξ and η, is an affine open set of Y (cf. [MO, (5.5)]). Under the bijection $\mathrm{Star}_\eta(\Gamma(\varpi)) \xrightarrow{\sim} \Gamma(\varpi + \boldsymbol{R}\eta)$ in (1.7), however, $\mathrm{Star}_\eta(\Xi)$ is sent to $\{\boldsymbol{R}\eta, \xi + \boldsymbol{R}\eta\}$. Hence U is easily seen to be isomorphic to the product of the affine line and an algebraic torus. Obviously, the restriction to U of the canonical homomorphism is not surjective.

We now prove Corollary 2.2. The second isomorphism follows easily from Lemma in (1.13). Thus it suffices to show that (2) of Theorem 2.1 is satisfied if Ξ is d-dimensional k-Cohen-Macaulay and if $H_d(\mathrm{Star}_\beta(\Xi), k) \neq 0$ for any $\beta \in \Xi_{d-1}$. Let η be an element of Ξ. If $\dim \eta = d$, then there is nothing to prove. If $\dim \eta \leqq d - 2$, then there exists $\alpha \in \Xi_d$ with $\alpha > \eta$ by the equidimensionality. Then there exists at least one $\beta \in \Xi_{d-1}$ with $\alpha > \beta > \eta$. It remains to consider the case $\dim \eta = d - 1$. If the cardinality of $\mathrm{Star}_\eta(\Xi)$ were two, then it would consist exactly of η and $\alpha \in \Xi_d$ with $\alpha > \eta$. Thus $H_d(\mathrm{Star}_\eta(\Xi), k) = 0$, a contradiction to the assumption.

5. Homogeneous components of the hyperextension modules. As a preparation for the proofs of Theorems 2.3 and 2.4 in the next section, we reduce the computation, reviewed in (1.14), of the hyperextension sheaves in question to that of certain combinatorial cohomology groups with coefficients in k.

Throughout, we fix a $\boldsymbol{Z}$-basis $\{m_1, \cdots, m_r\}$ of M and the *nonsingular* convex rational polyhedral cone $\varpi = \boldsymbol{R}_{\geqq 0} m_1 + \cdots + \boldsymbol{R}_{\geqq 0} m_r$, which thus satisfies $\varpi \cap (-\varpi) = \{0\}$. As in (1.9), we freely identify $\xi \in \Gamma(\varpi)$ with the subset $\{i; 1 \leqq i \leqq r, m_i \in \xi\}$. Thus, in particular, $\phi = \{0\}$ is identified with the empty set. The advantage in the nonsingular case lies in the fact that for $\xi, \eta \in \Gamma(\varpi)$, there exists a unique $\eta' \in \Gamma(\varpi)$ satisfying $\xi = (\xi \cap \eta) \cup \eta'$ and $(\xi \cap \eta) \cap \eta' = \{0\}$, i.e., η' is the set-theoretical difference

$\xi \setminus \eta$ in the above identification. Henceforth, we always denote η' by $\xi \setminus \eta$. The advantage also lies in the fact that $P = k[M \cap \varpi] = k[t_1, \cdots, t_r]$ is a polynomial k-algebra, where $t_i = e(m_i)$. Thus we can apply the computation process described in (1.14) to this P.

We also fix a subcomplex $\Xi \subset \Gamma(\varpi)$ and simply denote $J = J(\Xi)$ and $S = S(\Xi) = P/J$.

DEFINITION. For $\lambda \in \Gamma(\varpi)$, we denote by $|\lambda|$ the cardinality of λ and let $\boldsymbol{m}(\lambda) = \sum_{i \in \lambda} m_i$. For $m \in M \cap \varpi$, let $\operatorname{supp}(m) = \{i; 1 \leqq i \leqq r, \langle m, n_i \rangle \neq 0\}$, where $\{n_1, \cdots, n_r\}$ is the $\boldsymbol{Z}$-basis of N dual to $\{m_1, \cdots, m_r\}$. The *canonical decomposition* for $m \in M$ is the unique expression $m = m' - m''$ with $m', m'' \in M \cap \varpi$ such that $\operatorname{supp}(m')$ and $\operatorname{supp}(m'')$ are disjoint.

In particular, we have $\operatorname{supp}(\boldsymbol{m}(\lambda)) = \lambda$.

DEFINITION. For simplicity, we denote by $\Lambda = \Gamma(\varpi) \setminus \Xi$ the complement of Ξ in $\Gamma(\varpi)$. We denote $G = M \cap \varpi$ and $G' = \bigcup_{\xi \in \Xi} (M \cap \xi)$.

Hence $\{\boldsymbol{m}(\lambda)\}_{\lambda \in \Lambda}$ generates the semigroup ideal $G \setminus G'$ of G, i.e., $G \setminus G' = \{m \in M \cap \varpi; \operatorname{supp}(m) \notin \Xi\} = \bigcup_{\lambda \in \Lambda} (\boldsymbol{m}(\lambda) + G)$. Thus $\{e(\boldsymbol{m}(\lambda))\}_{\lambda \in \Lambda}$ generates J as an ideal of P.

PROPOSITION 5.1. *For $i \geqq 0$, let F_i be the M-graded free P-module with the basis $\operatorname{Sd}_i(\Lambda)$ and with $(\lambda_1 \lneqq \cdots \lneqq \lambda_i) \in \operatorname{Sd}_i(\Lambda)$ regarded as M-homogeneous of degree $\boldsymbol{m}(\lambda_i)$ for $i \geqq 1$, while $(\ ) \in \operatorname{Sd}_0(\Lambda)$ is regarded as M-homogeneous of degree 0. For $i \geqq 1$, $u_{i-1}\colon F_i \to F_{i-1}$ is the homomorphism of M-graded P-modules of degree 0 defined by $u_{i-1}((\lambda_1 \lneqq \cdots \lneqq \lambda_i)) = \sum_{1 \leqq l \leqq i-1} (-1)^{l+1} (\lambda_1 \lneqq \cdots \overset{l}{\vee} \cdots \lneqq \lambda_i) + (-1)^{i+1} e(\boldsymbol{m}(\lambda_i \setminus \lambda_{i-1}))(\lambda_1 \lneqq \cdots \lneqq \lambda_{i-1})$ for $(\lambda_1 \lneqq \cdots \lneqq \lambda_i)$ in $\operatorname{Sd}_i(\Lambda)$. The augmentation homomorphism $u_{-1}\colon F_0 \to S$ is defined by $u_{-1}((\ )) = 1$. The homomorphism $v\colon F_1 \to \Omega_P^1$ is defined by $v((\lambda)) = \sum_{i \in \lambda} e(\boldsymbol{m}(\lambda \setminus \{i\})) de(m_i)$ for (λ) in $\operatorname{Sd}_1(\Lambda)$, where d is the exterior differentiation for P. The homomorphism $w'\colon \wedge^2 F_1 \to F_1$ is defined by $w'((\lambda) \wedge (\mu)) = e(\boldsymbol{m}(\lambda))(\mu) - e(\boldsymbol{m}(\mu))(\lambda)$, for $(\lambda), (\mu) \in \operatorname{Sd}_1(\Lambda)$. Finally $w\colon \wedge^2 F_1 \to F_2$ is defined by $w((\lambda) \wedge (\mu)) = e(\boldsymbol{m}(\lambda \cap \mu))\{(\lambda < \lambda \cup \mu) - (\mu < \lambda \cup \mu)\}$, for $(\lambda), (\mu) \in \operatorname{Sd}_1(\Lambda)$, with the convention that $(\lambda < \lambda \cup \mu) = 0$ (resp. $(\mu < \lambda \cup \mu) = 0$) if $\lambda = \lambda \cup \mu$ (resp. if $\mu = \lambda \cup \mu$). Then u_{-1}, v, w, w' are M-homogeneous of degree zero by letting* $\deg de(m_i) = m_i$ *and* $\deg (\lambda) \wedge (\mu) = \boldsymbol{m}(\lambda) + \boldsymbol{m}(\mu)$, *and we have the following*:

(i) *$(F_\cdot, u_\cdot) = \{F_i, u_i; i \geqq 0\}$ is an acyclic complex of P-modules with u_{-1} inducing the isomorphism $H_0(F_\cdot, u_\cdot) \xrightarrow{\sim} S$, i.e.,*

$$\cdots \to F_i \to F_{i-1} \to \cdots \to F_1 \to F_0 \to S \to 0$$

is exact.

(ii) *$S \otimes_P v : S \otimes_P F_1 \to S \otimes_P \Omega_P^1$ factors through the canonical S-*

homomorphism $J/J^2 = S\otimes_P J \to S\otimes_P \Omega^1_P$.

(iii) $u_1\circ w = w'$.

PROOF. (ii) and (iii) are trivial. Let us show (i). The fact $u_{i-1}\circ u_i = 0$ for $i \geqq 1$ results from a straightforward computation. It suffices to check the acyclicity for the homogeneous part of degree m for each $m\in M$. If $m\notin G$, then the homogenous parts $(F_i)_m$ and S_m vanish. If $m\in G'$, then we have $(F_0)_m = ke(m)(\ )$, $S_m = ke(m)$ and $(F_i)_m = 0$ for $i\geqq 1$, and u_{-1} induces an isomorphism. Finally if $m\in G\setminus G'$, then $S_m = 0$, while for $i\geqq 0$, $(F_i)_m$ is the k-vector space with the basis consisting of $\{e(m - \boldsymbol{m}(\lambda_i))(\lambda_1 \lneqq \cdots \lneqq \lambda_i); (\lambda_1 \lneqq \cdots \lneqq \lambda_i)\in \mathrm{Sd}_i(\Lambda), \lambda_i\subset \mathrm{supp}(m)\}$. It is obviously k-isomorphic to $C_i(\mathrm{Sd}(\Lambda\cap\Gamma(\mathrm{supp}(m))), k)$ by sending $e(m - \boldsymbol{m}(\lambda_i))(\lambda_1 \lneqq \cdots \lneqq \lambda_i)$ to $(\lambda_1 \lneqq \cdots \lneqq \lambda_i)$. Checking $(u_i)_m$, we see easily that $(F_., u_.)_m$ is isomorphic to the complex $C_.(\mathrm{Sd}(\Lambda\cap \Gamma(\mathrm{supp}(m))), k)$. Since $m\in G\setminus G'$, we see that $\mathrm{supp}(m)$ is in Λ, hence $\Lambda\cap \Gamma(\mathrm{supp}(m))$ has the largest element $\mathrm{supp}(m)$. Thus by Lemma 3.2, we are done.

