Tôhoku Math. Journ.
34 (1982), 587-629.

HOLOMORPHIC STRUCTURES MODELED AFTER HYPERQUADRICS

SHOSHICHI KOBAYASHI* AND TAKUSHIRO OCHIAI*

(Received April 20, 1982)

1. Introduction. In our joint paper with Inoue [7], we studied holomorphic affine connections and affine structures on complex manifolds and classified all compact complex surfaces admitting such structures. In [12] we studied holomorphic projective connections and projective structures and classified all compact complex surfaces admitting such structures. The one case left open in [12] has been solved recently ([13]). Both of our papers were partly based on Gunning's earlier work [4].

In the present paper we shall study holomorphic geometric structures modeled after a hyperquadric. Leaving the precise definitions of holomorphic $CO(n; C)$-structure and quadric structure to § 2, we shall explain them by the following diagram:

Model space	*Infinitesimal structure*	*Local structure*
Affine space C^n	Affine connection	Affine structure
Projective space P_nC	Projective connection	Projective structure
Quadric Q_n	$CO(n; C)$-structure	Quadric structure

By a quadric Q_n we mean a non-singular hyperquadric in $P_{n+1}C$; it is a holomorphic analogue of a sphere. A holomorphic $CO(n; C)$-structure may be considered as a holomorphic conformal connection, and a quadric structure as a flat holomorphic conformal structure.

In § 2, § 3 and § 4, we shall discuss general results valid for all dimension. In the subsequent sections we determine all compact complex surfaces admitting holomorphic $CO(2; C)$-structures and quadric structures. The 2-dimensional case is somewhat exceptional as in the case of conformal differential geometry. This is because a non-singular quadric Q_2 is isomorphic to $P_1C \times P_1C$, i.e., reducible. Hence, a holomorphic $CO(2; C)$-structure is equivalent (modulo passing to a double covering) to a splitting of the holomorphic tangent bundle into a direct sum of two holomorphic line subbundles, which in turn, is equivalent to a pair of mutually transversal holomorphic foliations of dimension 1. We take a

* Partially supported by NSF Grant MCS 79-02552.

full advantage of this special situation to achieve the following classification.

The class of compact complex surfaces admitting holomorphic $CO(2;C)$-structure consists of the following:

(1) *the quadric $P_1C \times P_1C$;*

(2) *ruled surfaces of the form $\tilde{\Delta} \times_\rho P_1C$, where $\tilde{\Delta}$ is the universal covering space of an algebraic curve Δ and ρ is a homomorphism of $\pi_1(\Delta)$ into $\mathrm{Aut}(P_1C) = PGL(1)$, in other words, flat holomorphic fibre bundles over Δ with fibre P_1C;*

(3) *bielliptic (or hyperelliptic) surfaces;*

(4) *complex tori;*

(5) *minimal elliptic surfaces with $c_2 = 0$ and even first Betti number;*

(6) *surfaces with universal covering space $D \times D$ (bidisk);*

(7) *Hopf surfaces $(C^2 - 0)/\Gamma$, where Γ consists of linear transformations of the form*

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix};$$

(8) *Inoue surfaces S_U associated with $U \in SL(3; Z)$;*

These surfaces admit not only holomorphic $CO(2; C)$-structures but also quadric structures.

2. **Holomorphic $CO(n; C)$-structures.** Let M be an n-dimensional complex manifold. Let

$$CO(n; C) = \{cU;\ U \in O(n; C) \text{ and } c \in C^*\}, \tag{2.1}$$

where $O(n; C) = \{U \in GL(n; C);\ {}^tUU = 1\}$. Let $L(M)$ be the bundle of complex linear frames over M; it is a holomorphic principal bundle with structure group $GL(n; C)$. A holomorphic principal subbundle P of $L(M)$ with structure group $CO(n; C)$ is called a *holomorphic $CO(n; C)$-structure* on M.

Given a holomorphic $CO(n; C)$-structure P on M, we can cover M by small open sets U_α with local coordinate system $z_\alpha^1, \cdots, z_\alpha^n$ and find a holomorphic non-degenerate symmetric covariant tensor field

$$g_\alpha = \sum_{i,j} g_{\alpha ij}\, dz_\alpha^i dz_\alpha^j, \qquad \det(g_{\alpha ij}) \neq 0, \tag{2.2}$$

on each U_α in such a way that

$$g_\beta = f_{\beta\alpha} g_\alpha \quad \text{on} \quad U_\alpha \cap U_\beta, \tag{2.3}$$

where $f_{\beta\alpha}$ is a holomorphic function on $U_\alpha \cap U_\beta$ (without zeros).

Conversely, given $\{U_\alpha, g_\alpha\}$ satisfying the conditions above, we obtain a holomorphic conformal structure P on M. Two such $\{U_\alpha, g_\alpha\}$ and $\{U'_\lambda, g'_\lambda\}$ correspond to the same structure P if and only if $g'_\lambda = h_{\lambda\alpha} g_\alpha$ on $U_\alpha \cap U'_\lambda$, where $h_{\lambda\alpha}$ is a function holomorphic on $U_\alpha \cap U'_\lambda$.

From (2.3) we obtain

$$(2.4) \qquad \det(g_{\beta ij})(dz^1_\beta \wedge \cdots \wedge dz^n_\beta)^2 = f^n_{\beta\alpha} \det(g_{\alpha ij})(dz^1_\alpha \wedge \cdots \wedge dz^n_\alpha)^2 .$$

If we denote the canonical line bundle of M by K and the line bundle with transition functions $\{f_{\beta\alpha}\}$ by F, then (2.4) implies

$$(2.5) \qquad F^n = K^{-2} .$$

As an immediate consequence of (2.5), we have

PROPOSITION (2.6). *For a compact complex manifold M of dimension n to admit a holomorphic $CO(n; \boldsymbol{C})$-structure, it is necessary that $2c_1(M)$ be divisible by n.*

Since we shall be working in one coordinate neighborhood U_α, we drop the subscript α temporarily in the following calculation. As in the Riemannian case, to the given $g = \sum g_{ij}\, dz^i dz^j$ we associate a holomorphic affine connection Γ^i_{jk} in U by

$$(2.7) \qquad \Gamma^i_{jk} = (1/2) \sum g^{ih}(\partial g_{hj}/\partial z^k + \partial g_{hk}/\partial z^j - \partial g_{jk}/\partial z^h) .$$

Given a holomorphic $CO(n; \boldsymbol{C})$-structure P, g is defined only up to the multiple of a non-vanishing holomorphic function. If we replace g by $\tilde{g} = fg = \sum f g_{ij}\, dz^i dz^j$, then the corresponding affine connection $\tilde{\Gamma}^i_{jk}$ is related to Γ^i_{jk} by

$$(2.8) \qquad \tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + (1/2)\delta^i_j \rho_k + (1/2)\delta^i_k \rho_j - (1/2) \sum g^{ih} g_{jk} \rho_h ,$$

where

$$(2.9) \qquad \rho_k = \partial(\log f)/\partial z^k .$$

The formula (2.8) is classical in conformal differential geometry and can be verified by a direct calculation. We note that while $\log f$ is defined modulo $2\pi i m$, $m \in \boldsymbol{Z}$, its derivatives ρ_k are well defined. Setting $i = j$ in (2.8) and summing over i, we obtain

$$(2.10) \qquad \sum \tilde{\Gamma}^i_{ik} = \sum \Gamma^i_{ik} + (n/2)\rho_k .$$

Eliminate ρ_k from (2.8) using (2.10) and use the fact that $\tilde{g}^{ih}\tilde{g}_{jk} = g^{ih} g_{jk}$. Then we obtain

$$(2.11) \qquad \begin{aligned} &\Gamma^i_{jk} - (1/n)\delta^i_j \Gamma_k - (1/n)\delta^i_k \Gamma_j + (1/n) \sum g^{ih} g_{jk} \Gamma_h \\ &\quad = \tilde{\Gamma}^i_{jk} - (1/n)\delta^i_j \tilde{\Gamma}_k - (1/n)\delta^i_k \tilde{\Gamma}_j + (1/n) \sum \tilde{g}^{ih} \tilde{g}_{jk} \tilde{\Gamma}_h , \end{aligned}$$

where

$$\Gamma_k = \sum \Gamma^h_{hk} \quad \text{and} \quad \tilde{\Gamma}_k = \sum \tilde{\Gamma}^h_{hk} . \tag{2.12}$$

We denote the left side (and also the right side) of (2.11) by C^i_{jk}. Once the coordinate system $z^1, \cdots, z^n$ is fixed, C^i_{jk} depends only on the holomorphic $CO(n; C)$-structure P but not on the particular g.

We shall now study how C^i_{jk} changes under coordinate transformations. First, we note

$$\Gamma_k = (1/2) \sum g^{ih}(\partial g_{hi}/\partial z^k) = (1/2)(\partial(\log G)/\partial z^k) , \quad \text{where} \quad G = \det (g_{ij}) . \tag{2.13}$$

Now, we use two local coordinate systems $z^1_\alpha, \cdots, z^n_\alpha$ and $z^1_\beta, \cdots, z^n_\beta$, and we calculate $C^i_{\alpha jk}$ and $C^i_{\beta jk}$ with respect to these coordinate systems. Since g and fg give rise to the same C^i_{jk}, we may assume $g_\beta = g_\alpha$, i.e., $f_{\beta\alpha} = 1$ for the purpose of calculating C^i_{jk}. Then

$$\sum g_{\alpha ij}\, dz^i_\alpha dz^j_\alpha = \sum g_{\beta ij}\, dz^i_\beta dz^j_\beta \tag{2.14}$$

so that

$$G_\beta = \det (g_{\beta ij}) = J^2_{\beta\alpha} \det (g)_{\alpha ij} = J^2_{\beta\alpha} G_\alpha , \tag{2.15}$$

where

$$J_{\beta\alpha} = \det (\partial z^i_\alpha/\partial z^j_\beta) . \tag{2.16}$$

From (2.13) and (2.15), we obtain

$$\Gamma_{\beta k} = \sum \Gamma_{\alpha h}(\partial z^h_\alpha/\partial z^k_\beta) + \partial(\log J_{\beta\alpha})/\partial z^k_\beta . \tag{2.17}$$

From the definition (2.11) of C^i_{jk} and (2.17), it follows that

$$\begin{aligned} C^i_{\beta jk} = {} & \sum (\partial z^i_\beta/\partial z^a_\alpha) C^a_{\alpha bc} (\partial z^b_\alpha/\partial z^j_\beta)(\partial z^c_\alpha/\partial z^k_\beta) + \sum (\partial z^i_\beta/\partial z^a_\alpha)(\partial^2 z^a_\alpha/\partial z^j \partial z^k) \\ & - (1/n)(\delta^i_j \sigma_{\beta\alpha k} + \delta^i_k \sigma_{\beta\alpha j} - \sum g^{ih}_\beta g_{\beta jk} \sigma_{\beta\alpha h}) , \end{aligned} \tag{2.18}$$

where

$$\sigma_{\beta\alpha k} = \partial(\log J_{\beta\alpha})/\partial z^k_\beta . \tag{2.19}$$

We consider a non-singular hyperquadric Q_n in $P_{n+1}C$ defined in terms of the homogeneous coordinate system $\zeta^0, \zeta^1, \cdots, \zeta^{n+1}$ by the following equation:

$$-2\zeta^0\zeta^{n+1} + (\zeta^1)^2 + \cdots + (\zeta^n)^2 = 0 . \tag{2.20}$$

Let Q be the symmetric matrix of degree $n + 2$ corresponding to the quadritic form of (2.20):

$$(2.21)\qquad Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Let $G = O(n+2; \boldsymbol{C})$ be the group of complex matrices A of degree $n+2$ such that

$$(2.22)\qquad {}^tAQA = Q .$$

Its Lie algebra $\mathfrak{g} = \mathfrak{o}(n+2; \boldsymbol{C})$ consists of complex matrices A of degree $n+2$ satisfying

$$(2.23)\qquad {}^tAQ + QA = 0 .$$

Then it can be easily verified that $\mathfrak{g}$ is a graded Lie algebra

$$(2.24)\qquad \mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 , \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} ,$$

where

$$(2.25)\qquad \mathfrak{g}_{-1} = \left\{\begin{pmatrix} 0 & 0 & 0 \\ u & 0 & 0 \\ 0 & {}^tu & 0 \end{pmatrix}\right\}, \qquad \mathfrak{g}_0 = \left\{\begin{pmatrix} a & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & -a \end{pmatrix}\right\}, \qquad \mathfrak{g}_1 = \left\{\begin{pmatrix} 0 & {}^tv & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

where u and v are complex n-vectors, U is a complex skew-symmetric matrix of degree n and a is a complex number.

The group G acts transitively on the quadric Q_n. Let H be the isotropy subgroup leaving the point $p_0 = {}^t(1, 0, \cdots, 0) \in Q_n$ fixed. Then H consists of matrices of the form

$$(2.26)\qquad \begin{pmatrix} a & {}^tv & b \\ 0 & U & w \\ 0 & 0 & c \end{pmatrix}, \quad \text{where} \quad \begin{aligned} &a, b, c \in \boldsymbol{C} , \quad ac = 1 , \quad {}^tUU = I_n , \\ &v = a\,{}^tUw , \quad 2bc = {}^tww . \end{aligned}$$

Note that a, w, U determine b, c, v.