Since P is a polynomial k-algebra, $F_0\cong P$ and $F_1\to F_0\to S\to 0$ is exact, we can apply (1), (2) of (1.14).

LEMMA 5.2. *Let* $E^{\cdot} = (0\to E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} E^2 \xrightarrow{\delta} E^3 \to 0)$ *be the cochain complex of M-graded S-modules defined as follows*: $E^0 = \mathrm{Hom}_P(\Omega^1_P, S)$, $E^1 = \mathrm{Hom}_P(F_1, S)$, $E^2 = \mathrm{Hom}_P(F_2, S)$ *and* $E^3 = \mathrm{Hom}_P(F_3\oplus \wedge^2 F_1, S)$. *Moreover*, $\delta\colon E^i\to E^{i+1}$ *for* $i = 0, 1, 2$ *are the homomorphisms induced respectively by* v, u_1 *and* $(u_2, w)\colon F_3\oplus\wedge^2 F_1\to F_2$. *Then we have*

$$\mathrm{Ext}^i_S(L^S_., S) = H^i(E^{\cdot}) \qquad \textit{for} \quad i = 0, 1, 2\,.$$

PROOF. $\mathrm{Ext}^0_S(L^S_., S) = \mathrm{Der}_k(S) = \mathrm{Hom}_S(\Omega^1_S, S)$ is well-known to be the kernel of $\mathrm{Hom}_P(\Omega^1_P, S)\to \mathrm{Hom}_P(J, S)$, while $\mathrm{Ext}^1_S(L^S_., S)$ is its cokernel by (1.14) (1′). By Proposition 5.1 (i), (ii), u_0 induces an isomorphism from $\mathrm{Hom}_P(J, S)$ to the kernel of $u_1^*\colon \mathrm{Hom}_P(F_1, S)\to \mathrm{Hom}_P(F_2, S)$.

Finally, let Kosz be the Koszul complex built out of $u_0\colon F_1 \twoheadrightarrow J(\ )\subset F_0\cong P$. Then by (1.14) (2), we see that $\mathrm{Ext}^2_S(L^S_., S)$ is the cokernel of $\mathrm{Hom}_P(F_1, S)\to \mathrm{Hom}_P(H_1(\mathrm{Kosz}), S)$. By Proposition 5.1 (i), $H_1(\mathrm{Kosz}) = \ker(u_0)/\mathrm{Image}(w') = \mathrm{Image}(u_1)/\mathrm{Image}(w')$. Since $\mathrm{Image}(u_1)$ is isomorphic to $\mathrm{coker}(u_2)$ again by Proposition 5.1 (i) and $\mathrm{Image}(w')$ is a quotient of $\wedge^2 F_1$, we are done in view of Proposition 5.1 (iii).

DEFINITION. Let $\Gamma(\varpi)$, Ξ, Λ be as at the beginning of this section. For *disjoint* $s', s''\in\Gamma(\varpi)$, we define the cochain complex $A^{\cdot} = A^{\cdot}(\Xi, s', s'')$ of k-vector spaces as follows: $A^i = 0$ for $i\neq 0, 1, 2$. A^0 consists of such k-valued functions a on $\{1, \cdots, r\}$ that $a(j)\neq 0$ only if $\{j\} > s''$ and $s'\cup$

$(\{j\}\setminus s'')\in\Xi$. A^1 consists of such k-valued functions b on $\mathrm{Sd}_1(\Lambda)=\Lambda$ that $b(\lambda)\neq 0$ only if $\lambda>s''$ and $s'\cup(\lambda\setminus s'')\in\Xi$. A^2 consists of such k-valued functions c on $\mathrm{Sd}_2(\Lambda)$ that $c(\lambda_1\subsetneqq\lambda_2)\neq 0$ only if $\lambda_2>s''$ and $s'\cup(\lambda_2\setminus s'')\in\Xi$, that $c(\lambda_1\subsetneqq\lambda_2)=c(\lambda_1'\subsetneqq\lambda_2)$ for $(\lambda_1\subsetneqq\lambda_2)$, $(\lambda_1'\subsetneqq\lambda_2)\in\mathrm{Sd}_2(\Lambda)$ if $\lambda_1\not>s''$ and $\lambda_1'\not>s''$, and that the cocycle condition $c(\lambda_2\subsetneqq\lambda_3)-c(\lambda_1\subsetneqq\lambda_3)+c(\lambda_1\subsetneqq\lambda_2)=0$ is satisfied for all $(\lambda_1\subsetneqq\lambda_2\subsetneqq\lambda_3)\in\mathrm{Sd}_3(\Lambda)$ with $\lambda_3>s''$ and $s'\cup(\lambda_3\setminus s'')\in\Xi$. $\delta\colon A^0\to A^1$ sends $a\in A^0$ to $\delta a\in A^1$ defined by

$$(\delta a)(\lambda)=\begin{cases}\sum_{j\in\lambda}a(j) & \text{if}\quad \lambda>s''\quad\text{and}\quad s'\cup(\lambda\setminus s'')\in\Xi\\ 0 & \text{otherwise}\,.\end{cases}$$

$\delta\colon A^1\to A^2$ sends $b\in A^1$ to $\delta b\in A^2$ defined by

$$(\delta b)(\lambda_1\subsetneqq\lambda_2)=\begin{cases}b(\lambda_2)-b(\lambda_1) & \text{if}\quad \lambda_2>s''\quad\text{and}\quad s'\cup(\lambda_2\setminus s'')\in\Xi\\ 0 & \text{otherwise}\,.\end{cases}$$

With these definitions, we have:

PROPOSITION 5.3. *For $m\in M$ with the canonical decomposition $m=m'-m''$, let $s'=\mathrm{supp}(m')$ and $s''=\mathrm{supp}(m'')$. Then for $i=0,1,2$, the homogenenous part of $\mathrm{Ext}^i_S(L^S_\cdot,S)$ of degree m is of the form*

$$\mathrm{Ext}^i_S(L^S_\cdot,S)_m=\begin{cases}0 & \text{if}\quad m''\neq\boldsymbol{m}(s'')\\ H^i(A^\cdot(\Xi,s',s'')) & \text{if}\quad m''=\boldsymbol{m}(s'')\,.\end{cases}$$

PROOF. By Lemma 5.2. it is enough to study the homogeneous part of $H^i(E^\cdot)$. As usual, for M-graded P-modules F, F' of finite type, let us regard $\mathrm{Hom}_P(F,F')$ as an M-graded P-module by letting $f\colon F\to F'$ to be homogeneous of degree m if $f(F_\mu)\subset F'_{\mu+m}$ for all $\mu\in M$. Since $S_m=ke(m)$ if $m\in G'$ and $S_m=0$ otherwise, we see that E^0_m consists of the homomorphisms $\boldsymbol{a}$ sending $de(m_i)$ to $a(i)e(m+m_i)$ for $1\leqq i\leqq r$ with $a(i)\in k$, where $a(i)\neq 0$ only if $m+m_i\in G'$. E^1_m consists of the homomorphisms $\boldsymbol{b}$ sending $(\lambda)\in\mathrm{Sd}_1(\Lambda)$ to $b(\lambda)e(m+\boldsymbol{m}(\lambda))$ with $b(\lambda)\in k$, where $b(\lambda)\neq 0$ only if $m+\boldsymbol{m}(\lambda)\in G'$. E^2_m consists of the homomorphisms $\boldsymbol{c}$ sending $(\lambda_1\subsetneqq\lambda_2)\in\mathrm{Sd}_2(\Lambda)$ to $c(\lambda_1\subsetneqq\lambda_2)e(m+\boldsymbol{m}(\lambda_2))$ with $c(\lambda_1\subsetneqq\lambda_2)\in k$, where $c(\lambda_1\subsetneqq\lambda_2)\neq 0$ only if $m+\boldsymbol{m}(\lambda_2)\in G'$. Finally, E^3_m consists of the homomorphisms $\boldsymbol{h}$ sending $(\lambda_1\subsetneqq\lambda_2\subsetneqq\lambda_3)\in\mathrm{Sd}_3(\Lambda)$ to $h(\lambda_1\subsetneqq\lambda_2\subsetneqq\lambda_3)e(m+\boldsymbol{m}(\lambda_3))$ with $h(\lambda_1\subsetneqq\lambda_2\subsetneqq\lambda_3)\in k$, where $h(\lambda_1\subsetneqq\lambda_2\subsetneqq\lambda_3)\neq 0$ only if $m+\boldsymbol{m}(\lambda_3)\in G'$ and sending $(\lambda)\wedge(\mu)$ to $h((\lambda)\wedge(\mu))e(m+\boldsymbol{m}(\lambda)+\boldsymbol{m}(\mu))$ with $h((\lambda)\wedge(\mu))\in k$ where $h((\lambda)\wedge(\mu))\neq 0$ only if $m+\boldsymbol{m}(\lambda)+\boldsymbol{m}(\mu)\in G'$. Thus obviously $E^j_m=0$, hence $H^j(E^\cdot)_m=0$ for $j=0,1,2$, if $m''\neq\boldsymbol{m}(s'')$. Henceforth, we thus assume that $m''=\boldsymbol{m}(s'')$. Then by the definition of the canonical decomposition, we see the following: (i) $m+m_i\in G'$ if and only if $\{i\}>s''$