To see the action of H on the tangent space at p_0, i.e., the linear isotropy representation of H, we use the inhomogeneous coordinate system $z^1, \cdots, z^n, z^{n+1}$ of $P_{n+1}\boldsymbol{C}$ defined by $z^i = \zeta^i/\zeta^0$, $i = 1, \cdots, n+1$. Then the defining equation (2.20) for the quadric Q_n becomes

$$(2.27)\qquad 2z^{n+1} = (z^1)^2 + \cdots + (z^n)^2 = {}^tzz ,$$

where z denotes the vector ${}^t(z^1, \cdots, z^n)$. To see how the element of H given by (2.26) acts on Q_n, we calculate

$$(2.28)\qquad \begin{pmatrix} a & {}^tv & b \\ 0 & U & w \\ 0 & 0 & c \end{pmatrix}\begin{pmatrix} 1 \\ z \\ z^{n+1} \end{pmatrix} = \begin{pmatrix} a + {}^tvz + bz^{n+1} \\ Uz + wz^{n+1} \\ cz^{n+1} \end{pmatrix}.$$

Hence, the transformation is given by

$$z \mapsto \{Uz + (1/2)({}^tzz)w\}\{a + {}^tvz + (1/2)({}^tzz)b\}^{-1} . \tag{2.29}$$

Its differential at p_0, i.e., at $z = 0$, is given by

$$dz \mapsto cU\,dz . \tag{2.30}$$

Thus the linear isotropy representation λ of H is given by

$$\lambda : \begin{pmatrix} a & {}^tv & b \\ 0 & U & w \\ 0 & 0 & c \end{pmatrix} \mapsto c\cdot U . \tag{2.31}$$

Its kernel N consists of matrices of the form

$$\begin{pmatrix} \pm 1 & {}^tv & b \\ 0 & \pm I_n & v \\ 0 & 0 & \pm 1 \end{pmatrix}, \quad b = \pm(1/2)({}^tvv) . \tag{2.32}$$

It is not hard to see that $\mathfrak{g}_1$ is the Lie algebra of N and $\mathfrak{g}_0 + \mathfrak{g}_1$ is the Lie algebra of H while $\mathfrak{g}_0$ is the Lie algebra of the subgroup $G_0 \subset H$ consisting of matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & c \end{pmatrix}, \quad ac = 1 , \quad {}^tUU = I_n . \tag{2.33}$$

We shall now construct a holomorphic $CO(n; C)$-structure on the quadric Q_n. Let $(e_1, \cdots, e_n)$ be the frame at $p_0 \in Q_n$ given by $(\partial/\partial z^1)_{p_0}, \cdots, (\partial/\partial z^n)_{p_0}$. Let P be the subbundle of the bundle $L(Q_n)$ of complex linear frames of Q_n consisting of those frames which are obtained from $(e_1, \cdots, e_n)$ by translation by elements of $G = O(n + 2; C)$. Then P is a principal subbundle of $L(Q_n)$ with structure group $H/N = CO(n; C)$, (see (2.1) and (2.31)). Thus we have constructed a natural holomorphic $CO(n; C)$-structure on the quadric Q_n. The action of G on Q_n lifts naturally to the bundle $L(Q_n)$, and P is nothing but the G-orbit of the frame $(e_1, \cdots, e_n)$. It is then clear that the holomorphic $CO(n; C)$-structure P is invariant by G. Moreover G is the largest group of holomorphic transformations of Q_n which leaves P invariant.

The homogeneous space G/N is a principal bundle over G/H with structure group H/N. It is also clear that this bundle is naturally isomorphic to the bundle P.

We shall now construct a holomorphic non-degenerate symmetric covariant tensor field (2.2) associated to the holomorphic conformal structure P. Consider the tensor field

(2.34) $$f = -d\zeta^0 d\zeta^{n+1} - d\zeta^{n+1} d\zeta^0 + d\zeta^1 d\zeta^1 + \cdots + d\zeta^n d\zeta^n$$

on $C^{n+2} - \{0\}$. Let s be a local holomorphic section of the bundle $C^{n+2} - \{0\}$ over $P_{n+1}C$. Although s^*f depends on the section s, its restriction to Q_n is uniquely defined, independently of s, up to a multiplicative factor of non-vanishing holomorphic function. In fact, let $s' = \lambda s$ be another local holomorphic section. Since

(2.35) $$\begin{aligned} &-d(\lambda\zeta^0)d(\lambda\zeta^{n+1}) - d(\lambda\zeta^{n+1})d(\lambda\zeta^0) + \sum_{i=1}^{n} d(\lambda\zeta^i)d(\lambda\zeta^i) \\ &\quad = \lambda^2(-d\zeta^0 d\zeta^{n+1} - d\zeta^{n+1}d\zeta^0 + \textstyle\sum d\zeta^i d\zeta^i) \\ &\qquad + (\lambda d\lambda)d(-2\zeta^0\zeta^{n+1} + \textstyle\sum \zeta^i\zeta^i) + (d\lambda d\lambda)(-2\zeta^0\zeta^{n+1} + \textstyle\sum \zeta^i\zeta^i) , \end{aligned}$$

we obtain

(2.36) $$s'^* f|_{Q_n} = \lambda^2(s^* f|_{Q_n}) .$$

In the affine space $A_{n+1} \subset P_{n+1}C$ defined by $\zeta^0 \neq 0$, we use the inhomogeneous coordinate system $z^1, \cdots, z^{n+1}$ given by $z^i = \zeta^i/\zeta^0$. Let s be the cross section $A_{n+1} \to C^{n+2} - \{0\}$ defined by

(2.37) $$\zeta^0 = 1,\ \zeta^1 = z^1,\ \cdots,\ \zeta^{n+1} = z^{n+1} .$$

Since $Q_n \cap A_{n+1}$ is given by the equation (2.27), $(z^1, \cdots, z^n)$ can be taken as a coordinate system in $Q_n \cap A_{n+1}$. Then s^*f is given on $Q_n \cap A_{n+1}$ by

(2.38) $$dz^1 dz^1 + \cdots + dz^n dz^n .$$

Let M be an n-dimensional complex manifold and $P(M)$ a holomorphic $CO(n; C)$-structure on M. Let $P(Q_n)$ be the natural holomorphic $CO(n; C)$-structure on the quadric Q_n defined above. We say that the structure $P(M)$ is *flat* if it is locally isomorphic to $P(Q_n)$, i.e., if, for every point of M, there is a biholomorphic map h of a neighborhood U of that point into Q_n which induces an isomorphism $P(M)|_U \to P(Q_n)|_{h(U)}$. A flat $CO(n; C)$-structure $P(M)$ is called a *quadric structure* on M. It can be proved that M admits a quadric structure if and only if it is covered by coordinate charts $(U_\alpha, \varphi_\alpha)$ such that

(i) φ_α maps U_α biholomorphically onto an open subset of Q_n,

(ii) for every pair (α, β) with $U_\alpha \cap U_\beta \neq \emptyset$, the coordinate change

$$\varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is given by (the restriction of) an element of G.

We shall now consider the noncompact dual of Q_n. In $P_{n+1}C$, consider the domain B of Q_n defined in terms of the homogenous coordinate system $\eta^0, \cdots, \eta^{n+1}$ by

$$(2.39) \quad B = \left\{ [\eta^0 : \cdots : \eta^{n+1}] \in P_{n+1}C; \begin{array}{l} -(\eta^0)^2 + (\eta^1)^2 + \cdots + (\eta^n)^2 - (\eta^{n+1})^2 = 0 \\ -|\eta^0|^2 + |\eta^1|^2 + \cdots + |\eta^n|^2 - |\eta^{n+1}|^2 < 0 \end{array} \right\} .$$

Let t be the projective transformation of $P_{n+1}C$ defined by

$$(2.40) \quad t([\eta^0 : \cdots : \eta^{n+1}]) = [(i\eta^0 + \eta^{n+1})/\sqrt{2} : \eta^1 : \cdots : \eta^n : (-i\eta^0 + \eta^{n+1})/\sqrt{2}] .$$

Set $D = t(B)$. Then we have

$$(2.41) \quad D = \left\{ [\zeta^0 : \cdots : \zeta^{n+1}] \in P_{n+1}C; \begin{array}{l} -2\zeta^0\zeta^{n+1} + (\zeta^1)^2 + \cdots + (\zeta^n)^2 = 0 \\ -|\zeta^0 - \zeta^{n+1}|^2/2 + |\zeta^1|^2 + \cdots + |\zeta^n|^2 - |\zeta^0 + \zeta^{n+1}|^2/2 < 0 \end{array} \right\} .$$

Hence D is a domain in Q_n. Actually D is in $Q_n \cap A_{n+1}$. With respect to the coordinate $(z^1, \cdots, z^n)$ of $Q_n \cap A_{n+1}$ defined above, D can be identified with the bounded domain

$$(2.42) \quad \left\{ (z^1, \cdots, z^n) \in C^n; \sum_{k=1}^{n} |z^k|^2 < 1 + \left| \sum_{k=1}^{n} (z^k)^2 \right|^2 \right\} .$$

We know D is a symmetric bounded domain, called the *noncompact dual* of Q_n. We write H for the subgroup of $O(n+2; C)$ leaving the domain D invariant. Then H is the largest group of holomorphic transformations of the bounded domain D. The natural invariant quadric structure on Q_n constructed above induces a quadric structure on D which is clearly invariant by the subgroup H. If Γ is a discrete subgroup of H acting freely on D, then the quotient manifold $M = D/\Gamma$ carries a natural quadric structure induced from that of D.

3. Chern classes. Let M be an n-dimensional complex manifold with a holomorphic conformal structure $\{g_\alpha\}$. To calculate its Chern classes, we construct a C^∞ affine connection on M and compute its curvature tensor.

Since $d(\log f_{\beta\alpha})$ is a 1-cocycle, we can find a C^∞ form

$$(3.1) \quad \varphi_\alpha = \sum \varphi_{\alpha k} \, dz^k_\alpha$$

on each U_α such that

$$(3.2) \quad d(\log f_{\beta\alpha}) = \varphi_\beta - \varphi_\alpha$$

or equivalently

$$(3.3) \quad \rho_{\beta\alpha k} = \varphi_{\beta k} - \varphi_{\alpha k} .$$

We set

$$\Gamma_{\alpha jk}^{\ i} = C_{\alpha jk}^{\ i} - (\delta_j^i \varphi_{\alpha k} + \delta_k^i \varphi_{\alpha j} - \sum g_{\alpha jk} g_\alpha^{il} \varphi_{\alpha l})/2 . \tag{3.4}$$

Then $\Gamma_{\alpha jk}^{\ i}$ defines an affine connection globally on M.

Since we shall work within one coordinate neighborhood in the remainder of this section, we shall drop the subscript α in the following calculation. The curvature tensor is given by

$$R^i{}_{jAB} = \partial \Gamma_{jB}^i / \partial z^A - \partial \Gamma_{jA}^i / \partial z^B + \sum (\Gamma_{CA}^i \Gamma_{jB}^C - \Gamma_{CB}^i \Gamma_{jA}^C) . \tag{3.5}$$

Hence, (using the fact that C_{jk}^i are holomorphic), we obtain

$$R^i{}_{j\bar{k}\bar{h}} = 0 , \tag{3.6}$$

and

$$R^i{}_{jk\bar{h}} = -\partial \Gamma_{jk}^i / \partial \bar{z}^h = (\delta_j^i \varphi_{k\bar{h}} + \delta_k^i \varphi_{j\bar{h}} - \sum g_{jk} g^{il} \varphi_{l\bar{h}})/2 . \tag{3.7}$$

The curvature form is given by

$$\begin{aligned} \Omega_j^i &= \sum R^i{}_{jk\bar{h}}\, dz^k \wedge d\bar{z}^h + \cdots \\ &= -(\delta_j^i \bar{\partial}\varphi + \bar{\partial}\varphi_j \wedge dz^i - \sum g_{jk} g^{il} \bar{\partial}\varphi_l \wedge dz^k)/2 + \cdots , \end{aligned} \tag{3.8}$$

where the dots indicate terms of degree (2, 0). (By (3.6), there is no terms of degree (0, 2)).

The Chern forms c_i, $i = 1, \cdots, n$, are given by (see, for example [10])

$$\det (I + (\sqrt{-1}/2\pi)\Omega) = 1 + c_1 + \cdots + c_n . \tag{3.9}$$

It is clear from (3.6) that c_i involves only forms of degree $(i + m, i - m)$, $m \geqq 0$ and not those of degree $(i + m, i - m)$ for $m < 0$. We shall calculate, only the (i, i)-component $c^{(i,i)}$ of c_i. We substitute (3.8) into (3.9) and drop the terms indicated by dots. Then

$$\begin{aligned} &\det [(1 - (\sqrt{-1}/4\pi)\bar{\partial}\varphi)\delta_j^i - (\sqrt{-1}/4\pi)(\delta_k^i \bar{\partial}\varphi_j - g_{jk} g^{il} \bar{\partial}\varphi_l) \wedge dz^k] \\ &\qquad = \sum_{p=0}^{n} (1 - (\sqrt{-1}/4\pi)\bar{\partial}\varphi)^{n-p} (-\sqrt{-1}/4\pi)^p (1/p!) \Phi_p , \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \Phi_p = \sum \delta_{i_1 \cdots i_p}^{j_1 \cdots j_p} &(\delta_{k_1}^{i_1} \bar{\partial}\varphi_{j_1} - g_{j_1 k_1} g^{i_1 l_1} \bar{\partial}\varphi_{l_1}) \wedge dz^{k_1} \wedge \cdots \\ &\wedge (\delta_{k_p}^{i_p} \bar{\partial}\varphi_{j_p} - g_{j_p k_p} g^{i_p l_p} \bar{\partial}\varphi_{l_p}) \wedge dz^{k_p} . \end{aligned} \tag{3.11}$$

Given a point of M, we choose a local coordinate system so that $g_{ij} = \delta_{ij}$ at that point. Then a straightforward calculation shows

$$\Phi_p = \begin{cases} p!\, (\bar{\partial}\varphi)^p & \text{if } p \text{ is even} , \\ 0 & \text{if } p \text{ is odd} . \end{cases} \tag{3.12}$$

If we set

$$(3.13)\qquad h = (4\pi\sqrt{-1})^{-1}\bar{\partial}\varphi ,$$

then (3.10) may be written as follows:

$$(3.14)\qquad 1 + c^{(1,1)} + \cdots + c^{(n,n)} = \sum_{0\leq q\leq n/2} (1 + h)^{n-2q}h^{2q} .$$

Since $h^{n+1} = 0$, this may be rewritten as follows:

$$(3.15)\qquad 1 + c^{(1,1)} + \cdots + c^{(n,n)} = (1 + h)^{n+2}/(1 + 2h)$$

In particular,

$$(3.16)\qquad c^{(1,1)} = nh .$$

Substituting (3.16) back into (3.14) or (3.15), we can express $c^{(i,i)}$ in terms of $c^{(1,1)}$. Write

$$(3.17)\qquad \sum_{q=0}^{m} (1 + h)^{n-2q}h^{2q} = 1 + a_1h + a_2h^2 + \cdots + a_nh^n ,$$

where $a_1, \cdots, a_n$ are positive integers. (We can easily see that $a_1 = n$ and $a_n = m + 1$, where $n = 2m$ or $2m + 1$). Then

$$(3.18)\qquad c^{(r,r)} = a_r\cdot n^{-r}(c^{(1,1)})^r .$$

As we have stated above, c_r involves only forms of degree $(r + m, r - m)$, $m \geqq 0$. Hence, both $c_r - c^{(r,r)}$ and $c_1^r - (c^{(1,1)})^r$ involve only forms of degree $(r + m, r - m)$, $m > 0$. Hence, if Q_{n-r} is a $2(n - r)$-form involving only forms of degree $(n - r + k, n - r - k)$, $k \geqq 0$, then

$$(3.19)\qquad c_rQ_{n-r} = c^{(r,r)}Q_{n-r} , \qquad c_1^rQ_{n-r} = (c^{(1,1)})^rQ_{n-r} .$$

We have shown

THEOREM (3.20). *Let M be an n-dimensional complex manifold with a holomorphic $CO(n; \boldsymbol{C})$-structure and $c_i \in H^{2i}(M, \boldsymbol{R})$ its i-th Chern class. Then for every weighted homogeneous polynomial $Q_{n-r} = Q_{n-r}(c_1, \cdots, c_{n-r}) \in H^{2n-2r}(M, \boldsymbol{R})$ in Chern classes, we have*

$$c_rQ_{n-r} = a_rn^{-r}c_1^rQ_{n-r} \quad \textit{for} \quad r = 1, \cdots, n ,$$

where a_r is the positive integer defined by (3.17). *If M is moreover Kähler, then*

$$c_r = a_rn^{-r}c_1^r \quad \textit{for} \quad r = 1, \cdots, n .$$

For surfaces, whether Kähler or not, the only relation we have is

$$(3.21)\qquad 2c_2 = c_1^2 .$$

REMARK (3.22). Let D be the noncompact dual of Q_n (cf. § 2) and

Γ a discrete subgroup of H acting freely on D. Then we have shown in § 2, that the quotient manifold $M = D/\Gamma$ carries the natural quadric structure (and hence a holomorphic $CO(n; C)$-structure). In this case Theorem (3.20) above is known as Hirzebruch's proportionality principle ([5]).