and $s' \cup (\{i\} \setminus s'') \in \Xi$. (ii) $m + \boldsymbol{m}(\lambda) \in G'$ if and only if $\lambda > s''$ and $s' \cup (\lambda \setminus s'') \in \Xi$. (iii) $m + \boldsymbol{m}(\lambda) + \boldsymbol{m}(\mu) \in G'$ if and only if $\lambda \cup \mu > s''$ and $s' \cup (\lambda \cap \mu) \cup (\lambda \cup \mu \setminus s'') \in \Xi$. Hence we see easily that $E_m^0 \to A^0$ and $E_m^1 \to A^1$ sending $\boldsymbol{a}$ and $\boldsymbol{b}$ to their respective coefficients a and b are isomorphisms commuting with $\delta: E_m^0 \to E_m^1$ and $\delta: A^0 \to A^1$. Moreover, $Z^2(E^{\cdot})_m$ consists of $\boldsymbol{c}$ sending $(\lambda_1 \lneqq \lambda_2) \in \mathrm{Sd}_2(\Lambda)$ to $c(\lambda_1 \lneqq \lambda_2)e(m + \boldsymbol{m}(\lambda_2))$ such that (iv) $c(\lambda_1 \lneqq \lambda_2) \neq 0$ only if $\lambda_2 > s''$ and $s' \cup (\lambda_2 \setminus s'') \in \Xi$, that (v) $c(\lambda_2 \lneqq \lambda_3) - c(\lambda_1 \lneqq \lambda_3) + c(\lambda_1 \lneqq \lambda_2) = 0$ if $(\lambda_1 \lneqq \lambda_2 \lneqq \lambda_3) \in \mathrm{Sd}_3(\Lambda)$ satisfies $\lambda_3 > s''$ and $s' \cup (\lambda_3 \setminus s'') \in \Xi$ and that (vi) $c(\lambda < \lambda \cup \mu) = c(\mu < \lambda \cup \mu)$ if $\lambda \cup \mu > s''$ and $s' \cup (\lambda \cap \mu) \cup (\lambda \cup \mu \setminus s'') \in \Xi$. We claim that (iv), (v), (vi) are satisfied if and only if c belongs to A^2. Then we would be done, since $\delta: E_m^1 \to Z^2(E^{\cdot})_m$ is easily seen to correspond to $\delta: A^1 \to A^2$. If c belongs to A^2, then $\boldsymbol{c}$ belongs to $Z^2(E^{\cdot})_m$. Indeed, (iv) and (v) are obviously satisfied. Let us show (vi). For $(\lambda) \wedge (\mu)$ with $\lambda \cup \mu > s''$ and $s' \cup (\lambda \cap \mu) \cup (\lambda \cup \mu \setminus s'') \in \Xi$, we automatically have $\lambda \not> s''$ and $\mu \not> s''$, since Λ is the complement of Ξ. Hence $\lambda_1 = \lambda$, $\lambda_1' = \mu$ and $\lambda_2 = \lambda \cup \mu$ will do. Conversely, if $\boldsymbol{c}$ belongs to $Z^2(E^{\cdot})_m$, then c belongs to A^2. Indeed, in view of (iv) and (v), it suffices to show $c(\lambda_1 \lneqq \lambda_2) = c(\lambda_1' \lneqq \lambda_2)$ if $\lambda_2 > s''$, $s' \cup (\lambda_2 \setminus s'') \in \Xi$, $\lambda_1 \not> s''$ and $\lambda_1' \not> s''$. If $\lambda_1 \cup \lambda_1' \not> s''$, then $c(\lambda_1 \lneqq \lambda_2) = c(\lambda_1 \cup \lambda_1' \lneqq \lambda_2) = c(\lambda_1' \lneqq \lambda_2)$ by (v) applied to $(\lambda_1 \lneqq \lambda_1 \cup \lambda_1' \lneqq \lambda_2)$ and $(\lambda_1' \lneqq \lambda_1 \cup \lambda_1' \lneqq \lambda_2)$ in view of (iv). Suppose $\lambda_1 \cup \lambda_1' > s''$. If $s' \cup (\lambda_1 \cap \lambda_1') \cup (\lambda_1 \cup \lambda_1' \setminus s'') = \lambda_1'' \in \Xi$, then we are done by (vi) with $\lambda = \lambda_1$ and $\mu = \lambda_1'$. It remains to consider the case $\lambda_1 \cup \lambda_1' > s''$ and $\lambda_1'' \in \Lambda$. We then obviously have $s' \cup \lambda_2 \gneqq \lambda_1'' \cup \lambda_1 \not> s''$ and $\lambda_1'' \not> s''$, since $\lambda_1 \not> s''$ and $\lambda_2 > s''$. Hence again by (iv) and (v) applied to $(\lambda_1 < \lambda_1'' \cup \lambda_1 \lneqq s' \cup \lambda_2)$ and $(\lambda_1'' < \lambda_1'' \cup \lambda_1 \lneqq s' \cup \lambda_2)$, we have $c(\lambda_1 \lneqq s' \cup \lambda_2) = c(\lambda_1'' \cup \lambda_1 \lneqq s' \cup \lambda_2) = c(\lambda_1'' \lneqq s' \cup \lambda_2)$. Similarly, we have $c(\lambda_1' \lneqq s' \cup \lambda_2) = c(\lambda_1'' \cup \lambda_1' \lneqq s' \cup \lambda_2) = c(\lambda_1'' \lneqq s' \cup \lambda_2)$. Hence $c(\lambda_1 \lneqq s' \cup \lambda_2) = c(\lambda_1' \lneqq s' \cup \lambda_2)$. We are done, since $c(\lambda_1 \lneqq \lambda_2) = c(\lambda_1 \lneqq s' \cup \lambda_2) - c(\lambda_2 < s' \cup \lambda_2)$ and $c(\lambda_1' \lneqq \lambda_2) = c(\lambda_1' \lneqq s' \cup \lambda_2) - c(\lambda_2 < s' \cup \lambda_2)$ by (v) applied to $(\lambda_1 \lneqq \lambda_2 < s' \cup \lambda_2)$ and $(\lambda_1' \lneqq \lambda_2 < s' \cup \lambda_2)$.

We now study $H^j(A^{\cdot}(\Xi, s', s''))$ for $j = 0, 1, 2$ more closely. Note that $\Xi(s', s'')$ in a more general context and $\Xi'(s', s'')$ below were already defined in Propositions 3.6 and 3.8.

PROPOSITION 5.4. *Let $\Xi \subset \Gamma(\varpi)$ be as at the beginning of this section. For disjoint $s', s'' \in \Gamma(\varpi)$, we define local subcomplexes $\Xi'(s', s'') \subset \Xi(s', s'')$ of Ξ by*

$$\Xi(s', s'') = \{\xi \in \Xi;\ s' \cup \xi \in \Xi,\ \xi \cap s'' = \phi,\ \xi \cup s'' \notin \Xi\}$$

$$\Xi'(s', s'') = \{\xi \in \Xi;\ s' \cup \xi \in \Xi,\ \xi \cap s'' = \phi,\ \xi \cup s'' \setminus \{l\} \notin \Xi \quad \text{for some } l \in s''\}\,.$$

Then we have the following, where we simply denote $A^{\cdot} = A^{\cdot}(\Xi, s', s'')$.

(1) *If* $s' \notin \Xi$, *then* $H^j(A^\cdot) = 0$ *for* $j = 0, 1, 2$.

(2) *If* $s' \in \Xi$ *and* $s'' = \phi$, *then* $H^0(A^\cdot) = A^0$ *can be identified with the k-vector space of k-valued functions on* $\{i;\ 1 \leqq i \leqq r,\ s' \cup \{i\} \in \Xi\}$, *while* $H^j(A^\cdot) = 0$ *for* $j = 1, 2$.

(3) *If* $s' \in \Xi$ *and* $|s''| = 1$, *then we have canonical isomorphisms*

$$H^j(A^\cdot) \xrightarrow{\sim} H^j(\mathrm{Sd}(\Xi(s', s''), k) \qquad \textit{for} \quad j = 0, 1, 2 .$$

(4) *If* $s' \in \Xi$ *and* $|s''| \geqq 2$, *then* $H^0(A^\cdot) = 0$, $H^1(A^\cdot) = \ker[Z^1(\mathrm{Sd}(\Xi(s', s'')), k) \xrightarrow{\text{rest}} Z^1(\mathrm{Sd}(\Xi'(s', s'')), k)]$ *canonically, and the following sequence is exact*: $H^1(\mathrm{Sd}(\Xi(s', s'')), k) \xrightarrow{\text{rest}} H^1(\mathrm{Sd}(\Xi'(s', s'')), k) \to H^2(A^\cdot) \to H^2(\mathrm{Sd}(\Xi(s', s'')), k)$, *where* rest *are the restriction maps induced by the inclusion* $\Xi'(s', s'') \subset \Xi(s', s'')$.

PROOF. (1) and (2) are obvious, since there exists, for instance, no $\lambda \in \Lambda$ with $\lambda > s''$ and $s' \cup (\lambda \setminus s'') \in \Xi$ in these cases. Here, as before, Λ is the complement of Ξ in $\Gamma(\varpi)$. Let us now prove (3) and (4). We first observe that there is a homomorphism of cochain complexes

$$\varepsilon^\cdot : A^\cdot \to C^\cdot(\mathrm{Sd}(\Xi(s', s'')), k)$$

defined as follows: If s'' consists of one element i, then $a \in A^0$ is determined by its value $a(i)$. In this case, we let $(\varepsilon^0 a)((\ \ )) = a(i)$, hence ε^0 is an isomorphism. If $|s''| \geqq 2$, then obviously $A^0 = 0$ and we let $\varepsilon^0 = 0$. For $b \in A^1$ and $c \in A^2$, we let

$$\begin{aligned} (\varepsilon^1 b)(\xi) &= b(\xi \perp\!\!\!\perp s'') && \text{for} \quad (\xi) \in \mathrm{Sd}_1(\Xi(s', s'')) \\ (\varepsilon^2 c)(\xi_1 \lneqq \xi_2) &= c(\xi_1 \perp\!\!\!\perp s'' \lneqq \xi_2 \perp\!\!\!\perp s'') && \text{for} \quad (\xi_1 \lneqq \xi_2) \in \mathrm{Sd}_2(\Xi(s', s'')) . \end{aligned}$$

It is easy to check that $\varepsilon^\cdot$ commute with the coboundary maps. We see that ε^1 is an isomorphism, since the map from $\mathrm{Sd}_1(\Xi(s', s''))$ to $\{(\lambda) \in \mathrm{Sd}_1(\Lambda);\ \lambda > s'',\ s' \cup (\lambda \setminus s'') \in \Xi\}$ sending (ξ) to $(\lambda) = (\xi \perp\!\!\!\perp s'')$ is bijective. We also see that the image of ε^2 is contained in $Z^2(\mathrm{Sd}(\Xi(s', s'')), k)$.

Thus, using the snake lemma, we easily get an exact sequence

$$\begin{aligned} 0 \to Z^1(A^\cdot) \to Z^1(\mathrm{Sd}(\Xi(s', s'')), k) \to \ker(\varepsilon^2) \to\ & H^2(A^\cdot) \\ \to H^2(\mathrm{Sd}(\Xi(s', s'')), k) . & \end{aligned}$$

We now show that the third and the fourth arrows can be identified with

$$Z^1(\mathrm{Sd}(\Xi(s', s'')), k) \to Z^1(\mathrm{Sd}(\Xi'(s', s'')), k) \to H^2(A^\cdot) ,$$

where the arrow on the left hand side is the restriction map induced by the inclusion. Indeed, by definition, $\ker(\varepsilon^2)$ consists of such k-valued functions c on $\mathrm{Sd}_2(\Lambda)$ that

$$c(\lambda_1 \lneqq \lambda_2) \neq 0 \quad \text{only if} \quad \lambda_1 \not> s'', \lambda_2 > s'' \quad \text{and} \quad s' \cup (\lambda_2 \setminus s'') \in \Xi \ ,$$
$$c(\lambda_1 \lneqq \lambda_2) = c(\lambda_1' \lneqq \lambda_2) \quad \text{if} \quad \lambda_1 \not> s'' \quad \text{and} \quad \lambda_1' \not> s'' \ , \quad \text{and}$$
$$c(\lambda_1 \lneqq \lambda_2) = c(\lambda_1 \lneqq \lambda_3) \quad \text{if} \quad \lambda_1 \not> s'', \lambda_2 > s'' \quad \text{and} \quad \lambda_2 < \lambda_3 \ .$$

Then we obviously have the required isomorphism $\ker(\varepsilon^2) \xrightarrow{\sim} Z^1(\mathrm{Sd}(\Xi'(s', s'')), k)$ by sending such c to $\tilde{c}$ defined by

$$\tilde{c}(\xi) = c(\xi \perp\!\!\!\perp (s'' \setminus \{l\}) \lneqq \xi \perp\!\!\!\perp s'') \qquad \text{for} \quad \xi \in \Xi'(s', s'') \ ,$$

where l is any one of the elements of s'' satisfying $\xi \perp\!\!\!\perp (s'' \setminus \{l\}) \notin \Xi$ on which the value of c does not depend by the second defining condition for c. We thus get (3) and (4). Indeed, the restriction map $B^1(\mathrm{Sd}(\Xi(s', s'')), k) \to B^1(\mathrm{Sd}(\Xi'(s', s'')), k)$ is surjective. When $|s''| = 1$, we see, moreover, that $\Xi'(s', s'')$ is empty, hence $Z^1(\mathrm{Sd}(\Xi'(s', s'')), k) = 0$. On the other hand, ε^1 maps $B^1(A^{\cdot})$ onto $B^1(\mathrm{Sd}(\Xi(s', s'')), k)$ when $|s''| = 1$, while $B^1(A^{\cdot}) = 0$ when $|s''| \geqq 2$.

COROLLARY 5.5. *Let $\Xi \subset \Gamma(\varpi)$ be as at the beginning of this section. For disjoint $s', s'' \in \Gamma(\varpi)$, the following are equivalent*:

(1) $H^0(A^{\cdot}(\Xi, s', s'')) \neq 0$ *only if* $s' \in \Xi$ *and* $s'' = \phi$.

(2) $H^0(\mathrm{Sd}(\Xi(s', s'')), k) = 0$ *whenever* $s' \in \Xi$ *and* $|s''| = 1$.

(3) *Any* $s' \in \Xi$ *is the intersection of all the maximal elements* $\xi \in \Xi$ *satisfying* $\xi > s'$.

PROOF. By Proposition 5.4, (1) and (2) are equivalent. As we saw at the beginning of Section 3, $H^0(\mathrm{Sd}(\Xi(s', s'')), k) = 0$ if and only if $\Xi(s', s'')$ is nonempty. Now by the definition of $\Xi(s', s'')$, the equivalence of (2) and (3) is clear.

REMARK. We see easily that Corollary 5.5 is nothing but Theorem 2.1 in the special case when $Y(\Xi)$ has the nonsingular normalization.

6. The proofs of Theorems 2.3 and 2.4 and Corollary 2.5. For the proofs we may assume, by (1.9), that $\varpi = \boldsymbol{R}_{\geqq 0} m_1 + \cdots + \boldsymbol{R}_{\geqq 0} m_r$ for a $\boldsymbol{Z}$-basis $\{m_1, \cdots, m_r\}$ of M and that $\Xi \subset \Gamma(\varpi)$ is a d-dimensional k-*spherical* subcomplex (cf. (1.8)). In particular, $\phi = \{0\}$ is the smallest element of Ξ. We again fix these notations throughout this section and adopt the convention at the beginning of Section 5, e.g., we freely identify $\xi \in \Gamma(\varpi)$ with the subset $\{i; 1 \leqq i \leqq r, m_i \in \xi\}$. We also use the notations in (1.11) for simplicity.

For the proof of Theorem 2.3, we first need to analyze the consequences of Section 5, when Ξ is k-spherical. In this case, $H^1(A^{\cdot}(\Xi, s', s'')) \neq 0$ only if $s' \in \Xi$ and $|s''| \geqq 2$, by Propositions 3.6 and 5.4.

LEMMA 6.1. *Let $\Xi \subset \Gamma(\varpi)$ be as above. For $s' \in \Xi$ and $s'' \in \Gamma(\varpi)$ with $|s''| \geqq 2$ and $s' \cap s'' = \phi$, we have $H^1(A^\cdot(\Xi, s', s'')) = k$ if s' and s'' satisfy the following equivalent conditions* (i), (i′), (ii), (iii) *and* (iv). *Otherwise, we have $H^1(A^\cdot(\Xi, s', s'')) = 0$.*

(i) *$\Xi'(s', s'')$ is empty, i.e., for any $l \in s''$ and any $\xi \in \Xi$ with $s' \cup \xi \in \Xi$ and $\xi \cap s'' = \phi$, we have $\xi \cup (s'' \setminus \{l\}) \in \Xi$.*

(i′) *For any $\eta \in \Xi$ with $\eta > s'$ and for any $l \in s''$, we have $\eta \cup s'' \setminus \{l\} \in \Xi$.*

(ii) *For any $\alpha \in \Xi_d$ with $\alpha > s'$, we have $|s'' \setminus \alpha| \leqq 1$. Moreover, if $|s'' \setminus \alpha| = 1$, then $\alpha \cup s'' \setminus \{j\} \in \Xi$ for any $j \in \alpha \cap s''$.*

(iii) *We have* $\dim s' \leqq d-1$. *Moreover for any $\beta \in \Xi_{d-1}$ with $\beta > s'$, we have $|s'' \setminus \beta| \leqq 2$. If the equality is satisfied and $s'' \setminus \beta = \{j, j'\}$, then $\alpha = \beta \cup \{j\}$ and $\alpha' = \beta \cup \{j'\}$ are exactly the d-dimensional cones in Ξ containing β as a face. Moreover, in this case we have $\beta \cap s'' < \{l \in \beta;\ v(\beta \setminus \{l\}) = 3\}$.*

(iv) *Either* (1) *we have* $\dim s' = d-1$ *and $s'' = \{j, j'\}$ with $\alpha = s' \cup \{j\}$, $\alpha' = s' \cup \{j'\}$ being exactly the d-dimensional cones in Ξ containing s' as a face, or* (2) $\dim s' \leqq d-2$ *and we have the following: For any $\gamma \in \Xi_{d-2}$ with $\gamma > s'$, we have $|s'' \setminus \gamma| \leqq 3$. If $|s'' \setminus \gamma| = 3$ and $s'' \setminus \gamma = \{j, j', j''\}$, then necessarily $v(\gamma) = 3$ and $\beta = \gamma \cup \{j\}$, $\beta' = \gamma \cup \{j'\}$, $\beta'' = \gamma \cup \{j''\}$ are exactly the $(d-1)$-dimensional cones in Ξ containing γ as a face. If $|s'' \setminus \gamma| = 2$ and $s'' \setminus \gamma = \{j, j'\}$, then either $v(\gamma) = 3$, or $v(\gamma) = 4$ with $\beta = \gamma \cup \{j\}$, $\beta' = \gamma \cup \{j'\} \in \Xi$ and $\beta \cup \beta' \notin \Xi$.*

COROLLARY 6.2. *If Ξ and s', s'' satisfy the equivalent conditions* (i) *through* (iv) *in Lemma* 6.1, *then we have the following:* $\dim s' \leqq d-1$. *For $\beta \in \Xi_{d-1}$ with $\beta > s'$ and $|s'' \setminus \beta| = 2$, let α, α' be the d-dimensional cones in Ξ containing β as a face. Then for $d \geqq 2$, we have $\{l \in \beta;\ v(\beta \setminus \{l\}) \geqq 5\} < s'$ and $(\alpha \cup \alpha' \setminus \beta) < s'' < (\alpha \cup \alpha' \setminus \beta) \cup \{l \in \beta;\ v(\beta \setminus \{l\}) = 3\}$. Furthermore if* $\dim s' \leqq d-2$, *then for any two β, β' satisfying these conditions, there exist a sequence $\beta = \beta_0, \beta_1, \cdots, \beta_q = \beta'$ in Ξ_{d-1} with $\beta_i > s'$, $|s'' \setminus \beta_i| = 2$ and a sequence $\gamma_1, \cdots, \gamma_q$ in Ξ_{d-2} with $\gamma_i > s'$ and $\beta_{i-1} > \gamma_i < \beta_i$ such that either $v(\gamma_i) = 3$ and $|s'' \setminus \gamma_i| = 3$, or $v(\gamma_i) = 4$ with $\beta_{i-1} \cup \beta_i \notin \Xi$.*

PROOF OF LEMMA 6.1. By Proposition 5.4, we see that $H^1(A^\cdot(\Xi, s', s'')) = \ker[Z^1(\mathrm{Sd}(\Xi(s', s'')), k) \xrightarrow{\text{rest}} Z^1(\mathrm{Sd}(\Xi'(s', s'')), k)]$. Since Ξ is assumed to be k-spherical, we see by Proposition 3.6 that $Z^1(\mathrm{Sd}(\Xi(s', s''), k) = B^1(\mathrm{Sd}(\Xi(s', s''), k) \cong k$. By definition, rest is not the zero map if $\Xi'(s', s'')$ is nonempty. Thus we have $H^1(A^\cdot(\Xi, s', s'')) = k$ (resp. $= 0$) if $\Xi'(s', s'')$ is empty (resp. nonempty).