4. Einstein-Kähler manifolds. In this section we shall prove the following

THEOREM (4.1). *Let M be a compact n-dimensional Einstein-Kähler manifold admitting a holomorphic $CO(n; C)$-structure. Then M is either a hyperquadric, or flat, or covered by the noncompact dual of a hyperquadric as described in § 2 according as the Ricci tensor is positive,* 0 *or negative.*

Let a holomorphic $CO(n; C)$-structure is given by $\{g_\alpha\}$ as in (2.2). Let $S^k T^*$ denote the symmetric k-th tensor power of the cotangent bundle $T^* = T^*M$. Let F be the line bundle defined by $\{f_{\alpha\beta}\}$, (see (2.3)). Then $\{g_\alpha\}$ may be considered as a holomorphic section of $F \otimes S^2 T^*$. We shall denote this section by g. Then $g^n = g \otimes \cdots \otimes g$ is a section of $F^n \otimes (S^2 T^*)^{\otimes n}$. By symmetrizing g^n we obtain a section $g^{(n)}$ of $F^n \otimes S^{2n} T^*$. Since $F^n = K^{-2}$ by (2.5) (where K is the canonical line bundle of M), $g^{(n)}$ is a section of $K^{-2} \otimes S^{2n} T^*$. In particular, $g^{(n)}$ is a holomorphic tensor field of covariant degree $2n$ and contravariant degree $2n$. On a compact Einstein-Kähler manifold such a holomorphic tensor field is parallel (by Theorem 1 in [9]). We lift this parallel tensor field to the universal covering manifold $\tilde{M}$ of M and shall show that $\tilde{M}$ is either a hyperquadric or its noncompact dual according as the Ricci tensor is positive or negative. (The Ricci flat case will be considered separately).

We shall wite $K^{-2} \otimes S^{2n} T^*$ for $K^{-2} \otimes S^{2n} T^*(\tilde{M})$ and denote the lift of $g^{(n)}$ to $\tilde{M}$ by the same symbol $g^{(n)}$. Let $\tilde{M} = M_1 \times \cdots \times M_r$ be the de Rham decomposition of $\tilde{M}$ into Kähler manifolds $M_1, \cdots, M_r$ with irreducible holonomy group. (Since the Ricci tensor is definite, there is no Euclidean factor in the decomposion and the Ricci tensors of $M_1, \cdots, M_r$ are either all positive or negative definite.) If we write $T_i^* = T^* M_i$ and denote the canonical line bundle of M_i by K_i, then under a natural identification we have

$$K^{-2} \otimes S^{2n} T^* = \sum (K_1^{-2} \otimes S^{m_1} T_1^*) \otimes \cdots \otimes (K_r^{-2} \otimes S^{m_r} T_r^*) , \tag{4.2}$$

where the summation is taken over all partitions $2n = m_1 + \cdots + m_r$. We shall now restrict (4.2) to one point of $\tilde{M}$. Thus we regard (4.2) as an isomorphism between the fibres of the two bundles at one point. We consider $g^{(n)}$ as an element of that particular fibre which is invariant by the holonomy group rather than a parallel section of the tensor bundle.

Let $\Phi, \Phi_1, \cdots, \Phi_r$ be the holonomy groups of $M, M_1, \cdots, M_r$. Then $\Phi = \Phi_1 \times \cdots \times \Phi_r$ in a natural manner. If we denote in (4.2) the subspaces consisting of elements invariant by these holonomy groups by the superscript $(\cdots)^I$, then we obtain

$$(4.3) \qquad (K^{-2} \otimes S^{2n}T^*)^I = \sum (K_1^{-2} \otimes S^{m_1}T_1^*)^I \otimes \cdots \otimes (K_r^{-2} \otimes S^{m_r}T_r^*)^I .$$

We claim that $(K_i^{-2} \otimes S^{m_i}T_i^*)^I = 0$ unless M_i is a symmetric space. In fact, (by the argument in [9]),

LEMMA (4.4). *If M is a Kähler manifold with irreducible holonomy, then*

$$(K^q \otimes S^m T^*)^I = 0 \quad \textit{for all} \quad q \quad \textit{and} \quad m > 0$$

unless M is a symmetric space.

Since (4.4) is not stated exactly in this form in [9], we shall sketch its proof. Since M is not symmetric and has nonzero Ricci tensor, its holonomy group is either $U(n)$ or $Sp(n/2) \times U(1)$ by Berger's holonomy theorem. But these groups act irreducibly on $K^q \otimes S^m T^*$.

Now we claim that $M_1, \cdots, M_r$ are all symmetric. Since $g = \{g_\alpha\}$ is non-degenerate, the element $g^{(n)}$ of the left hand side of (4.3) involves all factors $M_1, \cdots, M_r$. If one of them, say M_1, is not symmetric, then there would be no terms involving $(K_1^{-2} \otimes S^{m_1}T_1^*)^I$ in the right hand side of (4.3). This is a contradiction.

We shall show now either $\tilde{M} = M_1$, i.e., $\tilde{M}$ is already irreducible, or $\tilde{M} = P_1C \times P_1C$ or $\tilde{M} = D \times D$, where D denotes the unit disk. By (3.20), the ratio between all Chern numbers of M with a holomorphic $CO(n; C)$-structure depends only on the dimension n and does not depend on a particular M. This ratio can be determined, for example, from the hyperquadric. In particular, the n-dimensional hyperquadric has

$$\text{arithmetic genus} = 1 , \quad c_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even} . \end{cases}$$

We consider first the case where the Ricci tensor is positive so that M itself is simply connected. In this case, the arithmetic genus of M is 1 and hence the Euler number c_n is $n+1$ or $n+2$. If we denote the complex dimension of M_i by n_i, then its Euler number is at least $n_i + 1$ since M_i is of compact type. Hence $n + 2 \geqq (n_1 + 1) \cdots (n_r + 1)$, where $n = n_1 + \cdots + n_r$. But this is possible only when $r = 1$ or $r = 2$ with $n_1 = n_2 = 1$. When the Ricci tensor is negative we consider the compact dual of M and apply Hirzebruch's proportionality principle (cf. Remark (3.22)). This proves our assertion that either $\tilde{M}$ is irreducible or $\tilde{M} = P_1C \times P_1C$ or $\tilde{M} = D \times D$.

Assume that $\tilde{M}$ is irreducible. Again we consider first the case the Ricci tensor is positive. Then $c_n > n + 2$ unless M is either the projective space P_nC (in which case $c_n = n + 1$) or the hyperquadric. The projective space can be eliminated by considering the Chern class c_2. (For the hyperquadric $c_2 = ((n^2 - n + 2)/2n^2)c_1^2$ while $c_2 = (n/2(n + 1))c_1^2$ for P_nC.) The case of negative Ricci tensor can be reduced to the positive case by the proportionality principle.

We shall now consider the remaining case, i.e., the Ricci flat case. Since $c_1 = 0$, $c_2 = 0$ by (3.20). But we know that a compact Kähler manifold with vanishing Ricci tensor and $c_2 = 0$ is flat, (see [7] as well as [17]). This completes the proof of (4.1).

COROLLARY (4.5). *Let M be a compact n-dimensional Kähler manifold admitting a holomorphic $CO(n; \boldsymbol{C})$-structure. If $c_1 < 0$ (i.e., if the canonical bundle is ample), then the universal covering space of M is the noncompact dual of the hyperquadric. If $c_1 = 0$ in $H^2(M; \boldsymbol{R})$, then M has a complex torus as a unramified covering space.*

PROOF. The case $c_1 < 0$ follows from the theorem of Aubin [1] and Yau [20] that such a manifold admits an Einstein-Kähler metric. The case $c_1 = 0$ follows from the theorem of Yau [20] that such a manifold admits a Ricci flat Kähler metric. q.e.d.

Although a compact Kähler manifold with $c_1 > 0$ may not admit an Einstein-Kähler metric, we can still say something. Since a compact Kähler manifold with $c_1 > 0$ admits a Kähler metric with positive Ricci tensor [20], it is simply connected, [8]. The standard argument using the development (cf. § 4 of [12]) implies the following:

THEOREM (4.6). *Let M be an n-dimensional compact Kähler manifold with $c_1 > 0$. If it admits a quadric structure, it is biholomorphic to a nonsingular hyperquadric Q_n in $P_{n+1}\boldsymbol{C}$.*

When n is odd, we can say more.

THEOREM (4.7). *Let M be an n-dimensional compact Kähler manifold with $c_1 > 0$. If n is odd and if M admits a holomorphic $CO(n; \boldsymbol{C})$-structure, then M is biholomorphic to a nonsingular hyperquadric Q_n in $P_{n+1}\boldsymbol{C}$.*

PROOF. By (2.5), the canonical bundle K satisfies the relationship

$K^{-2} = F^n$, where F is a line bundle. Let α be the characteristic class of F. Then $2c_1 = n\alpha$ (in $H^{1,1}(M: \boldsymbol{Z})$). If n is odd, there is an element β in $H^{1,1}(M: \boldsymbol{Z})$ such that $c_1 = n\beta$. Since c_1 is positive, so is β. By the characterization of a nonsingular hyperquadric given in [11], M is biholomorphic to Q_n. q.e.d.

It would be natural to raise the question whether a compact Kähler manifold with $c_1 > 0$ admitting a holomorphic $CO(n; \boldsymbol{C})$-structure is biholomorphic to Q^n. In dimension 2, the condition $c_1 > 0$ implies the rationality and, as we shall see later, the only rational surface admitting a holomorphic $CO(n; \boldsymbol{C})$-structure is the quadric $Q_2 = P_1\boldsymbol{C} \times P_1\boldsymbol{C}$.

5. **Compact complex surfaces.** Let M be a complex surface with a holomorphic $CO(2; \boldsymbol{C})$-structure $\{g_\alpha\}$, where $g_\alpha = \sum g_{\alpha ij}\, dz^i_\alpha dz^j_\alpha$ in U_α. At each point $x_\alpha \in U_\alpha \subset M$, the equation

$$g_\alpha(X, X) = 0 \tag{5.1}$$

defines two lines L'_x and L''_x in the tangent plane T_xM. Since we cannot distinguish L'_x and L''_x, we may not be able to choose L'_x continuously on M. However, on a double covering space $\tilde{M}$ of M, we can obtain holomorphic line subbundles L' and L'' of $T\tilde{M}$. Thus, a holomorphic $CO(2; \boldsymbol{C})$-structure on M gives rise to a splitting $T\tilde{M} = L' \oplus L''$.

Conversely, given a splitting

$$TM = L' \oplus L'' \tag{5.2}$$

of the tangent bundle into line subbundles L' and L'', we can obtain a holomorphic $CO(2; \boldsymbol{C})$-structure on M by setting

$$g_\alpha(L', L') = g_\alpha(L'', L'') = 0 , \qquad g_\alpha(e', e'') = 1 , \tag{5.3}$$

where e' and e'' are arbitrarily chosen local holomorphic sections spanning L' and L'' over U_α. The structure is independent of the choice of e', e''.

Since every 1-dimensional holomorphic distribution is integrable, L' and L'' are integrable and define foliations. Hence,

LEMMA (5.4). *A splitting $TM = L' \oplus L''$ on a complex surface M is equivalent to a pair of mutually transversal 1-dimensional holomorphic foliations on M.*

In other words, on such a surfance M we can choose a system of coordinate charts $\{U_\alpha; (z^1_\alpha, z^2_\alpha)\}$ such that

$$z^1_\alpha = f^1_{\alpha\beta}(z^1_\beta) , \qquad z^2_\alpha = f^2_{\alpha\beta}(z^2_\beta) \tag{5.5}$$

so that $\partial/\partial z^1_\alpha$ and $\partial/\partial z^2_\alpha$ span L' and L'', respectively. With respect to such a coordinate system, g_α is of the following form (see (5.3)):

$$g_\alpha = 2g_{\alpha 12}\, dz^1_\alpha dz^2_\alpha \,. \tag{5.6}$$

Without loss of generality we may assume that $g_{\alpha 12} = 1$ so that

$$g_\alpha = 2\, dz^1_\alpha dz^2_\alpha \,. \tag{5.7}$$

LEMMA (5.8). *Let M be a compact complex surface with a splitting $TM = L' \oplus L''$. If f' and f'' denote the characteristic classes of the line bundles L' and L'', then*

$$c_1(M) = f' + f'' \,, \qquad c_2(M) = f' \cdot f'' \,, \qquad f'^2 = f''^2 = 0 \,.$$

PROOF. The first two equalities are obvious. The third follows from the vanishing theorem of Bott for integrable distributions, [3].

We shall now show that a complex surface admitting a holomorphic $CO(2; \boldsymbol{C})$-structure is free of exceptional curves. The following lemma will be used also in studying Hopf surfaces.

LEMMA (5.9). *Given a holomorphic $CO(2; \boldsymbol{C})$-structure $\{g_\alpha\}$ on the punctured unit ball*

$$B^* = \{(z^1, z^2) \in \boldsymbol{C}^2;\ 0 < |z^1|^2 + |z^2|^2 < 1\}$$

in $\boldsymbol{C}^2$, there is a globally defined holomorphic quadratic form $g = \sum g_{ij} dz^i dz^j$ on B^* such that $g = f_\alpha g_\alpha$ on U_α, where f_α is a holomorphic function on U_α.