The equivalence of (i) and (i′) is clear, while (i′) $\Rightarrow$ (ii) is obvious. On the other hand, (ii) implies (i′). Indeed, for any $\eta \in \Xi$ with $\eta > s'$ and $\eta \cup s'' \notin \Xi$, there certainly exists $\alpha \in \Xi_d$ with $\alpha > \eta$, since Ξ is equidimensional. We then necessarily have $\alpha > s'$ and $\alpha \cup s'' \notin \Xi$. Hence by (ii), we have $s'' \setminus \alpha = \{j'\}$ for some j'. Let $j \in s''$. If $j = j'$, then $\eta \cup s'' \setminus \{j\} < \alpha \cup s'' \setminus \{j\} = \alpha \in \Xi$. If $j \neq j'$, then $j \in \alpha \cap s''$. Hence by (ii) we have $\eta \cup s'' \setminus \{j\} < \alpha \cup s'' \setminus \{j\} \in \Xi$.

Let us show (ii) $\Rightarrow$ (iii). First of all, we have $\dim s' \leqq d - 1$. Indeed, since $|s''| \geqq 2$ and $s' \cap s'' = \phi$, we would otherwise have $\dim s' = d$ and $|s'' \setminus s'| = |s''| = 2$, a contradiction to (ii). For any $\beta \in \Xi_{d-1}$ with $\beta > s'$, let $\alpha \neq \alpha'$ be the d-dimensional cones in Ξ containing β as a face. If $\alpha > s''$ or $\alpha' > s''$, then $|s'' \setminus \beta| \leqq 1$. We may thus assume $\alpha \cup s'' \notin \Xi$ and $\alpha' \cup s'' \notin \Xi$. Hence by (ii) there exist $j, j' \in s''$ with $s'' \setminus \alpha = \{j'\}$ and $s'' \setminus \alpha' = \{j\}$. Since $\alpha \cap \alpha' = \beta$, we have $s'' \setminus \beta = \{j, j'\}$. If $j = j'$, then there is nothing further to prove. If $j \neq j'$, then $j \in \alpha$ and $j' \in \alpha'$. Consequently, $\alpha = \beta \cup \{j\}$ and $\alpha' = \beta \cup \{j'\}$. Furthermore for $l \in \beta \cap s''$, let $\gamma = \beta \setminus \{l\}$ be v-valent. If $v \geqq 4$, then there certainly exists d-dimensional $\alpha'' > \gamma$ different from α, α' with $\alpha'' \ni j$ but $\alpha'' \not\ni j', l$. Hence $s'' \setminus \alpha'' = \{j', l\}$, a contradiction to (ii).

Let us now show (iii) $\Rightarrow$ (iv). If $\dim s' = d-1$, then since $|s''| \geqq 2$ and $s' \cap s'' = \phi$, we have $|s'' \setminus s'| = |s''| = 2$ by (iii). Thus we obviously have (iv) (1). Now suppose $\dim s' \leqq d - 2$. Let $\gamma \in \Xi_{d-2}$ with $\gamma > s'$ be v-valent. If $|s'' \setminus \gamma| \geqq 4$, then for any one of the $(d-1)$-dimensional $\beta > \gamma$, we have $|s'' \setminus \beta| \geqq 3$, a contradiction to (iii). Now suppose $|s'' \setminus \gamma| = 3$. Then for any one of the $(d-1)$-dimensional cones $\beta > \gamma$, we necessarily have $|s'' \setminus \beta| = 2$. Hence γ should be 3-valent. Finally, suppose $|s'' \setminus \gamma| = 2$ with $s'' \setminus \gamma = \{j, j'\}$. If $v = 3$, then there is nothing more to prove. If $v = 4$, then $\gamma \cup \{j, j'\} \notin \Xi$, since otherwise there would exist $\beta > \gamma$ with $\beta \not\ni j, j'$ and $\beta \cup \{j\} \in \Xi$, $\beta \cup \{j'\} \notin \Xi$, a contradiction to (iii). If $v \geqq 5$, then again there exists $\beta > \gamma$ with $\beta \not\ni j, j'$, $\beta \cup \{j\} \in \Xi$, $\beta \cup \{j'\} \notin \Xi$, a contradiction.

It remains to show (iv) $\Rightarrow$ (ii). Let $\alpha \in \Xi_d$ satisfy $\alpha > s'$. If $\alpha > s''$, then there is nothing to prove. Thus we may assume $\alpha \cup s'' \notin \Xi$. If $\dim s' = d - 1$, then $s'' = \{j, j'\}$ and $\alpha = s' \cup \{j\}$, say. Hence $s'' \setminus \alpha = \{j'\}$ and $\alpha \cup s'' \setminus \{j\} = s' \cup \{j'\} \in \Xi$. Suppose $\dim s' \leqq d - 2$. Then there exists $\gamma \in \Xi_{d-2}$ with $\alpha > \gamma > s'$. Hence $|s'' \setminus \gamma| \geqq |s'' \setminus \alpha| \geqq 1$. Consequently if $s'' \setminus \gamma = \{j'\}$, then $s'' \setminus \alpha = \{j'\}$. Since $|s''| \geqq 2$ and $s' \cap s'' = \phi$, there exists $j \in \alpha \cap s''$ with $\alpha \setminus \{j\} > s'$. Thus there certainly exists $(d-2)$-dimensional γ' satisfying $\alpha \setminus \{j\} > \gamma' > s'$, hence $|s'' \setminus \gamma'| \geqq 2$. Thus replacing γ by γ', we may from the outset assume the existence of $\gamma \in \Xi_{d-2}$ with $\alpha > \gamma > s'$ and $|s'' \setminus \gamma| \geqq 2$. Then by (iv), either γ is 3-valent, or is 4-valent with $\gamma \cup$

$s'' \notin \Xi$. Since $\alpha \cup s'' \notin \Xi$ by our assumption, we can easily check that $\alpha \cup s'' \setminus \{j\} \in \Xi$ for any $j \in \alpha \cap s''$.

PROOF OF COROLLARY 6.2. The first part can be seen as follows: For $\beta \in \Xi_{d-1}$ with $\beta > s'$ and $|s'' \setminus \beta| = 2$, let α, α' be the distinct d-dimensional cones in Ξ containing β as a face. Then by Lemma 6.1 (iii), we have $s'' > \alpha \cup \alpha' \setminus \beta$. Let $l \in \beta$ with $\gamma = \beta \setminus \{l\}$ being v-valent. If $l \in s''$, then $l \notin s'$, hence $\gamma > s'$ and $|s'' \setminus \gamma| = 3$. Thus $v = 3$ by Lemma 6.1 (iv). Suppose $v \geqq 5$. Then again by Lemma 6.1 (iv), we cannot have $\gamma > s'$. Hence $l \in s'$.

Let us now prove the latter half of Corollary 6.2. Let us call β, β' with $\beta > s'$, $\beta' > s'$ and $|s'' \setminus \beta| = |s'' \setminus \beta'| = 2$ *equivalent*, if there exist a sequence $\beta = \beta_0, \cdots, \beta_q = \beta'$ in Ξ_{d-1} and a sequence $\gamma_1, \cdots, \gamma_q$ in Ξ_{d-2} satisfying the following conditions: $\beta_i > s'$, $|s'' \setminus \beta_i| = 2$, $\beta_{i-1} > \gamma_i < \beta_i$, $\gamma_i > s'$ and either $v(\gamma_i) = 3$ and $|s'' \setminus \gamma_i| = 3$, or $v(\gamma_i) = 4$ with $\beta_{i-1} \cup \beta_i \notin \Xi$. Ξ. We need to show that any two β's satisfying $\beta > s'$ and $|s'' \setminus \beta| = 2$ are equivalent in this sense.