PROOF. Let F be the line bundle given by the transition functions $\{f_{\alpha\beta}\}$ defined by $g_\alpha = f_{\alpha\beta} g_\beta$. By (2.5), $F^2 = K^{-2}$, where K is the canonical line bundle of B^*. Since K on B^* is trivial, so is F^2. From the simple connectedness of B^* it follows that F itself is trivial. Hence, $f_{\alpha\beta} = f_\alpha^{-1} f_\beta$, where f_α is an invertible holomorphic function on U_α. Then $f_\alpha g_\alpha = f_\beta g_\beta$ on $U_\alpha \cap U_\beta$, which defines a global form g. q.e.d.

LEMMA (5.10). *Let M be a complex surface and $\tilde{M}$ the surface obtained by blowing up a point, say o, of M. If $\tilde{M}$ admits a holomorphic $CO(2; \boldsymbol{C})$-structure, so does M.*

PROOF. Let $p\colon \tilde{M} \to M$ be the natural projection and $C = p^{-1}(o)$. The given holomorphic $CO(2; \boldsymbol{C})$-structure on $\tilde{M}$ induces a holomorphic $CO(2; \boldsymbol{C})$-structure on $M - \{o\}$. Let B be a neighborhood of o in M and $B^* = B - \{o\}$. By (5.9), the induced holomorphic $CO(2; \boldsymbol{C})$-structure on B^* can be given by a single quadratic form $g = \sum g_{ij}\, dz^i dz^j$. Since g is holomorphic, it extends through o by Hartogs' theorem. Since both $\det(g_{ij})$ and $\det(g_{ij})^{-1}$ are holomorphic and extend through o, $\det(g_{ij})$ remains nonzero even at the point o. Hence the extended g is everwhere nondegenerate. q.e.d.

THEOREM (5.11). *A complex surface admitting a holomorphic $CO(2; \boldsymbol{C})$-structure is free of exceptional curves of the first kind.*

PROOF. Let M and $\tilde{M}$ be as in (5.10). Assume that $\tilde{M}$ admits a holomorphic $CO(2; \boldsymbol{C})$-structure. With the notation in the proof of (5.10), let $g = \sum g_{ij} dz^i dz^j$ be a form on B defining the induced $CO(2; \boldsymbol{C})$-structure on $B \subset M$. The pull-back $p^*(g)$ defines the given holomorphic $CO(2;\boldsymbol{C})$-structure on $p^{-1}(B^*) = p^{-1}(B) - C$ while it degenerates at each point of C since p collapses C into a single point. This is a contrdiction. q.e.d.

REMARK (5.12). If we assume M to be compact, we can use (3.21) to obtain (5.11). Since $c_2(\tilde{M}) = c_2(M) + 1$ and $c_1(\tilde{M})^2 = c_1(M)^2 - 1$, (3.21) cannot hold for both M and $\tilde{M}$ at the same time. This is the argument used by Gunning [4] for holomorphic affine and projective connections.

Using the splitting $TM = L' + L''$ we can strengthen (5.11).

THEOREM (5.13). *Let M be a complex surface admitting a $CO(2; \boldsymbol{C})$-structure. Let C be a nonsingular rational curve in M and N_C its normal line bundle. Let H be the hyperplane line bundle over C (so that every line bundle over C is of the form H^k, $k \in \boldsymbol{Z}$). Then $N_C = H^k$, where $k \geqq 2$ or $k = 0$.*

PROOF. Taking a double covering space $\tilde{M}$ of M and lifting C to $\tilde{M}$ if necessary, we may assume that the $CO(2; \boldsymbol{C})$-structure on M gives rise to a splitting $TM = L' \oplus L''$. Consider first the case where C is tangent to L' (or L''). Then C is a leaf of the foliation defined by L'. The holonomy of the leaf C is discrete by the general theory. Since C is simply connected, the holonomy of C is trivial. Hence the normal bundle N_C is trivial. Assume that C is not tangent to L' (nor to L''). Let X be a holomorphic vector field of C with two isolated zeros. We write $X = X' + X''$ so that $X' \in L'$ and $X'' \in L''$. Let s be the section of the normal bundle N_C obtained by projecting X' to N_C. Then s is a nontrivial section with at least two zeros. Hence, $N_C = H^k$ with $k \geqq 2$. q.e.d.

COROLLARY (5.14). *A complex surface M with a holomorphic $CO(2; \boldsymbol{C})$-structure cannot contain a nonsingular rational curve with self-intersection $C \cdot C < 0$ or $C \cdot C = 1$.*

6. Elliptic surfaces. We shall determine the elliptic surfaces admitting $CO(2; \boldsymbol{C})$-structures. Let M be an elliptic surface with a $CO(2; \boldsymbol{C})$-structure. Then it is free of exceptional curves of the first kind and hence $c_1^2 = 0$. Therefore, $c_2 = 0$ by (3.21). Since the Euler number c_2 of M is the sum of the Euler numbers of all singular fibres of M, it follows that there are no singular fibres except multiple fibres, (see [14]).

LEMMA (6.1). *Let Δ be a compact Riemann surface of genus g, and $a_1, \cdots, a_r$ be r distinct points of Δ with multiplicities $m_1, \cdots, m_r > 1$. Assume $(g, r) \neq (0, 1), (0, 2)$. Then*

(1) *There exists a (ramified) covering $\pi: \tilde{\Delta} \to \Delta = \tilde{\Delta}/\Gamma$ such that*

(a) *$\tilde{\Delta}$ is simply connected and Γ is a group acting properly discontinuously on $\tilde{\Delta}$;*

(b) *$\pi: \tilde{\Delta} - \pi^{-1}(\{a_i\}) \to \Delta - \{a_i\}$ is an unramified covering;*

(c) *π is ramified with ramification index $m_i - 1$ at each point of $\pi^{-1}(a_i)$.*

(2) *There exists a normal subgroup Γ_0 of Γ of finite index such that*

(d) *Γ_0 acts freely on $\tilde{\Delta}$;*

(e) *$\Delta_0 = \tilde{\Delta}/\Gamma_0 \to \Delta$ is a (ramified) covering satisfying* (b) *and* (c).

PROOF. (1) Set $U = \Delta - \{a_i\}$ and $\tilde{U} \to U$ be the universal covering with covering group $\tilde{\Gamma}$. Then $\tilde{\Gamma}$ is a group with generators $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, S_1, \cdots, S_r$ with one relation

$$(*) \qquad \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}S_1 \cdots S_r = 1 .$$

Let Γ be the group with the same set of generators and additional relations

$$(**) \qquad S_1^{m_1} = \cdots = S_r^{m_r} = 1 .$$

Let N be the kernel of the natural homomorphism $\tilde{\Gamma} \to \Gamma$; it is the normal subgroup of $\tilde{\Gamma}$ generated by $S_1^{m_1}, \cdots, S_r^{m_r}$. Let $\tilde{\Delta}$ be the Riemann surface obtained from $\tilde{U}/N$ by filling r points corresponding to $a_1, \cdots, a_r$. Then $\tilde{\Delta}$ satisfies (a), (b) and (c). (We note that if $(g, r) \neq (0, 1), (0, 2)$ then $\tilde{U}$ is biholomorphic to the upper half-plane and the action of Γ on $\tilde{U}/N$ extends to the compactification $\tilde{\Delta}$ by Picard's theorem).

(2) Given a group Γ with generators $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, S_1, \cdots, S_r$ and relations $(*)$ and $(**)$, the theorem of Bundgaard-Nielsen [22] and Fox [24] conjectured by Fenchel [23] states that there exists a normal subgroup $\Gamma_0 \subset \Gamma$ of finite index with no torsion (i.e., with no elements of finite order). Since Γ acts properly discontinuously on $\tilde{\Delta}$, the torsion-free subgroup Γ_0 acts freely on $\tilde{\Delta}$. q.e.d.

LEMMA (6.2). *If $M \to \Delta$ is a holomorphic fibre bundle over a simply connected Δ with an elliptic curve as fibre, then it is a principal bundle with group T.*

PROOF. Let A be the group of holomorphic transformations of T. The translations of T form a normal subgroup, denoted also by T, such that A/T is finite. Since the base manifold Δ is simply connected, the structure group A of the bundle M reduces to its identity component T. Hence, M is a principal T-bundle. q.e.d.

LEMMA (6.3). *Let $\Phi: M \to \Delta$ be an elliptic surface, free of exceptional curve of the first kind, with multiple singular fibres of multiplicities $m_1, \cdots, m_r$ at $a_1, \cdots, a_r \in \Delta$ and no other singular fibres. Assume that $c_2(M) = 0$ and exclude the case $\Delta = P_1C$ and $r = 1$ or 2. Then there exists an elliptic surface $\tilde{\Phi}: \tilde{M} \to \tilde{\Delta}$ with a commutative diagram*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{p} & M \\ {\scriptstyle\tilde{\Phi}}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \tilde{\Delta} & \longrightarrow & \Delta \end{array}$$

such that

(1) *$\pi: \tilde{\Delta} \to \Delta = \tilde{\Delta}/\Gamma$ is a (ramified) simply connected covering as described in* (6.1);

(2) *$\tilde{\Phi}: \tilde{M} \to \tilde{\Delta}$ is a principal T-bundle;*

(3) *$p: \tilde{M} \to M$ is an unramified normal covering with covering group Γ, and the group Γ acts on $\tilde{M}$ as bundle automorphisms (but not necessarily as principal bundle automorphisms which commute with the action of T);*

(4) *There exists a normal subgroup $\hat{\Gamma} \subset \Gamma$ of finite index acting on $\tilde{\Delta}$ freely and on $\tilde{M}$ as principal bundle automorphisms. (Set $\hat{M} = \tilde{M}/\hat{\Gamma}$ and $\hat{\Delta} = \tilde{\Delta}/\hat{\Gamma}$. Then $\hat{\Phi}: \hat{M} \to \hat{\Delta}$ is a holomorphic principal T-bundle over a compact Riemann surface $\hat{\Delta}$).*

PROOF. We construct $\pi: \tilde{\Delta} \to \Delta$ as in (6.1). We consider the pull-back $M' = \pi^*M$ and the commutative diagram:

$$\begin{array}{ccc} M' & \xrightarrow{p'} & M \\ {\scriptstyle\Phi'}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \tilde{\Delta} & \xrightarrow{\pi} & \Delta \end{array}$$

Then M' has no singularities outside the curves obtained by pulling back the singular fibres $\Phi^{-1}(a_i)$. Each of these curves $\Phi^{-1}(a_i) \times b_{i\lambda}$, $(b_{i\lambda} \in \pi^{-1}(a_i))$, is a multiple curve of multiplicity m_i. In fact in a neighborhood of each point of $\Phi^{-1}(a_i) \times b_{i\lambda}$, M' is composed of m_i non-singular sheets passing through $\Phi^{-1}(a_i) \times b_{i\lambda}$. By separating these sheets, we obtain a non-singular elliptic surface $\tilde{\Phi}: \tilde{M} \to \tilde{\Delta}$ with a commutative diagram:

$$\begin{array}{ccccc} \tilde{M} & \longrightarrow & M' & \xrightarrow{p'} & M \\ \tilde{\Phi}\downarrow & & \Phi'\downarrow & & \downarrow\Phi \\ \tilde{\Delta} & \longrightarrow & \tilde{\Delta} & \xrightarrow{\pi} & \Delta \end{array}$$

The action of Γ on $\tilde{\Delta}$ induces an action of Γ on $M' \subset \tilde{\Delta} \times M$ and then an action of Γ on $\tilde{M}$. Let Γ_0 be a normal subgroup of Γ of finite index as described in (6.1). Then $\tilde{M}/\Gamma_0 \to \tilde{\Delta}/\Gamma_0$ is an elliptic surface over a compact Riemann surface $\tilde{\Delta}/\Gamma_0$ with no singular fibres. It follows that it is a holomorphic fibre bundle (with a fixed elliptic curve T as fibre). Hence, $\tilde{M}$ is also a holomorphic fibre bundle over $\tilde{\Delta}$ with fibre T. Since $\tilde{\Delta}$ is simply connected, $\tilde{M}$ is a holomorphic principal T-bundle over $\tilde{\Delta}$, (see (6.2)).

If $\tilde{\Delta} = P_1C$, then we take as Γ the trivial group consisting of the identity only. If $\tilde{\Delta} = C$ or $\tilde{\Delta} = H$ (upper half-plane), then $\tilde{M}$ is a product bundle $\tilde{M} = \tilde{\Delta} \times T$. Since $\mathrm{Aut}\,(T)/T$ is finite, the subgroup Γ' of Γ consisting of elements which act as principal bundle automorphisms on $\tilde{M}$ is a normal subgroup of finite index in Γ. Let Γ_0 be as in (6.1), and set $\hat{\Gamma} = \Gamma' \cap \Gamma_0$.

LEMMA (6.4). *Let $\Phi\colon M \to \Delta$ be a holomorphic principal bundle over a compact Riemann surface Δ with structure group T, where T is an elliptic curve. Let V be a vertical vector field on M defined by the action of T.*

(1) *If $b_1(M)$ is even, then there exists a holomorphic 1-form $\omega \in H^0(M, \Omega^1)$ such that $\omega(V) = 1$, and*

$$\dim H^0(M, \Omega^1) - 1 = \mathrm{genus}\,(\Delta) = \dim H^0(M, \Omega^2)$$

(2) *If $b_1(M)$ is odd, then*

$$\dim H^0(M, \Omega^1) = \mathrm{genus}\,(\Delta) = \dim H^0(M, \Omega^2)\,.$$

PROOF. Let (x, t) be a local coordinate system for the bundle M, where x is a local coordinate for the base Δ and t is a local coordinate for the fibre T. Let $\theta = Adx + Bdt \in H^0(M, \Omega^1)$, where A and B are holomorphic functions of (x, t). Since $B = \theta(V)$ is holomorphic on M, it is constant. Since θ is closed, A is a function of x only. Hence, $\Phi^*(H^0(\Delta, \Omega^1))$ consists of $\theta \in H^0(M, \Omega^1)$ with $B = 0$. This implies that $\Phi^*(H^0(\Delta, \Omega^1))$ is either equal to $H^0(M, \Omega^1)$ or of codimension 1 in $H^0(M, \Omega^1)$ so that

$$h^{1,0} - 1 = \dim H^0(M, \Omega^1) - 1 \leqq \mathrm{genus}\,(\Delta) \leqq \dim H^0(M, \Omega^1) = h^{1,0}\,.$$

Since $\Phi\colon M \to \Delta$ is a principal T-bundle, for every $\theta = Adx \in \Phi^*H^0(\Delta, \Omega^1)$

we have a globally well defined 2-form $\omega = Adx \wedge dt \in H^0(M, \Omega^2)$. Conversely, every holomorphic 2-form $\omega = Adx \wedge dt \in H^0(M, \Omega^2)$ comes from a holomorphic 1-form $\theta = \iota_V\omega = Adx \in \Phi^* H^0(\Delta, \Omega^1)$. This establishes an isomorphism between $H^0(\Delta, \Omega^1)$ and $H^0(M, \Omega^2)$ so that

$$\text{genus}(\Delta) = \dim H^0(M, \Omega^2) = h^{2,0} .$$

By Noether's formula, $12(1 - h^{0,1} + h^{0,2}) = c_1^2 + c_2 = 0$. When b_1 is even, $h^{0,1} = h^{1,0}$ and $h^{2,0} = h^{0,2} = h^{1,0} - 1$. When b_1 is odd, $h^{1,0} = h^{0,1} - 1$ and $h^{2,0} = h^{0,2} = h^{1,0}$. q.e.d.