Replacing Ξ by $\mathrm{Star}_{s'}(\Xi)$, we may assume $s' = \phi$, hence $d \geqq 2$ by assumption. We have $\tilde{\Xi} = \{\eta \in \Xi; \eta \cup s'' \notin \Xi\} = \Xi \setminus \Phi$, with $\Phi = \{\eta \in \Xi; \eta \cup s'' \in \Xi\}$. If $s'' \notin \Xi$, then Φ is empty. If $s'' \in \Xi$, then Φ is k-semispherical with respect to s''. Since $s'' \neq \phi$, Φ is k-homologically trivial (cf. (1.8)). On the other hand, Ξ is k-spherical, hence $H_d(\tilde{\Xi}, k) = k$. Moreover, for any $\beta \in \tilde{\Xi}_{d-1}$, there exist exactly two $\alpha, \alpha' \in \tilde{\Xi}_d$ with $\alpha > \beta < \alpha'$, since $\tilde{\Xi}$ is star closed in the k-spherical Ξ. Thus for any $\beta, \beta' \in \tilde{\Xi}_{d-1}$, there exist a sequence $\beta = \beta_0, \cdots, \beta_q = \beta'$ in $\tilde{\Xi}_{d-1}$ and a sequence $\alpha_1, \cdots, \alpha_q$ in $\tilde{\Xi}_d$ such that $\beta_{i-1} < \alpha_i > \beta_i$. Note that $d \geqq 2$ by assumption and that $v(\beta'' \setminus \{l\}) = 3$ or 4 for any $\beta'' \in \tilde{\Xi}_{d-1}$ and any $l \in \beta''$ satisfying $|s'' \setminus \beta''| = 2$. We are thus reduced to showing the following (1) and (2):

(1) If β, β', α in $\tilde{\Xi}$ satisfy $\beta < \alpha > \beta'$ and $|s'' \setminus \beta| = |s'' \setminus \beta'| = 2$, then β and β' are equivalent. Indeed, if $\beta \neq \beta'$, then $\gamma = \beta \cap \beta'$ is 3-valent by Lemma 6.1 (iv), since $|s'' \setminus \gamma| \geqq 3$. Hence $|s'' \setminus \gamma| = 3$, and β is equivalent to β'

(2) If α, α', β'' in $\tilde{\Xi}$ satisfy $\alpha \neq \alpha'$, $\alpha > \beta'' < \alpha'$ and $|s'' \setminus \beta''| \neq 2$, then there exist $\beta < \alpha$ and $\beta' < \alpha'$ with $|s'' \setminus \beta| = |s'' \setminus \beta'| = 2$ such that β and β' are equivalent. Indeed, since $|s'' \setminus \alpha| = |s'' \setminus \alpha'| = 1$ by Lemma 6.1 (ii) and $|s'' \setminus \beta''| \neq 2$, we necessarily have $s'' \setminus \alpha = s'' \setminus \alpha' = s'' \setminus \beta''$ by Lemma 6.1 (iii). Since $|s''| \geqq 2$, there thus exists $j \in s'' \cap \beta''$. Let $\gamma = \beta'' \setminus \{j\}$. Then $\gamma \cup s'' = \beta'' \cup s'' \notin \Xi$ and $|s'' \setminus \gamma| = 2$. Hence by Lemma 6.1 (iv), we easily see that $v(\gamma) = 4$. Then certainly, $\beta = \alpha \setminus \{j\}$ and $\beta' = \alpha' \setminus \{j\}$ satisfy $|s'' \setminus \beta| = |s'' \setminus \beta'| = 2$ and $\beta \cup \beta' \notin \Xi$, hence they are equivalent.

PROOF OF THEOREM 2.3. By (1.9), we may assume $\Xi \subset \Gamma(\varpi)$ to be as at the beginning of this section. Let us work on the ring level for $S = S(\Xi)$. By Lemma 5.2, we have $\mathrm{Ext}^1_S(L^S_\cdot, S) = H^1(E^\cdot)$. Let $\{(\lambda)^*; \lambda \in \Lambda\}$ be the P-basis of $(F_1)^* = \mathrm{Hom}_P(F_1, P)$ dual to the P-basis $\mathrm{Sd}_1(\Lambda)$ of F_1.

For $\beta \in \Xi_{d-1}$, let $\alpha, \alpha' \in \Xi_d$ be the distinct cones satisfying $\alpha > \beta$ and $\alpha' > \beta$. Then $e(\boldsymbol{m}(\alpha \cup \alpha'))$ is an element of $J = J(\Xi)$. Thus we have a degree zero homomorphism of M-graded P-modules $Pe(\boldsymbol{m}(\alpha \cup \alpha')) \to J$, which has the lifting $Pe(\boldsymbol{m}(\alpha \cup \alpha')) \to F_1$ sending $e(\boldsymbol{m}(\alpha \cup \alpha'))$ to $(\alpha \cup \alpha')$. Consider the induced homomorphisms $P/\mathfrak{p}(\beta) \otimes_S Z^1(E^\cdot) = \mathrm{Hom}_P(J, P/\mathfrak{p}(\beta)) \to P/\mathfrak{p}(\beta) \otimes_P E^1 = \mathrm{Hom}_P(F_1, P/\mathfrak{p}(\beta)) \to \mathrm{Hom}_P(Pe(\boldsymbol{m}(\alpha \cup \alpha')), P/\mathfrak{p}(\beta))$, the term on the extreme right hand side of which is a free $(P/\mathfrak{p}(\beta))$-module of rank one with the basis $h(\beta)$ defined by $h(\beta)(e(\boldsymbol{m}(\alpha \cup \alpha'))) = 1$. Thus $\deg h(\beta) = -\boldsymbol{m}(\alpha \cup \alpha')$ (cf. the beginning of the proof of Proposition 5.3). Obviously, the homomorphism $P/\mathfrak{p}(\beta) \otimes_S E^0 = \mathrm{Hom}_P(\Omega^1_P, P/\mathfrak{p}(\beta)) \to \mathrm{Hom}_P(J, P/\mathfrak{p}(\beta))$ composed with the above homomorphisms gives the zero map. Thus we have homomorphisms $P/\mathfrak{p}(\beta) \otimes_S H^1(E^\cdot) \to P/\mathfrak{p}(\beta) \otimes_S E^1/B^1(E^\cdot) \to (P/\mathfrak{p}(\beta))h(\beta)$. The image of the composite homomorphism is contained in $(P/\mathfrak{p}(\beta))g(\beta)$, where $g(\beta) = e(\boldsymbol{m}(\{l \in \beta;\ \boldsymbol{v}(\beta \setminus \{l\}) \geqq 4\}) + \boldsymbol{m}(\{l \in \beta;\ \boldsymbol{v}(\beta \setminus \{l\}) \geqq 5\}))h(\beta)$. Indeed, by Propositions 5.3, 5.4 and 3.6, the image of the first homomorphism consists of the k-linear combinations of elements of the form

$$\sum_{\xi \in \Xi(s', s'')} e(m')e(\boldsymbol{m}(\xi))(1 \otimes (\xi \amalg s'')^*) ,$$

with $|s''| \geqq 2$ and $\mathrm{supp}(m') = s'$, where 1 is the unit element of $P/\mathfrak{p}(\beta)$. Moreover, $s' < \beta$ and $\xi < \beta$ should be satisfied. The element $e(m')e(\boldsymbol{m}(\xi))(1 \otimes (\xi \amalg s'')^*)$ is mapped to a nonzero element of $(P/\mathfrak{p}(\beta))h(\beta)$ only if $\xi \amalg s'' < \alpha \cup \alpha'$ and $\alpha \cup \alpha' \setminus (\xi \amalg s'') < \beta$. Then we have $\Xi \not\ni \xi \amalg s'' < \beta \cup s'' < \alpha \cup \alpha'$, hence $\beta \cup s'' = \alpha \cup \alpha'$ and $|s'' \setminus \beta| = 2$. Thus by Corollary 6.2, we necessarily have $\{l \in \beta;\ \boldsymbol{v}(\beta \setminus \{l\} \geqq 5\}) < s' < \beta$ and $\alpha \cup \alpha' \setminus \beta < s'' < \alpha \cup \alpha' \setminus \{l \in \beta;\ \boldsymbol{v}(\beta \setminus \{l\}) \geqq 4\}$. The element $e(m')e(\boldsymbol{m}(\xi))(1 \otimes (\xi \amalg s'')^*)$ is then mapped to $e(m')e(\boldsymbol{m}(\xi))e(\boldsymbol{m}(\alpha \cup \alpha' \setminus \xi \amalg s''))h(\beta) = e(m')e(\boldsymbol{m}(\alpha \cup \alpha' \setminus s''))h(\beta)$, which is thus a multiple of $g(\beta)$.

Consequently, we have a homomorphism

$$H^1(E^\cdot) \to \bigoplus_{\beta \in \Xi_{d-1}} G(\beta) ,$$

where $G(\beta) = (P/\mathfrak{p}(\beta))g(\beta)$. Obviously, the $\mathcal{O}_Y$-module associated to $G(\beta)$ is $\mathcal{G}(\beta)$. On the other hand, the $\mathcal{O}_Y$-module $\mathcal{G}(\gamma)$ for $\gamma \in \Xi_{d-2}$ is associated to the S-module $G(\gamma)$ described as follows: If $\boldsymbol{v}(\gamma) = 4$ and if $\beta_1, \beta_2, \beta_3, \beta_4 > \gamma$ are distinct with $\beta_1 \cup \beta_3, \beta_2 \cup \beta_4 \notin \Xi$, then $\mathcal{G}(\gamma, \beta_i)$ is associated to the free $(P/\mathfrak{p}(\gamma))$-module $G(\gamma, \beta_i)$ of rank one with the base $g(\gamma, \beta_i)$ satisfying $\deg g(\gamma, \beta_1) = \deg g(\gamma, \beta_3) = -\boldsymbol{m}(\beta_2 \cup \beta_4)$ and $\deg g(\gamma, \beta_2) = \deg g(\gamma, \beta_4) =$

$-\boldsymbol{m}(\beta_1 \cup \beta_3)$. Similarly, if $\boldsymbol{v}(\gamma) = 3$ and if $\beta_1, \beta_2, \beta_3 > \gamma$ are distinct, then $\mathscr{G}(\gamma, \beta_i)$ is associated to the free $(P/\mathfrak{p}(\gamma))$-module $G(\gamma, \beta_i)$ of rank one with the base $g(\gamma, \beta_i)$ satisfying $\deg g(\gamma, \beta_i) = -\boldsymbol{m}(\beta_1 \cup \beta_2 \cup \beta_3)$. Then the S-module $G(\gamma)$ is defined exactly as in the case of $\mathscr{G}(\gamma)$. $G(\gamma) = 0$ for $\boldsymbol{v}(\gamma) \geqq 5$.