LEMMA (6.5). *Let $\Phi: M \to \Delta$ and $\Phi': M' \to \Delta'$ be two elliptic surfaces such that M' is a normal unramified covering of M. Then $b_1(M')$ is even if and only if $b_1(M)$ is even.*

PROOF. According to Miyaoka [21], an elliptic surface admits a Kähler metric if (and only if) its first Betti number b_1 is even. If M is Kähler, clearly M' is also Kähler. If M' is Kähler, by averaging its Kähler metric by the action of the covering group, we obtain a Kähler metric on M. q.e.d.

LEMMA (6.6). *Assume in (6.3) that $b_2(M)$ is even. Then*

$$\tilde{M} = \tilde{\Delta} \times T ,$$

and there is a representation $\rho: \Gamma \to \mathrm{Aut}(T)$ such that the action of Γ on $\tilde{M} = \tilde{\Delta} \times T$ is given by

$$\gamma(z, t) = (\gamma(z), \rho(\gamma)t) \quad \textit{for} \quad (z, t) \in \tilde{\Delta} \times T \quad \textit{and} \quad \gamma \in \Gamma .$$

PROOF. We exclude first the case where $\Delta = P_1C$ and the number r of singular (modified) fibres is at most 2. Then we have the following commutative diagram described in (6.3)

$$\begin{array}{ccccc} \tilde{M} & \longrightarrow & \tilde{M}/\hat{\Gamma} & \longrightarrow & M \\ \downarrow{\scriptstyle \tilde{\Phi}} & & \downarrow{\scriptstyle \hat{\Phi}} & & \downarrow \\ \tilde{\Delta} & \longrightarrow & \tilde{\Delta}/\hat{\Gamma} & \longrightarrow & \Delta \end{array} .$$

We consider the natural representation of the covering group $\Gamma/\hat{\Gamma}$ of $\tilde{M}/\hat{\Gamma} \to M$ on $H^0(\tilde{M}/\hat{\Gamma}, \Omega^1)$. Since $\Gamma/\hat{\Gamma}$ is a finite group, the invariant subspace $\hat{\Phi}^*(H^0(\tilde{\Delta}/\hat{\Gamma}, \Omega^1))$ has a complementary invariant subspace W:

$$H^0(\tilde{M}/\hat{\Gamma}, \Omega^1) = \hat{\Phi}^*(H^0(\tilde{\Delta}/\hat{\Gamma}, \Omega^1)) + W .$$

Since $\tilde{M}/\hat{\Gamma} \to M$ is a finite unramified normal covering and $b_1(M)$ is even, $b_1(\tilde{M}/\hat{\Gamma})$ is also even by (6.5). Since $\tilde{M}/\tilde{\Gamma} \to \tilde{\Delta}/\hat{\Gamma}$ is a principal T-bundle and $b_1(\tilde{M}/\hat{\Gamma})$ is even, by (6.4) we have $\dim W = 1$. Hence, there is a holomorphic 1-form $\omega \in W$ such that $\omega(V) = 1$ where V is the vertical vector field on $\tilde{M}/\hat{\Gamma}$ defined by the action of T. Since W is invariant by $\Gamma/\hat{\Gamma}$, we have

$$\sigma^*\omega = \chi(\sigma)\omega \quad \text{for} \quad \sigma \in \Gamma/\hat{\Gamma} ,$$

where $\chi\colon \Gamma/\hat{\Gamma} \to \boldsymbol{C}^*$ is a character.

Since $\omega(V) = 1$ and $\mathscr{L}_V\omega := d\cdot\iota_V\omega + \iota_V d\omega = 0$, it follows that ω is a connection form for the principal T-bundle $\tilde{M}/\hat{\Gamma} \to \tilde{\Delta}/\hat{\Gamma}$. Since ω is holomorphic and the base space is of complex dimension 1, the curvature form vanishes, i.e., the connection is flat. Let $\tilde{\omega}$ be the connection form for the bundle $\tilde{M} \to \tilde{\Delta}$ induced by ω. Let (z, t) denote the coordinate for $\tilde{\Delta} \times T$. Then $\tilde{M}$ is isomorphic to the product bundle $\tilde{\Delta} \times T$ in such a way that $\tilde{\omega} = dt$. Let $\tilde{\chi}\colon \Gamma \to \boldsymbol{C}^*$ denote the character induced by $\chi\colon \Gamma/\hat{\Gamma} \to \boldsymbol{C}^*$. Then

$$\gamma^*\tilde{\omega} = \chi(\gamma)\tilde{\omega} \text{ for } \gamma \in \Gamma \quad \text{or} \quad \gamma^* dt = \chi(\gamma)dt \text{ for } \gamma \in \Gamma .$$

This implies

$$\gamma(z, t) = (\gamma(z), \rho(\gamma)t) \quad \text{for} \quad (z, t) \in \tilde{\Delta} \times T , \quad \gamma \in \Gamma ,$$

where $\rho\colon \Gamma \to \operatorname{Aut}(T)$ is a representation. q.e.d.

In order to consider the excluded cases ($\Delta = P_1C$ and $r = 1, 2$), we use the following result of Kodaira [15]. (The definition of logarithmic transformation is given later).

THEOREM (6.7). *An elliptic surface M over a curve Δ with multiple singular fibres of multiplicity $m_1, \cdots, m_r$ at $a_1, \cdots, a_r \in \Delta$ and no other singular fibres is obtained from a holomorphic bundle S over Δ with an elliptic fibre T by logarithmic transformations at $a_1, \cdots, a_r$.*

To explain what a logarithmic transformation at a_i is, we set $a = a_i$ and $m = m_i$ and take a neighborhood $D = \{|z| < 1\}$ in terms of a local coordinate z such that $z(a) = 0$. We may further assume that D contains no other a_j's, and that $S|D$ is a product bundle $D \times T$. Let the elliptic curve T be given by $T = \boldsymbol{C}/(1, \tau)$, where $(1, \tau)$ denotes the lattice generated by 1 and $\tau \in \boldsymbol{C}$ with positive imaginary part. We use w as coordinate in T as well as in $\boldsymbol{C}$. Fix a complex number β such that $[\beta]$ is an element of T of order m. Let $g\colon D \times T \to D \times T$ be defined by

$$g(z, w) = (\rho z, w + [\beta]) , \quad \text{where} \quad \rho = e^{2\pi i/m} .$$

Then g generates a cyclic group (g) of order m acting freely on $D \times T$. The quotient space $(D \times T)/(g)$ is a fibre space over D with projection Φ induced by $\Phi(x, w) = z^m$. We replace $S|D$ by $(D \times T)/(g)$, using the following identification of $D^* \times T$ with $(D^* \times T)/(g)$, where $D^* = D - \{0\}$. Let $\Lambda: D^* \times T \to D^* \times T$ be defined by

$$\Lambda(z, w) = (z^m, w - (m\beta/2\pi i) \log z) .$$

Then Λ induces an isomorphism $\lambda: (D^* \times T)/(g) \to D^* \times T$. This process, denoted by $L_a(m, \beta)$, is called a logarithmic transformation of S at a.

Suppose now that $p: M \to \Delta$ has multiple fibre at a_j with multiplicity m_j ($j = 1, \cdots, r$). When $\Delta = P_1C$, M can be written as follows (see pp. 685–687 of [15] for the argument as well as for the notation):

$$M = L_{a_r}(m_r, \beta_r) \cdots L_{a_1}(m_1, \beta_1)(P^1C \times T) , \quad (m_j \geqq 2) ,$$

where $T = C/(1, \tau)$. And $b_1(M)$ is even if and only if $\beta_1 + \cdots + \beta_r = 0$. Assume $\Delta = P_1C$, $b_1(M)$ is even and M admits a holomorphic $CO(2; C)$-structure. If $r = 1$, then $\beta_1 = 0$ and $M \to P_1C$ is a fibre bundle, contradicting the assumption that it has multiple fibres. If $r = 2$, set $d =$ g.c.d. (m_1, m_2) with $m_1 = m_1'd$ and $m_2 = m_2'd$. Then M has a finite covering $\tilde{M}$ given by

$$\tilde{M} = L_{a_2}(m_2', \beta_2 d)L_{a_1}(m_1', \beta_1 d)(P^1C \times T) ,$$

(see the argument given in [15, p. 689, lines 7-15]). Since $am_1 + bm_2 = d$ for some integers a, b and since $\beta_1 + \beta_2 = 0$, we have $\beta_1 d = a\beta_1 m_1 + b\beta_1 m_2 = a\beta_1 m_1 - b\beta_2 m_2 \in (1, \tau)$ and $\beta_2 d = -\beta_1 d \in (1, \tau)$. Hence, $\tilde{M}$ is a fibre bundle over P_1C. As we have shown above, the holomorphic connection form ω given by (6.4) is integrable and, hence $\tilde{M} = P_1C \times T$.

By the argument above and (6.5), we have established the following

THEOREM (6.8). *Let $\Phi: M \to \Delta$ be an elliptic surface free from exceptional curves of the first kind. If $c_2(M) = 0$ and $b_1(M)$ is even, then*

$$M = \tilde{\Delta} \times_\rho T ,$$

where $\tilde{\Delta}$ ($=P_1C$, C or the upper half-plane H) is a normal ramified covering of Δ with covering group Γ so that (i) $\Delta = \tilde{\Delta}/\Gamma$, (ii) $\rho: \Gamma \to \mathrm{Aut}(T)$ *is a representation and* (iii) *Γ acts freely on $\tilde{\Delta} \times T$.*

COROLLARY (6.9). *Let $\Phi: M \to \Delta$ be an elliptic surface satisfying the assumption of* (6.8). *Then it admits a holomorphic $CO(2; C)$-structure.*

Next, we shall show that if $\Phi: M \to \Delta$ is an elliptic surface with $b_1(M)$ odd, then M admits no holomorphic $CO(2; C)$-structure unless $\Delta = P_1C$. At the same time, we shall obtain some information on $CO(2; C)$-structures of M when $b_1(M)$ is even.

Let $\Phi: M \to \Delta$ be an elliptic surface free from exceptional curves of the first kind such that $c_2(M) = 0$. Exclude the case $\Delta = P_1C$. In (6.3) we proved that there is an elliptic surface $\hat{\Phi}: \hat{M} \to \hat{\Delta}$ with the commutative diagram

$$\begin{array}{ccc} \hat{M} & \longrightarrow & M \\ {\scriptstyle\hat{\Phi}}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \hat{\Delta} & \longrightarrow & \Delta \end{array}$$

where $\hat{\Delta} = \tilde{\Delta}/\Gamma$ and $\hat{M} = \tilde{M}/\hat{\Gamma}$ in the notation of (6.3). Since $\hat{M} \to M$ is an unramified covering, if M admits a holomorphic $CO(2; \boldsymbol{C})$-structure so does $\hat{M}$. Since $\hat{\Phi}: \hat{M} \to \Delta$ is a principal T-bundle, we shall assume that $\Phi: M \to \Delta$ itself is a principal T-bundle.

LEMMA (6.10). *Let $\Phi: M \to \Delta$ be a holomophic principal T-bundle. Then the tangent bundle TM admits a splitting $TM = L' \oplus L''$ such that L' is the line bundle in the fibre direction and L'' is a line bundle transversal to L' if and only if the first Betti number b_1 is even.*

PROOF. Let V be the vector field defined by the T-action on M. Given L'', we define a holomorphic 1-form ω on M by $\omega(L'') = 0$ and $\omega(V) = 1$. Conversely, given a holomorphic 1-form ω such that $\omega(V) = 1$, we define L'' by $\omega = 0$.

This gives a one-to-one correspondence between the set of L'' transversal to L' and the set of holomorphic 1-forms ω satisfying $\omega(V) = 1$. From (6.4) it is clear that such a holomorphic 1-form ω exists if and only if $b_1(M)$ is even.

Lemma (6.10) does not mean that an elliptic surface M with odd b_1 admits no holomorphic $CO(2; \boldsymbol{C})$-structures since there might exist a splitting $TM = L' \oplus L''$ where neither L' nor L'' is in the fibre direction. To look into this possibility, we prove the following.

LEMMA (6.11). *Let M be as in* (6.10). *Let $\mathfrak{a}$ and $\mathfrak{b}$ be the Lie algebras of holomorphic vector fields on M and Δ, respectively. Let $\mathfrak{v}$ be the 1-dimensional subalgebra of $\mathfrak{a}$ generated by the vertical vector field V. Then we have a natural exact sequence*:

$$0 \to \mathfrak{v} \to \mathfrak{a} \to \mathfrak{b} .$$

If $\mathfrak{v} = \mathfrak{a}$, then for any splitting $TM = L' \oplus L''$ either L' or L'' is vertical.

PROOF. Given a holomorphic vector field X on M, let $f_t = \exp(tX)$ be the 1-parameter group of holomorphic transformations generated by X. For a small value of t, each fibre $M_u = \Phi^{-1}(u)$, $u \in \Delta$, is mapped into a coordinate neighborhood around u in Δ by $\Phi \cdot f_t$. Since a holomorphic map of a compact complex space into a coordinate neighborhood is constant, it follows that f_t is fibre-preserving and induces a transformation f_t' on Δ. Let X' be the holomorphic vector field on Δ such that $f_t' = \exp(tX')$. This defines a natural homomorphism $X \in \mathfrak{a} \mapsto X' \in \mathfrak{b}$. The kernel of this homomorphism consists of vertical holomorphic vector fields. Since the vertical holomorphic vector field V never vanishes, every vertical holomorphic vector field is a (function and hence constant) multiple of V. This establishes the first half of (6.11).

If V is contained in neither L' nor L'', the decomposition $V = V' + V''$, where V' is in L'' and V'' is in L'', yields two linearly independent vector fields V' and V'', contradicting the assumption that $\dim \mathfrak{a} = \dim \mathfrak{b} = 1$. q.e.d.

LEMMA (6.12). *Let M be as in* (6.10). *If the genus of Δ is at least* 2, *then for any splitting $TM = L' \oplus L''$, either L' or L'' is vertical. If the genus of Δ is* 1 *and if there is a splitting of TM, then there is a splitting $TM = L' \oplus L''$ such that L' is vertical.*

PROOF. If the genus of Δ is at least 2, then $\mathfrak{b} = 0$ in (6.11) and the result follows from (6.11). Assume that the genus of Δ is 1. Given an arbitrary splitting $TM = L' \oplus L''$, decompose $V = V' + V''$, where V' is in L' and V'' is in L''. If neither L' nor L'' is vertical at some point, V'' is not vertical at some point. Let W be the holomorphic vector field on Δ induced by V''. Then W is nonzero at some point since V'' is not vertical. Since Δ is a torus, W is nonzero everywhere. Hence, V'' is non-vertical everywhere. Then L'' is transversal to the vertical line bundle everywhere. So we have only to replace L' by the vertical line subbundle of TM. Then we have a desired splitting of TM. q.e.d.

The unramified covering space $\hat{M} = \tilde{M}/\tilde{\Gamma}$ of M in (6.3) admits a holomorphic $CO(2; \boldsymbol{C})$-structure if M does. Since the genus of $\tilde{\Delta}/\tilde{\Gamma}$ is greater than or equal to that of Δ, combining (6.5), (6.10) and (6.12) we obtain

THEOREM (6.13). *Let $\Phi\colon M \to \Delta$ be an elliptic surface free from exceptional curves of the first kind such that $c_2(M) = 0$ and $b_1(M)$ is odd. If the genus of Δ is positive, then M admits no holomorphic $CO(2; \boldsymbol{C})$-structures.*

We shall now consider the case where the genus of Δ is 0, i.e., $\Delta = P_1\boldsymbol{C}$.

THEOREM (6.14). *Let M be an elliptic surface over $\Delta = P_1\boldsymbol{C}$ with odd first Betti number. If it admits a holomorphic $CO(2; \boldsymbol{C})$-structure, then it must be a Hopf surface.*

PROOF. It suffices to show that $\hat{M} = \tilde{M}/\tilde{\Gamma}$ in (6.3) is a Hopf surface. We may therefore assume that $M \to \Delta$ is a principal T-bundle. We consider first the case $r > 2$. Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{v}$ be as in (6.11). If $\dim \mathfrak{a} = 1$, i.e., $\mathfrak{v} = \mathfrak{a}$, then M admits no holomorphic $CO(2: \boldsymbol{C})$-structure by (6.10) and (6.11). Hence, there is a holomorphic vector field $X \in \mathfrak{a}$, not contained in $\mathfrak{v}$. Its projection X' to the base curve $\Delta = P_1\boldsymbol{C}$ is a nonzero holomorphic vector field. Being a holomorphic vector field on $P_1\boldsymbol{C}$, X' vanishes at some point but no more than two points of $P_1\boldsymbol{C}$.

Let ω be a holomorphic 1-form on M. Then $\omega(X)$ is constant. Since $\omega(V) = 0$ by (6.4), $\omega(X)$ vanishes at a point where X is vertical, i.e., a point which projects to a zero of X'. Hence, $\omega(X)$ vanishes identically and $\omega = 0$. This shows that $h^{1,0} = 0$. Since $b_1 = 2h^{1,0} + 1 = 1$, M belongs to Class VII_0 in Kodaira's classification of surfaces, [15]. (Class VII_0 consists of minimal surfaces with $b_1 = 1$ and $P_g = 0$).

By integrating X we see that the fibre at a nonzero point of X' is biholomorphic to all nearby fibres. Since X' vanishes at no more than two points of $\Delta = P_1\boldsymbol{C}$, M has at most two singular fibres.

An elliptic surface of Class VII_0 with at most two singular fibres is a Hopf surface, i.e., has $\boldsymbol{C}^2 - \{0\}$ as its universal covering space [15]. q.e.d.

In the next section, we shall study Hopf surfaces.

7. Hopf surfaces. Throughout this section we shall denote the natural coordinate system (z^1, z^2) in $\boldsymbol{C}^2$ by (z, w) whenever convenient to do so.

A compact complex surface M is called a Hopf surface if its universal covering space is biholomorphic to $\boldsymbol{C}^2 - \{0\}$. A Hopf surface is said to be primary if its fundamental group is infinite cyclic. Every Hopf surface has a primary Hopf surface as a finite unramified covering. Every primary Hopf surface M is biholomorphic to a surface of the form $(\boldsymbol{C}^2 - \{0\})/(\sigma)$, where (σ) denotes the infinite cyclic group of transformations genearated by a transformation σ of the form (see [15])

$$\sigma(z, w) = (\alpha z + \lambda w^m, \beta w) \tag{7.1}$$

with

$$\alpha, \beta, \lambda \in \boldsymbol{C}\,, \qquad 0 < |\alpha| \leqq |\beta| < 1\,, \qquad (\alpha - \beta^m)\lambda = 0\,. \tag{7.2}$$

We shall determine which Hopf surfaces admit holomorphic $CO(2; \boldsymbol{C})$-structure. Let M be a primary Hopf surface $(\boldsymbol{C}^2 - \{0\})/(\sigma)$ with a holomorphic $CO(2; \boldsymbol{C})$-structure. A holomorphic $CO(2; \boldsymbol{C})$-structure on M may be regarded as a σ-invariant holomorphic $CO(2; \boldsymbol{C})$-structure on $\boldsymbol{C}^2 - \{0\}$. By (5.9), a holomorphic $CO(2; \boldsymbol{C})$-structure on $\boldsymbol{C}^2 - \{0\}$ is given by a globally defined quadratic from $g = \sum g_{ij}\, dz^i dz^j$ on $\boldsymbol{C}^2 - \{0\}$. Since $\boldsymbol{C}^2 - \{0\}$ is simply connected we can divide g by a globally defined $(\det(g_{ij}))^{1/2}$ and assume that $\det(g_{ij}) = 1$.

We represent $g = \sum g_{ij}\, dz^i dz^j$ by a matrix

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}. \tag{7.3}$$

Since

$$\begin{pmatrix} \sigma^* dz \\ \sigma^* dw \end{pmatrix} = \begin{pmatrix} \alpha & mw^{m-1} \\ 0 & \beta \end{pmatrix} \begin{pmatrix} dz \\ dw \end{pmatrix}, \tag{7.4}$$

$\sigma^* g$ is represented by

$$\begin{pmatrix} \alpha & 0 \\ \lambda m w^{m-1} & \beta \end{pmatrix} \begin{pmatrix} g^\sigma_{11} & g^\sigma_{12} \\ g^\sigma_{21} & g^\sigma_{22} \end{pmatrix} \begin{pmatrix} \alpha & \lambda m w^{m-1} \\ 0 & \beta \end{pmatrix}, \tag{7.5}$$

where $g^\sigma_{ij}(\zeta) = g_{ij}(\sigma(\zeta))$, $\zeta = (z, w) = (z^1, z^2)$.

The holomorphic $CO(2; \boldsymbol{C})$-structure on $C^2 - \{0\}$ defined by g is invariant by σ if and only if $\sigma^* g = fg$, where f is a holomorphic function without zeros. Comparing the martrices (7.3) and (7.4) and using the condition $\det(g_{ij}) = 1$, we obtain

$$f^2 = (\alpha\beta)^2 \tag{7.6}$$

and

$$f g_{11}(\zeta) = \alpha^2 g_{11}(\sigma(\zeta)). \tag{7.7}$$

Hence,

$$g_{11}(\zeta) = \pm(\alpha/\beta) g_{11}(\sigma(\zeta)). \tag{7.8}$$

Iterating this process, we obtain

$$g_{11}(\zeta) = \pm(\alpha/\beta)^n g_{11}(\sigma^n(\zeta)). \tag{7.9}$$

By Hartogs' theorem, g_{11} extends through the origin o. Hence

$$g_{11}(\zeta) = \lim_{n\to\infty} \pm(\alpha/\beta)^n g_{11}(\sigma^n(\zeta)) = 0 \quad \text{if} \quad |\alpha| < |\beta|. \tag{7.10}$$

We shall first consider the case $g_{11} = 0$ (which is satisfied if $|\alpha| < |\beta|$ by (7.10)). Then $1 = \det(g_{ij}) = -g_{12}g_{21}$ and $g_{12} = \pm\sqrt{-1}$. Comparing (7.3) with (7.5), we obtain

$$f = \alpha\beta, \qquad \alpha g_{22} = 2 g_{12} h + \beta g^\sigma_{22}, \tag{7.11}$$

where $g_{12} = \pm\sqrt{-1}$ and $h = mw^{m-1}$. Hence,

$$\alpha\, \partial g_{22}/\partial z = \alpha\beta\, \partial g^\sigma_{22}/\partial z. \tag{7.12}$$

By iterating this process, we obtain

$$\partial g_{22}(\zeta)/\partial z = \beta^n \partial g_{22}(\sigma^n(\zeta))/\partial z . \tag{7.13}$$

Then, as in (7.10), we conclude

$$\partial g_{22}/\partial z = 0 , \tag{7.14}$$

i.e., g_{22} is a function of w only.

From (7.11) we obtain

$$\alpha \, \partial^m g_{22}/(\partial w)^m = \beta^{m+1} \partial^m g_{22}/(\partial w)^m . \tag{7.15}$$

Assume $\lambda \neq 0$ so that $\alpha = \beta^m$. Then

$$\partial^m g_{22}/(\partial w)^m = \beta \, \partial^m g_{22}/(\partial w)^m . \tag{7.16}$$

In the same way as we derived (7.14) from (7.12), we obtain

$$\partial^m g_{22}/(\partial w)^m = 0 . \tag{7.17}$$

Hence, g_{22} is a polynomial of degree $m - 1$ in w, i.e.,

$$g_{22} = a_0 + a_1 w + \cdots + a_{m-1} w^{m-1} . \tag{7.18}$$

Substituting (7.18) into (7.11), we obtain contradiction. We have thus shown

LEMMA (7.19). *If $|\alpha| < |\beta|$ and $\lambda \neq 0$, then there is no σ-invariant holomorphic $CO(2; C)$-structures on $C^2 - \{0\}$.*

We shall now consider the case where $|\alpha| < |\beta|$ and $\lambda = 0$. We already know that $f = \alpha\beta$, $g_{11} = 0$ and $g_{12} = \pm\sqrt{-1}$. Since $\lambda = 0$ in (7.5), the σ-invariance $\sigma^* g = fg$ implies

$$\begin{pmatrix} 0 & \alpha\beta g_{12} \\ \alpha\beta g_{21} & \beta^2 g_{22} \end{pmatrix} = \begin{pmatrix} 0 & f g_{12} \\ f g_{21} & f g_{22}^\sigma \end{pmatrix} .$$

Hence,

$$g_{22} = (\beta/\alpha) g_{22}^\sigma . \tag{7.20}$$

By differentiating (7.20) with respect to z, we obtain

$$\partial g_{22}/\partial z = \beta \, \partial g_{22}^\sigma/\partial z . \tag{7.21}$$

As in (7.14) we conclude that $\partial g_{22}/\partial z = 0$, i.e., g_{22} is a function of w only. Let n be a larger integer such that $|\beta|^{n+1} < |\alpha|$. Then from (see (7.15)) $\partial^n g_{22}/(\partial w)^n = (\beta^{n+1}/\alpha) \partial^n g_{22}/(\partial w)^n$ we conclude that g_{22} is a polynomial of degree at most $n - 1$ in w. Substitute that polynomial into (7.20). Then we see that g_{22} is a monomial $g_{22} = a w^k$ in w if $\alpha = \beta^{k+1}$ and $g_{22} = 0$ if there is no such relation between α and β. Hence, LEMMA (7.22). *If $|\alpha| < |\beta|$ and $\lambda = 0$, then there exist σ-invariant holomorphic $CO(2; \boldsymbol{C})$-structures on $\boldsymbol{C}^2 - \{0\}$. They are given by*

$$g_{11} = 0\,, \qquad g_{12} = g_{21} = \text{constant} \neq 0\,,$$
$$g_{22} = \begin{cases} \text{a monomial of degree } k \text{ in } w & \text{if } \alpha = \beta^{k+1} \\ 0 & \text{otherwise}\,. \end{cases}$$

We shall now consider the case $|\alpha| = |\beta|$. By (7.2) we have either $\lambda = 0$ or $m = 1$. From (7.8) we have

$$|g_{11}| = |g_{11}^\sigma|\,. \tag{7.23}$$

Hence, $|g_{11}|$ may be considered as a function on M and is constant by the maximum principle. Hence, g_{11} itself is constant.

Assume $\lambda = 0$. The σ-invariance $\sigma^* g = fg$ implies

$$\begin{pmatrix} fg_{11} & fg_{12} \\ fg_{21} & fg_{22} \end{pmatrix} = \begin{pmatrix} \alpha^2 g_{11} & \alpha\beta g_{12} \\ \alpha\beta g_{21} & \beta^2 g_{22} \end{pmatrix}. \tag{7.24}$$

From (7.6) and (7.24) we obtain $|g_{22}| = |g_{22}^\sigma|$. By the same argument as above, g_{22} is constant. Similarly, g_{12} is also constant. Thus we have

LEMMA (7.25). *If $|\alpha| = |\beta|$ and $\lambda = 0$, then there exist σ-invariant holomorphic $CO(2; \boldsymbol{C})$-structures on $\boldsymbol{C}^2 - \{0\}$.*

(i) *If $\alpha = \beta$, then any non-degenerate constant matrix (g_{ij}) gives such a structure.*

(ii) *If $\alpha = -\beta$, then (g_{ij}) must be a constant matrix of the form*

$$\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & g_{12} \\ g_{21} & 0 \end{pmatrix}.$$

(iii) *If $\alpha \neq \pm\beta$, then (g_{ij}) must be a constant matrix of the form*

$$\begin{pmatrix} 0 & g_{12} \\ g_{21} & 0 \end{pmatrix}.$$

These exhaust all σ-invariant holomorphic $CO(2; \boldsymbol{C})$-structures on $\boldsymbol{C}^2 - \{0\}$ when $|\alpha| = |\beta|$, $\lambda = 0$.