For $\boldsymbol{v}(\gamma) = 4$ and $\beta > \gamma$, we easily see that the (β, γ)-component of the homomorphism ε in Theorem 2.3 is induced by the map sending $g(\beta)$ to $e(\boldsymbol{m}(\{l \in \beta; \boldsymbol{v}(\beta \setminus \{l\}) \geqq 5\}) + \boldsymbol{m}(\{l \in \gamma; \boldsymbol{v}(\beta \setminus \{l\}) \geqq 4\}))g(\gamma, \beta)$, since $\{l \in \beta; \boldsymbol{v}(\beta \setminus \{l\}) \neq 4\}$ is obviously contained in γ. Similarly for $\boldsymbol{v}(\gamma) = 3$ and $\beta > \gamma$, we see that the (β, γ)-component of the homomorphism ε is induced by the map sending $g(\beta)$ to $e(\boldsymbol{m}(\{l \in \beta; \boldsymbol{v}(\beta \setminus \{l\}) \geqq 5\}) + \boldsymbol{m}(\{l \in \beta; \boldsymbol{v}(\beta \setminus \{l\}) \geqq 4\}))g(\gamma, \beta)$, since $\{l \in \beta; \boldsymbol{v}(\beta \setminus \{l\}) \geqq 4\}$ is contained in γ.

For each $m \in M$, let $m = m' - m''$ be the canonical decomposition with $s' = \mathrm{supp}(m')$ and $s'' = \mathrm{supp}(m'')$. Then by Proposition 5.3 and Lemma 5.2, we have $\mathrm{Ext}^1_S(L^S_{\cdot}, S)_m = H^1(E^{\cdot})_m = H^1(A^{\cdot}(\Xi, s', s''))$ if $m'' = \boldsymbol{m}(s'')$, and is zero otherwise. By Propositions 3.6 and 5.4 and by Lemma 6.1, we have $H^1(A^{\cdot}(\Xi, s', s'')) = k$ if and only if $s' \in \Xi$, $|s''| \geqq 2$ and the equivalent conditions (iii) and (iv) of Lemma 6.1 are satisfied. Otherwise it is zero. Thus by Corollary 6.2, we see that the homogeneous component of degree m of

$$0 \to H^1(E^{\cdot}) \to \bigoplus_{\beta \in \Xi_{d-1}} G(\beta) \to \bigoplus_{\gamma \in \Xi_{d-2}} G(\gamma)$$

is exact and we are done.

Proof of Theorem 2.4. Let us work on the ring level $S = S(\Xi)$. For $m \in M$, let $m = m' - m''$ be the canonical decomposition with $\mathrm{supp}(m') = s'$ and $\mathrm{supp}(m'') = s''$. By Proposition 5.3, we have $\mathrm{Ext}^2_S(L^S_{\cdot}, S)_m = H^2(A^{\cdot}(\Xi, s', s''))$ if $m'' = \boldsymbol{m}(s'')$ and is zero otherwise. Since Ξ is k-spherical, we have $H^i(\mathrm{Sd}(\Xi(s', s'')), k) = 0$ for $i = 1, 2$ by Proposition 3.6. Hence by Proposition 5.4, we have $H^2(A^{\cdot}(\Xi, s', s'') = H^1(\mathrm{Sd}(\Xi'(s', s'')), k)$ if $s' \in \Xi$ and $|s''| \geqq 2$ and is zero otherwise. We are done, since $H^1(\mathrm{Sd}(\Xi'(s', s'')), k) = H_{d-1}(\Phi(s', s''), k)$ by Proposition 3.8.

Proof of Corollary 2.5. By Theorem 2.4, we need to compute $H_{d-1}(\Phi(s', s''), k)$ for $s' \in \Xi$ and $s'' \in \Gamma(\varpi)$ satisfying $s' \cap s'' = \phi$ and $|s''| \geqq 2$. The result (1) for $d \leqq 1$ is obvious.

(2) If $d = 2$, then Δ is a decomposition of the circle into r arcs by the vertices $1, 2, \cdots, r$ arranged in this order. $H_1(\Phi(s', s''), k)$ vanishes if $s' \neq \phi$, by the result (1) applied to $\mathrm{Star}_{s'}(\Xi)$ in view of (1.7). Moreover $\Phi(\phi, s'')$ is empty if $|s''| \geqq 3$. Thus we need to look at $H_1(\Phi(\phi, s''), k)$ when $|s''| = 2$, which, by an easy computation, is seen to be nonzero if

and only if $s'' = \{i, j\}$ with $j - i \not\equiv 0, \pm 1, \pm 2 \pmod r$. Such s'' exists only when $r \geqq 6$.

(3) If $d = 3$, then Δ is a triangulation of the 2-sphere. $H_2(\Phi(s', s''), k)$ vanishes if $|s'| \geqq 2$ again by (1) applied to $\mathrm{Star}_{s'}(\Xi)$. If $|s'| = 1$, we see that $H_2(\Phi(s', s'')), k)$ vanishes for all s'' if and only if $v(s') \leqq 5$ by (2) applied to $\mathrm{Star}_{s'}(\Xi)$.

Thus we first show the following:

PROPOSITION 6.3. *There are exactly eleven combinatorially different triangulations Δ of the 2-sphere with only five or less edges being incident to each vertex. Their stereographic projection onto the plane from one of the vertices look like the diagrams in Figure 1 immediately after Corollary 2.5.*

PROOF. Here is a sketch of the proof. Let Δ_0 be the set of the vertices of Δ. Let us call $\gamma \in \Delta_0$ v-valent and write $v(\gamma) = v$ if there are exactly v edges incident to the vertex γ. Depending on the cases, let us choose an appropriate $\gamma_0 \in \Delta_0$ and let $\gamma_1, \cdots, \gamma_v \in \Delta_0$ be the vertices adjacent to γ_0 in this circular order. Then the triangles in Δ not containing γ_0, and their faces, give rise to a triangulation Δ', not subdividing the circumference, of the v-gon $\gamma_1\gamma_2 \cdots \gamma_v\gamma_1$. At each vertex γ_i, there are exactly $v(\gamma_i) - 1 \leqq 4$ edges of Δ'. Each interior vertex $\gamma \in \Delta'$ has $v(\gamma) \leqq 5$ edges. Drawing the picture in each of the following three cases, we can easily classify such Δ'.

Case (1). There exists a 3-valent γ_0 in Δ_0. We thus have $3 \leqq v(\gamma_1) \leqq v(\gamma_2) \leqq v(\gamma_3) \leqq 5$. Then Δ is combinatorially equivalent to the following.

$$
\begin{aligned}
&v(\gamma_1) = 3 && \Longrightarrow [4\text{–}1] \\
&v(\gamma_1) = v(\gamma_2) = 4 && \Longrightarrow [5\text{–}1] \\
&v(\gamma_1) = 4,\ v(\gamma_2) = 5 && \Longrightarrow [6\text{–}2] \\
&v(\gamma_1) = 5 && \Longrightarrow [7\text{–}5] \quad \text{or} \quad [8\text{–}14]\,.
\end{aligned}
$$

Case (2). There exists no 3-valent vertex but there exists a 5-valent γ_0. Since $v(\gamma_i) = 4$ or 5 for $i = 1, \cdots, 5$, we need to consider only the following cases, by renumbering the vertices, if necessary.

$$
\begin{aligned}
&v(\gamma_1) = v(\gamma_2) = v(\gamma_3) = 4 && \Longrightarrow [7\text{–}1] \\
&v(\gamma_1) = v(\gamma_2) = 4,\ v(\gamma_3) = v(\gamma_5) = 5 && \Longrightarrow [8\text{–}13] \\
&v(r_1) = v(\gamma_3) = 4,\ v(\gamma_2) = v(\gamma_4) = v(\gamma_5) = 5 && \Longrightarrow [\ 9\ ] \\
&v(\gamma_1) = 4,\ v(\gamma_2) = v(\gamma_3) = v(\gamma_4) = v(\gamma_5) = 5 && \Longrightarrow [10] \\
&v(\gamma_i) = 5 \quad \text{for all} \quad i && \Longrightarrow [12]\,.
\end{aligned}
$$

Case (3). Every $\gamma \in \Delta_0$ is 4-valent. Then [6–1] is the only possibility.

PROOF OF COROLLARY 2.5 CONTINUED. By Theorem 2.4, Proposition 6.3 and what we have seen so far, we have

$$\mathscr{E}xt^2_{\mathscr{O}_Y}(L^Y_\cdot, \mathscr{O}_Y) = 0$$

if and only if Δ is among the diagrams in Figure 1 and satisfies

$$H_2(\Phi(\phi, s''), k) = 0 \qquad \text{for} \quad |s''| \geqq 2 \ .$$

For simplicity, we denote

$$\Phi = \Phi(\phi, s'') = \{\xi \in \Xi;\ \xi \cup s'' \setminus \{l\} \in \Xi \quad \text{for all} \quad l \in s''\}$$

in the rest of the proof.

We may assume $s''\setminus\{l\} \in \Xi$ for all $l \in s''$, since otherwise Φ is empty. Thus in particular we have $2 \leqq |s''| \leqq 4$. We consider the following five cases separately and prove the following:

$H_2(\Phi, k) = 0$ if we are in

$$\begin{aligned}
&\text{case (1)} \quad |s''| = 4\ , \\
&\text{case (2)} \quad |s''| = 3 \quad \text{and} \quad s'' \in \Xi\ , \\
&\text{case (3)} \quad |s''| = 3 \quad \text{and} \quad s'' \notin \Xi\ , \quad \text{or} \\
&\text{case (4)} \quad |s''| = 2 \quad \text{and} \quad s'' \in \Xi\ .
\end{aligned}$$

There exists some s'' with $H_2(\Phi, k) \neq 0$ in

$$\text{case (5)} \quad |s''| = 2 \quad \text{and} \quad s'' \notin \Xi$$

if and only if Δ is [8–14].