We shall consider the last remaining case where $|\alpha| = |\beta|$, $m = 1$ and $\lambda \neq 0$. By (7.2) we have $\alpha = \beta$. In this case, the σ-invariance $\sigma^* g = fg$ is equivalent to

$$\begin{aligned} \begin{pmatrix} fg_{11} & fg_{12} \\ fg_{21} & fg_{22} \end{pmatrix} &= \begin{pmatrix} \alpha & 0 \\ \lambda & \alpha \end{pmatrix} \begin{pmatrix} g_{11}^\sigma & g_{12}^\sigma \\ g_{21}^\sigma & g_{22}^\sigma \end{pmatrix} \begin{pmatrix} \alpha & \lambda \\ 0 & \alpha \end{pmatrix} \\ &= \begin{pmatrix} \alpha^2 g_{11}^\sigma & \alpha\lambda g_{11}^\sigma + \alpha^2 g_{12}^\sigma \\ \alpha\lambda_{11}^\sigma + \alpha^2 g_{21}^\sigma & \lambda^2 g_{11}^\sigma + 2\alpha\lambda g_{12}^\sigma + \alpha^2 g_{22}^\sigma \end{pmatrix}. \end{aligned} \tag{7.26}$$

We have shown already that g_{11} is constant. Assume $g_{11} \neq 0$. Then $f = \alpha^2$ from (7.26). Also from (7.26) we obtain

$$g_{12} = (\lambda/\alpha)g_{11} + g^{\sigma}_{12} . \tag{7.27}$$

Differentiating (7.27), we obtain

$$\partial g_{12}/\partial z = \alpha\, \partial g^{\sigma}_{12}/\partial z , \qquad \partial g_{12}/\partial w = \alpha\, \partial^{\sigma}_{12}/\partial w . \tag{7.28}$$

By the argument we have used several times, these partial derivatives are zero and g_{12} is constant. This contradicts (7.27). Hence, $g_{11} = 0$.

Since $1 = \det(g_{ij}) = -g_{12}g_{21}$, we obtain $g_{12} = g_{21} = \pm\sqrt{-1}$. From (7.26) we obtain $f = \alpha^2$ and

$$g_{22} = (2\lambda/\alpha)g_{12} + g^{\sigma}_{22} . \tag{7.29}$$

In the same way as we proved that g_{12} is constant, we can show that g_{22} is constant. This contradicts (7.29). Hence,

(7.30) *If* $|\alpha| = |\beta|$, $m = 1$ *and* $\lambda \neq 0$, *then there is no* σ*-invariant holomorphic* $CO(2; \boldsymbol{C})$*-structures on* $\boldsymbol{C}^2 - \{0\}$.

We have shown that a primary Hopf surface $(\boldsymbol{C}^2 - \{0\})/(\sigma)$ admitting a holomorphic $CO(2; \boldsymbol{C})$-structure must satisfy $\lambda = 0$, i.e., σ is of the form

$$\sigma(z, w) = (\alpha z, \beta w) \quad \text{with} \quad 0 < |\alpha| \leqq |\beta| < 1 . \tag{7.31}$$

It is clear that such a primary Hopf surface admits an obvious holomorphic $CO(2; \boldsymbol{C})$-structure (which is, in fact, a quadric structure and gives rise to a splitting $TM = L' \oplus L''$). We shall now examine Hopf surfaces covered by such a primary Hopf surface.

Let $M = (\boldsymbol{C}^2 - \{0\})/\Gamma$ be a Hopf surface covered by a primary Hopf surface $\tilde{M} = (\boldsymbol{C}^2 - \{0\})/(\sigma)$, where σ is of the form (7.31). Then (σ) is a subgroup of finite index in Γ. Moreover, a suitable power σ^q of σ is in the center of Γ, [15].

Let τ be an element of Γ given by

$$\tau(z, w) = (f^1(z, w), f^2(z, w)) . \tag{7.32}$$

If n is a multiple of q, then τ commutes with σ^n and we have

$$f^1(\alpha^n z, \beta^n w) = \alpha^n f^1(z, w) , \qquad f^2(\alpha^n z, \beta^n w) = \beta^n f^2(z, w) . \tag{7.33}$$

By differentiating the first equation with respect to z, we obtain

$$(\partial f^1/\partial z)(\alpha^n z, \beta^n w) = (\partial f^1/\partial z)(z, w) . \tag{7.34}$$

Letting $n \to \infty$, we see that the right hand side is equal to the constant $(\partial f^1/\partial z)(0, 0)$. Similarly, $\partial f^2/\partial w$ is also constant. Hence,

$$(7.35)\qquad f^1(z, w) = A(w)z + B(w)\ , \qquad f^2(z, w) = C(z) + D(z)w\ .$$

Since τ commutes with σ^n (where n is a multiple of q), we obtain

$$(7.36)\qquad \begin{aligned} A(\beta^n w)\alpha^n z + B(\beta^n w) &= \alpha^n A(w)z + \alpha^n B(w)\ , \\ C(\alpha^n z) + D(\alpha^n z)\beta^n w &= \beta^n C(z) + \beta^n D(z)w\ . \end{aligned}$$

From (7.36) we see immediately that both A and D are constant. Expanding $B(w)$ and $C(z)$ into power series and using the condition $0 < |\alpha| \leqq |\beta| < 1$, we arrive at the following possibilities:

$$(7.37)\qquad \tau(z, w) = (ad, dw) \quad \text{if} \quad \alpha^q \neq \beta^{qk} \text{ for all integers } k > 0\ ,$$

$$(7.38)\qquad \tau(z, w) = (az + bw^k, dw) \quad \text{if} \quad \alpha^q = \beta^{qk} \text{ for some integer } k \geqq 2\ ,$$

$$(7.39)\qquad \tau(z, w) = (az + bw, cz + dw) \quad \text{if} \quad \alpha^q = \beta^q\ .$$

In case (7.37), the natural splitting for the tangent bundle of $C^2 - \{0\}$ given by the coordinate system is invariant by the group Γ.

In case (7.38), we shall show that if $b \neq 0$, then $C^2 - \{0\}$ admits no holomorphic $CO(2; C)$-structures invariant by the element τ. Since (σ) is a subgroup of finite index in Γ, some power of τ, say τ^t, is equal to σ^s. (replacing τ by τ^{-1} if necessary we may assume that t is positive and s is non-negative). Then

$$(7.40)\qquad a^t = \alpha^s\ , \qquad d^t = \beta^s\ .$$

Since $\alpha^q = \beta^{qk}$ with $k \geqq 2$ in this case, we have $|\alpha| < |\beta|$. Since $b \neq 0$, τ^t cannot be the identity element and hence s is positive. From (7.40) we obtain $|a| < |d|$. Thus we are almost in the same situation as in (7.19). The difference here is that we have

$$(7.41)\qquad a^{tq} = d^{tqk}$$

instead of $\alpha = \beta^m$. Following the computation from (7.3) through (7.14), we see that if the $CO(2; C)$-structure is invariant by τ, then $g_{11} = 0$, $g_{12} = g_{21} = \pm\sqrt{-1}$ and g_{22} is a function of w only. As in (7.15), we obtain

$$(7.42)\qquad a\,\partial^k g_{22}/(\partial w)^k = d^{k+1}\,\partial^k g^{\tau}_{22}/(\partial w)^k\ .$$

From (7.41) and (7.42) we obtain

$$(7.43)\qquad (\partial^k g_{22}/(\partial w)^k)^{qt} = d^{qt}(\partial^k g^{\tau}_{22}/(\partial w)^k)^{qt}\ .$$

In the same way as we derived (7.14) from (7.12), we obtain

$$(7.44)\qquad \partial^k g_{22}/(\partial w)^k = 0\ .$$

Hence, g_{22} is a polynomial of degree $k - 1$ in w, i.e., (7.45) $$g_{22} = a_0 + a_1 w + \cdots + a_{k-1} w^{k-1} .$$

Now, we are in the same situation as in (7.18) and obtain the desired result that there is no holomorphic $CO(2; \boldsymbol{C})$-structures on $\boldsymbol{C}^2 - \{0\}$ invariant by τ.

We have shown that in case (7.38) a holomorphic $CO(2; \boldsymbol{C})$-structure exists on $M = (\boldsymbol{C}^2 - \{0\})/\Gamma$ if and only if every element τ of Γ is of the form

(7.46) $$\tau(z, w) = (ad, dw) ,$$

i.e., $b = 0$.

We consider now case (7.39). Let V be the vector field on $\boldsymbol{C}^2 - \{0\}$ defined by

(7.47) $$V = z\,\partial/\partial z + w\,\partial/\partial w .$$

Since it is invariant by any linear transformation of $\boldsymbol{C}^2$, it may be considered as a vector field on $\tilde{M} = (\boldsymbol{C}^2 - \{0\})/(\sigma)$ or $M = (\boldsymbol{C}^2 - \{0\})/\Gamma$. Assuming that M admits a holomorphic $CO(2; \boldsymbol{C})$-structure, consider the induced holomorphic $CO(2; \boldsymbol{C})$-structure on $\boldsymbol{C}^2 - \{0\}$ invariant by Γ. Since $\boldsymbol{C}^2 - \{0\}$ is simply connected, this $CO(2; \boldsymbol{C})$-structure is given by a splitting $T(\boldsymbol{C}^2 - \{0\}) = L' \oplus L''$ of the tangent bundle of $\boldsymbol{C}^2 - \{0\}$. Then every element of Γ leaves both L' and L'' invariant or interchanges them.

We claim that V is neither in L' nor in L''. Assume that V is in L'. Since σ leaves V invariant, it leaves both L' and L'' invariant (instead of interchanging them). Hence we obtain the induced splitting $T\tilde{M} = L' \oplus L''$ denoted by the same symbols as the splitting $T(\boldsymbol{C}^2 - \{0\}) = L' \oplus L''$. On the other hand, $\tilde{M}$ is an elliptic surface over $P_1\boldsymbol{C}$ with odd first Betti number and, by (6.8), does not admit a splitting $T\tilde{M} = L' \oplus L''$ such that L' is in the fibre direction, i.e., in the direction of V in this case. This is a contradiction.

Since V is neither in L' nor in L'', the decomposition

(7.48) $$V = V' + V'' , \quad (V' \in L',\ V'' \in L'')$$

yields two nonzero vector fields V' and V'' on $\boldsymbol{C}^2 - \{0\}$. Every element of Γ either leaves both V' and V'' invariant or interchanges them.

We shall prove next that V' is of the following form:

(7.49) $$V' = (\lambda_1 z + \lambda_2 w)\,\partial/\partial z + (\mu_1 z + \mu_2 w)\,\partial/\partial w .$$

We write $V' = \xi^1(z, w)\,\partial/\partial z + \xi^2(z, w)\,\partial/\partial w$. Since σ either leaves V' and V'' invariant or interchanges them, σ^2 leaves V' and V'' invariant. Let $n = 2q$ so that σ^n leaves V' invariant and $\alpha^n = \beta^n$. Then

$$\alpha^n \xi^1(z, w) = \xi^1(\alpha^n z, \beta^n w)\ , \qquad \beta^n \xi^2(z, w) = \xi^2(\alpha^n z, \beta^n w)\ . \tag{7.50}$$

Differentiating (7.50) with respect to z and w, we obtain (using $\alpha^n = \beta^n$)

$$\begin{aligned}(\partial\xi^i/\partial z)(z, w) &= (\partial\xi^i/\partial z)(\alpha^n z, \beta^n w)\ ,\\ (\partial\xi^i/\partial w)(z, w) &= (\partial\xi^i/\partial w)(\alpha^n z, \beta^n w)\ .\end{aligned} \tag{7.51}$$

Hence

$$\begin{aligned}(\partial\xi^i/\partial z)(z, w) &= (\partial\xi^i/\partial z)(\alpha^{pn} z, \beta^{pn} w)\ ,\\ (\partial\xi^i/\partial w)(z, w) &= (\partial\xi^i/\partial w)(\alpha^{pn} z, \beta^{pn} w)\ ,\end{aligned} \qquad p = 1, 2, \cdots\ . \tag{7.52}$$

Letting $p \to \infty$, we see that the left hand side of (7.52) is constant. It follows that ξ^i is linear in z, w, i.e., V' is of the form (7.49).

We associate to vector fields V, V', V'' the following matrices or linear transformations of $\boldsymbol{C}^2$:

$$V: \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad V': \begin{pmatrix}\lambda_1 & \lambda_2\\ \mu_1 & \mu_2\end{pmatrix}, \qquad V'': \begin{pmatrix}1-\lambda & -\lambda_2\\ -\mu_1 & 1-\mu_2\end{pmatrix}. \tag{7.53}$$

Then a linear transformation of $\boldsymbol{C}^2$ leaves the vector fields V' and V'' invariant if and only if it commutes with the corresponding linear transformations given in (7.53). By a linear change of coordinates, we reduce the matrics in (7.53) into the following canonical forms:

$$V': \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}, \qquad V'': \begin{pmatrix}1-\lambda & -1\\ 0 & 1-\lambda\end{pmatrix}, \tag{7.54}$$

or

$$V': \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}, \qquad V'': \begin{pmatrix}1-\lambda & 0\\ 0 & 1-\mu\end{pmatrix} \quad \text{with} \quad \lambda \neq \mu\ . \tag{7.55}$$

We note that $\lambda \neq \mu$ since V' is not a scalar multiple of V.

In case (7.54), a linear transformation of $\boldsymbol{C}^2$ leaves V' and V'' invariant if and only if it is of the form

$$\begin{pmatrix}a & b\\ 0 & a\end{pmatrix} \tag{7.56}$$

while it interchanges V' and V'' if and only if it is of the form

$$\begin{pmatrix}a & b\\ 0 & -a\end{pmatrix} \quad \text{with} \quad \lambda = 1/2\ . \tag{7.57}$$

By (7.30), in order for a matrix of the form (7.56) or (7.57) to leave a holomorphic $CO(2; \boldsymbol{C})$-structure on $\boldsymbol{C}^2 - \{0\}$ invariant, it is necessary that $b = 0$. Hence, every element of Γ must be of the form

$$(7.58)\qquad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

according as it leaves V' and V'' invariant or interchanges them. It is clear that, conversely, if every element of Γ is a matrix of the form (7.58), then the natural $CO(2; \boldsymbol{C})$-structure on $\boldsymbol{C}^2 - \{0\}$ is invariant by Γ.

In case (7.55), a linear transformation of $\boldsymbol{C}^2$ leaves V' and V'' invariant if and only if it is of the form

$$(7.59)\qquad \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

while it interchanges V' and V'' if and only if it is of the form

$$(7.60)\qquad \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \quad \text{with} \quad \lambda + \mu = 1 .$$

Hence every element of Γ must be of the form (7.59) or (7.60) according as it leaves V' and V'' invariant or it interchanges them. It is clear that, conversely, if every element of Γ is of the form (7.59) or (7.60), then the natural $CO(2; \boldsymbol{C})$-structure on $\boldsymbol{C}^2 - \{0\}$ is invariant by Γ.