Case (1). $|s''| = 4$. Since $s''\setminus\{l\} \in \Xi$ for all $l \in s''$, we see easily that Δ is [4–1] and $\Phi = \Xi$. Thus by the k-sphericity of Ξ, we have $H_2(\Phi, k) = 0$ (cf. (1.8)).

Case (2). $|s''| = 3$ and $s'' \in \Xi$. Suppose there exists $\gamma \in \Phi_1$ with $\gamma \not< s''$. Then Δ is [4–1], since $s'' \in \Xi$ and $\gamma \cup s''\setminus\{l\} \in \Xi$ for all $l \in s''$. Hence $\Phi = \Xi$ and we are done again. On the contrary, suppose every $\gamma \in \Phi_1$ is a face of s''. Then Φ consists of all the faces of s'', since Φ is a subcomplex of Ξ. Thus Φ is homologically trivial (cf. Ishida [I_3, the comment immediately after Corollary 2.3]).

Case (3). $|s''| = 3$ and $s'' \notin \Xi$. Let $s'' = \gamma_1 \cup \gamma_2 \cup \gamma_3$ with $\gamma_1, \gamma_2, \gamma_3 \in \Xi_1$. Because of the conditions $\gamma_1 \cup \gamma_2, \gamma_2 \cup \gamma_3, \gamma_3 \cup \gamma_1 \in \Xi$ and $\gamma_1 \cup \gamma_2 \cup \gamma_3 \notin \Xi$, we see that there are only four possibilities [5–1], [6–2], [7–5] and [8–14] for Δ (cf. Figure 2).

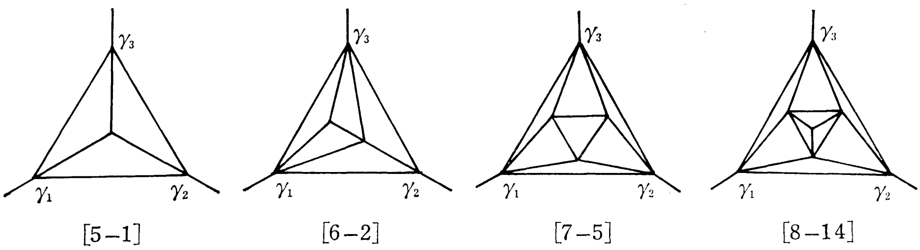


FIGURE 2

In case [5–1], we have $\Phi=\Xi$, hence we are done. In the other three cases, let γ_0 correspond to the vertex of Δ at infinity. Then Φ coincides with $\{\xi\in\Xi;\ \xi\cup\gamma_0\in\Xi\}$, which is k-semispherical with respect to γ_0, hence is k-homologically trivial (cf. (1.8)).

Case (4). $|s''|=2$ and $s''\in\Xi$. Let $s''=\gamma_1\cup\gamma_2$ with γ_1,γ_2 in Ξ_1. Clearly, Φ contains $\mathscr{U}=\{\xi\in\Xi;\ s''\cup\xi\in\Xi\}$. Since obviously $\Phi_3\subset\mathscr{U}$, there are the following three possibilities.

(i) $\Phi=\mathscr{U}$. Since this is k-semispherical with respect to s'', we are done.

(ii) There exists $\gamma_3\in\Phi_1$ not contained in $\mathscr{U}$. Since we have $\gamma_1\cup\gamma_2,\gamma_2\cup\gamma_3,\gamma_3\cup\gamma_1\in\Xi$ and $\gamma_1\cup\gamma_2\cup\gamma_3\notin\Xi$, we have the five possibilities in Figure 3.

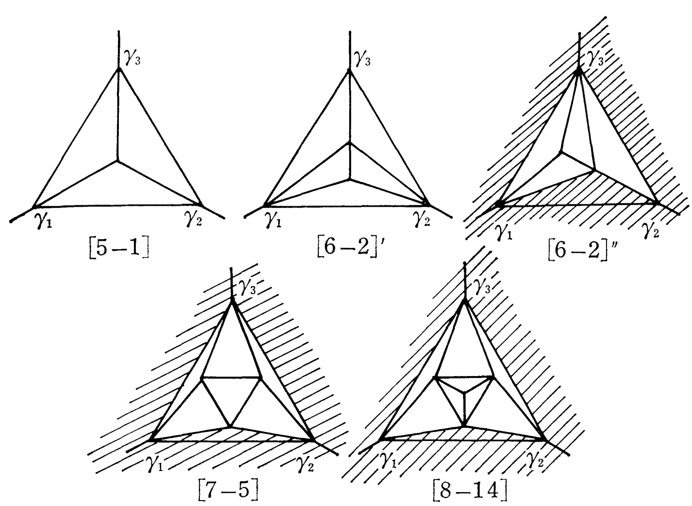


FIGURE 3

In the cases [5–1] and [6–2]′, we again have $\Phi=\Xi$ and we are done. In the other cases [6–2]″, [7–5] and [8–14], Φ corresponds to a triangulation of a simply connected closed subset of the 2-sphere (the shaded part of each of the diagrams). Hence Φ is homologically trivial.

(iii) Φ_1 is contained in $\mathscr{U}$ and there exists $\beta \in \Phi_2$ not in $\mathscr{U}$. Let $\beta = \gamma_3 \cup \gamma_4$ with $\gamma_3, \gamma_4 \in \Xi_1$. We have $\gamma_1 \cup \gamma_3 \cup \gamma_4, \gamma_2 \cup \gamma_3 \cup \gamma_4 \in \Xi$, since $\beta \in \Phi_2$. On the other hand, we have $\gamma_1 \cup \gamma_2 \cup \gamma_3, \gamma_1 \cup \gamma_2 \cup \gamma_4 \in \Xi$, since $\gamma_3, \gamma_4 \in \Phi_1 \subset \mathscr{U}$. Furthermore, $\beta \cup s'' = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \notin \Xi$, hence $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are mutually distinct and Δ is [4–1]. Thus $\Phi = \Xi$ and we are done again.

Case (5). $|s''| = 2$ and $s'' \notin \Xi$. Let Ψ be the subcomplex of Φ defined by $\Psi = \{\xi \in \Phi; \xi \cap s'' = \phi\}$. Then we claim that there exists an isomorphism $H_i(\Phi, k) \xrightarrow{\sim} H_{i-1}(\Psi, k)$ for all i. Indeed, let $\Phi' = \{\xi \in \Phi; \xi \cup \gamma_1 \in \Xi\}$ and $\Phi'' = \{\xi \in \Phi; \xi \cup \gamma_2 \in \Xi\}$. Since $s'' \notin \Xi$, we see easily that $\Phi = \Phi' \cup \Phi''$ and $\Psi = \Phi' \cap \Phi''$. In view of the Mayer-Vietoris exact sequence, it suffices to show the k-homological triviality of Φ' and Φ''. But, for instance, the map $\Psi \to \Phi' \setminus \Psi$ sending ξ in Ψ to $\xi \cup \gamma_1$ is bijective, hence so are the connecting homomorphisms $H_i(\Phi' \setminus \Psi, k) \to H_{i-1}(\Psi, k)$ in the long exact sequence arising from the inclusion $\Psi \subset \Phi'$. Hence Φ' is k-homologically trivial.

Thus it remains to pick up those Δ's in Proposition 6.3 for which $\Psi = \{\xi \in \Xi; \xi \cap s'' = \phi, \xi \cup s'' \setminus \{l\} \in \Xi$ for all $l \in s''\}$ satisfies the condition $H_1(\Psi, k) = 0$ whenever $\gamma_1, \gamma_2 \in \Xi$ and $\gamma_1 \cup \gamma_2 \notin \Xi$. The vanishing of $H_1(\Psi, k)$ means that either Ψ consists of ϕ only or corresponds to a triangulation of a closed connected subset of Δ.

We claim that each $\gamma \in \Psi_1$ is a face of some $\beta \in \Psi_2$. Indeed, since $\gamma \cup \gamma_1, \gamma \cup \gamma_2 \in \Xi_2, \gamma_1 \cup \gamma_2 \notin \Xi$ and $v(\gamma) \leqq 5$, we have $v(\gamma) \neq 3$ and there

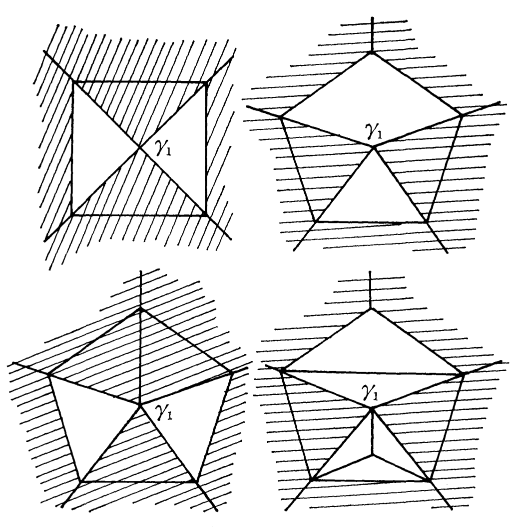


FIGURE 4

exists $\gamma_3 \in \Xi_1$ with $\gamma \cup \gamma_1 \cup \gamma_3$, $\gamma \cup \gamma_2 \cup \gamma_3 \in \Xi_3$. Hence $\beta = \gamma \cup \gamma_3 \in \Psi_2$ will do.

Thus if $v(\gamma_2) = 3$, then either $\Psi = \{\phi\}$ or Ψ corresponds to a triangulation of a closed connected subset. Hence $H_1(\Psi, k) = 0$.

Furthermore, if $H_1(\Psi, k) \neq 0$, then Δ is obtained by a triangulation of the nonshaded part of one of the three possibilities, i.e., the first three diagrams in Figure 4, where γ_2 corresponds to the vertex at infinity. In view of the condition $v(\gamma_1) \leqq 5$, the only possible triangulation for which $H_1(\Psi, k) \neq 0$ is the fourth diagram in Figure 4, which is easily seen to be equivalent to [8–14].

Thus we have completed the proof of Corollary 2.5.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, 980
JAPAN