We have established

THEOREM (7.61). *A Hopf surface $M = (\boldsymbol{C}^2 - \{0\})/\Gamma$ admits a holomorphic $CO(2; \boldsymbol{C})$-structure if and only if every element of Γ is a linear transformation of the form*

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} .$$

8. **Surfaces of Class** VII_0. Throughout this section we shall denote the natural coordinate system (z^1, z^2) in $\boldsymbol{C}^2$ by (z, w) whenever convenient to do so. A compact complex surface M is said to be in Class VII_0 if it is free of exceptional curves of the first kind, $b_1 = 1$ and $p_g = 0$. Then $q = 1$. (In general, $2q = b_1 + 1$ when b_1 is odd, [15]). By Noether's formula,

$$(8.1)\qquad c_1^2 + c_2 = 12(1 - q) = 0 .$$

Since c_2 is the Euler number and $b_1 = 1$,

$$(8.2)\qquad c_2 = b_2 .$$

Hence,

LEMMA (8.3). *If a surface of Class* VII_0 *satisfies* $c_1^2 = 2c_2$, *in particular, if it admits a holomorphic* $CO(2; \boldsymbol{C})$*-structure, then*

$$b_2 = 0 .$$

The surfaces of Class VII_0 with $b_2 = 0$ can be classified as follows:

(i) Hopf surfaces;

(ii) non-Hopf, elliptic surfaces with $b_1 = 1$, $b_2 = 0$;

(iii) non-Hopf, non-elliptic surfaces with $b_1 = 1$, $b_2 = 0$ and a line bundle F such that $H^0(M, \Omega^1(F)) \neq 0$;

(iv) non-Hopf, non-elliptic surfaces with $b_1 = 1$, $b_2 = 0$ such that $H^0(M, \Omega^1(F))) = 0$ for all line bundles F.

Moreover the above classification is invariant under passing to an unramified covering.

We have already considered Case (i) in § 7 and Case (ii) in § 6.

LEMMA (8.4). *A surface of Class* VII_0 *satisfying* (iv) *above admits no holomorphic* $CO(2; \boldsymbol{C})$*-structures.*

PROOF. Assuming that M admits a holomorphic $CO(2; \boldsymbol{C})$-structure, let $TM = L' \oplus L''$ as in § 5 (taking a double covering if necessary). Then the cotangent bundle is given by $L'^{-1} \oplus L''^{-1}$. Hence,

$$\Omega^1(L') = \mathcal{O}((L'^{-1} \oplus L''^{-1}) \otimes L') = \mathcal{O}(1) \oplus \mathcal{O}(L''^{-1} \otimes L') ,$$

which clearly admits a non-trivial holomorphic section. This contradicts the last condition in (iv). q.e.d.

We shall now consider Case (iii). According to Inoue [6], a surface M satisfying (iii) belongs to one of the following three classes:

(a) *Surfaces* S_U. Let $U = (u_{ij}) \in SL(3; \boldsymbol{Z})$ be a unimodular matrix with eigenvalues α, β, $\bar{\beta}$ such that $\alpha > 1$, $\beta \neq \bar{\beta}$. Choose a real eigenvector (a_1, a_2, a_3) and an eigenvector (b_1, b_2, b_3) of U corresponding to α and β, respectively. Let G_U be the group of holomorphic transformations of $H \times \boldsymbol{C}$ generated by

$$\sigma_0 \colon (z, w) \mapsto (\alpha z, \beta w) ,$$
$$\sigma_i \colon (z, w) \mapsto (z + a_i, w + b_i) , \quad i = 1, 2, 3 .$$

Let $M = S_U = (H \times \boldsymbol{C})/G_U$. From the construction of M it is clear that TM admits a splitting $TM = L' \oplus L''$, where L' and L'' are spanned by $\partial/\partial z$ and $\partial/\partial w$, respectively. It is also clear that this $CO(2; \boldsymbol{C})$-structure comes from a quadric structure.

We shall show that M admits no other $CO(2; \boldsymbol{C})$-structure. In fact, let $g = \sum g_{jk}\, dz^j dz^k$ define a holomorphic $CO(2; \boldsymbol{C})$-structure on M, i.e., a G_U-invariant $CO(2; \boldsymbol{C})$-structure on $H \times \boldsymbol{C}$ so that

$$\sigma_i^* g = f_i g , \quad i = 0, 1, 2, 3 , \tag{8.1}$$

where each f_i is a holomorphic function with no zeros. Because of the simple connectedness of $H \times \boldsymbol{C}$, we may assume as in § 7 that

$$\det(g_{jk}) = 1 . \tag{8.2}$$

Then the invariance condition (8.1) is equivalent to

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} g_{11}^{\sigma_0} & g_{12}^{\sigma_0} \\ g_{21}^{\sigma_0} & g_{22}^{\sigma_0} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = f_0 \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$
$$\begin{pmatrix} g_{11}^{\sigma_i} & g_{12}^{\sigma_i} \\ g_{21}^{\sigma_i} & g_{22}^{\sigma_i} \end{pmatrix} = f_i \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} . \tag{8.1$'$}$$

From these and (8.2) it follows

$$(\alpha\beta)^2 = (f_0)^2 , \qquad 1 = (f_i)^2 \quad (i = 1, 2, 3) . \tag{8.3}$$

From (8.3), we see that $|g_{12}|$ is invariant by G_U and hence g_{12} is a constant function. From (8.1)$'$ it then follows that

$$\begin{aligned} \alpha^2 g_{11}^{\sigma_0} &= f_0 g_{11} , \qquad g_{11}^{\sigma_i} = f_i g_{11} \quad (i = 1, 2, 3) , \\ \beta^2 g_{22}^{\sigma_0} &= f_0 g_{22} , \qquad g_{22}^{\sigma_i} = f_i g_{22} \quad (i = 1, 2, 3) . \end{aligned} \tag{8.1$''$}$$

Differentiating (8.1)$''$ with respect to w, we obtain

$$\alpha^2\beta^2(\partial^2 g_{11}/\partial w^2)^{\sigma_0} = f_0\,\partial^2 g_{11}/\partial w^2 , \qquad (\partial^2 g_{11}/\partial w^2)^{\sigma_i} = f_i\,\partial^2 g_{11}/\partial w^2 \quad (i = 1, 2, 3) .$$

Hence $((\partial^2 g_{11}/\partial w^2) dz \wedge dw)^2$ is invariant by G_U and hence is a section of K^2 on M. On the other hand, $H^0(M; K^2) = 0$ by Inoue [6]. Hence $\partial^2 g_{11}/\partial w^2 = 0$. Similarly, we have $\partial^2 g_{22}/\partial z^2 = 0$. So put

$$g_{11}(z, w) = A(z)w + B(z) , \qquad g_{22}(z, w) = C(w)z + D(w) ,$$

where $A(z)$, $B(z)$ (resp. $C(w)$, $D(w)$) are holomorphic on H (resp. C). From (8.1)$''$ we obtain

$$\begin{aligned} \alpha^2\{A(\alpha z)\beta w + B(\alpha z)\} &= f_0\{A(z)w + B(z)\} , \\ A(z + a_1)(w + b_1) + B(z + a_1) &= f_1\{A(z)w + B(z)\} . \end{aligned}$$

Hence,

$$\alpha^2\beta A(\alpha z) = f_0(z) , \qquad A(z + a_1) = f_1 A(z) , \tag{8.1$_A$}$$

$$\alpha^2 B(\alpha z) = f_0 B(z) , \qquad b_1 A(z + a_1) + B(z + a_1) = f_1 B(z) . \tag{8.1$_B$}$$

Without loss of generality, we may assume $a_1 = 1$. From (8.1)$_A$ we obtain

$$A(\alpha^k z + 2\alpha^k) = (f_0/\alpha^2\beta)^k A(z + 2) = (f_0/\alpha^2\beta)^k A(z) = A(\alpha^k z) \quad \text{for} \quad k \in \boldsymbol{Z} .$$

Hence, $A(z + 2\alpha^k) = A(z)$ for $k \in \boldsymbol{Z}$. This means that A is constant on the infinite sequence $\{z + 2\alpha^k\}$, $k = -1, -2, \cdots$, converging to z. Hence,

A is constant on H. From $(8.1)_A$, we have $(\alpha^2\beta - f_0)A = 0$. If $A \neq 0$, then $\alpha^4\beta^2 = f_0^2 = \alpha^2\beta^2$ and hence $\alpha^2 = 1$, contradicting the assumption $\alpha > 1$. We conclude $A = 0$. From $(8.1)_B$, we have

$$\alpha^2 B(\alpha z) = f_0 B(z) , \qquad B(z + a_1) = f_1 B(z) ,$$

and obtain "$B =$ constant" in a similar manner. If $B \neq 0$, then $\alpha^4 = f_0^2 = \alpha^2\beta^2$ and hence $\alpha^2 = \beta^2$, contradicting the assumption $\alpha > 1$ and $\alpha\beta\bar{\beta} = 1$. Hence, $B = 0$. This proves $g_{11} = 0$.

Similarly, from $(8.1)''$ we obtain

$$\beta^2\{C(\beta w)\alpha z + D(\beta w)\} = f_0\{C(w)z + D(w)\} ,$$
$$C(w + b_1)(z + a_1) + D(w + b_1) = f_1\{C(w)z + D(w)\} ,$$

and hence

$$(8.1)_C \qquad \alpha\beta^2 C(\beta w) = f_0 C(w) ,$$

$$(8.1)_D \qquad C(w + b_1) = f_1 C(w) .$$

Without loss of generality, we may assume $b_1 = 0$. In the same way as above, we conclude $C = D = 0$, i.e., $g_{22} = 0$. q.e.d.

(b) *Surfaces* $S^{(+)}_{N,p,q,r;t}$. Let $N = (n_{jk}) \in SL(2; \boldsymbol{Z})$ be a unimodular matrix with two real eigenvalues α, $1/\alpha$ with $\alpha > 1$. Choose real eigenvectors (a_1, a_2), (b_1, b_2) of N corresponding to α and $1/\alpha$ respectively and fix integers p, q, r $(r \neq 0)$ and a complex number t. Let (c_1, c_2) be the solution of

$$(c_1 c_2) = (c_1, c_2)\,{}^tN + (e_1, e_2) + (1/r)(b_1a_2 - b_2a_1)(p, q) ,$$

where

$$e_i = (1/2)n_{i1}(n_{i1} - 1)a_1b_1 + (1/2)n_{i2}(n_{i2} - 1)a_2b_2 + n_{i1}n_{i2}b_1a_2 .$$

Let $G = G^{(+)}_{N,p,q,r;t}$ be the group of holomorphic transformations of $H \times \boldsymbol{C}$, generated by

$$\sigma_0\colon (z, w) \mapsto (\alpha z, w + t) ,$$
$$\sigma_i\colon (z, w) \mapsto (z + a_i, w + b_i z + c_i) , \quad i = 1, 2 ,$$
$$\sigma_3\colon (z, w) \mapsto (z, w + (1/r)(b_1a_2 - b_2a_1))$$

and define $M = S^{(+)}_{N,p,q,r;t} = (H \times \boldsymbol{C})/G$.

We shall show that M admits no holomorphic $CO(2; \boldsymbol{C})$-structures. Let $g = \sum g_{jk}\,dz^j dz^k$ define a holomorphic $CO(2; \boldsymbol{C})$-structure on M, i.e., a G-invariant holomorphic $CO(2; \boldsymbol{C})$-structure on $H \times \boldsymbol{C}$ so that

$$(8.4) \qquad \sigma_i^* g = f_i g , \qquad i = 0, 1, 2, 3 ,$$

where each f_i is a holomorphic function with no zeros. Because of the simple connectedness of $H \times C$, we may assume as in § 7 that

$$\det(g_{jk}) = 1 . \tag{8.5}$$

Then the invariance condition (8.4) is equivalent to

$$\begin{aligned} &\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}g_{11}^{\sigma_0} & g_{12}^{\sigma_0}\\ g_{21}^{\sigma_0} & g_{22}^{\sigma_0}\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} = f_0\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} ,\\ &\begin{pmatrix}1 & b_i\\ 0 & 1\end{pmatrix}\begin{pmatrix}g_{11}^{\sigma_i} & g_{12}^{\sigma_i}\\ g_{21}^{\sigma_i} & g_{22}^{\sigma_i}\end{pmatrix}\begin{pmatrix}1 & 0\\ b_i & 1\end{pmatrix} = f_i\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} ,\\ &\begin{pmatrix}g_{11}^{\sigma_3} & g_{12}^{\sigma_3}\\ g_{21}^{\sigma_3} & g_{22}^{\sigma_3}\end{pmatrix} = f_3\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} . \end{aligned} \tag{8.4$'$}$$

From these and (8.5) it follows that

$$\alpha^2 = f_0^2 , \qquad 1 = f_i^2 , \quad (i = 1, 2) , \qquad 1 = f_3^2 . \tag{8.6}$$

We see now easily that $(g_{22}\, dz \wedge dw)^2$ is invariant by G and hence is a section of K^2 on M. On the other hand, $H^0(M; K^2) = 0$ by Inoue [6]. Hence, $g_{22} = 0$. Similarly, the function $(g_{12})^2$ is invariant by G and hence is constant on M. From (8.4)′ it then follows that

$$\alpha = f_0 , \qquad 1 = f_i , \quad (i = 1, 2) , \qquad f_3 = 1 .$$

From (8.4)′ we obtain

$$\alpha g_{11}^{\sigma_0} = g_{11} , \qquad g_{11}^{\sigma_i} + 2b_i g_{12} = g_{11} , \qquad g_{11}^{\sigma_3} = g_{11} . \tag{8.7}$$

Differentiating (8.7) with respect to w, we obtain

$$\partial g_{11}/\partial w = \alpha\, \partial g_{11}^{\sigma_i}/\partial w , \qquad \partial g_{11}/\partial w = \partial g_{11}^{\sigma_i}/\partial w , \qquad \partial g_{11}/\partial w = \partial g_{11}^{\sigma_i}/\partial w .$$

Hence, $(\partial g_{11}/\partial w)(\partial/\partial z \wedge \partial/\partial w)$ is a globally defined holomorphic section of K^{-1}. But, according to Inoue [6], K^{-1} has no holomorphic sections. Hence, $\partial g_{11}/\partial w = 0$, i.e., g_{11} is a function of z only. Now differentiating (8.7) with respect to z, we obtain

$$\partial g_{11}/\partial z = \alpha^2\, \partial g_{11}^{\sigma_0}/\partial z , \qquad \partial g_{11}/\partial z = \partial g_{11}^{\sigma_i}/\partial z , \qquad \partial g_{11}/\partial z = \partial g_{11}^{\sigma_3}/\partial z .$$

It follows that $(\partial g_{11}/\partial z)(\partial/\partial z \wedge \partial/\partial w)^2$ is a globally defined holomorphic section of K^{-2} and hence $\partial g_{11}/\partial z = 0$. We have shown that g_{11} is constant. In particular, $g_{11}^{\sigma_0} = g_{11}$. From (8.7) and $\alpha > 1$, we obtain $g_{11} = 0$. Since $b_i \neq 0$ for $i = 1$ or 2, (8.7) implies $g_{12} = 0$. This is a contradiction.

(c) *Surfaces* $S^{(-)}_{N,p,q,r}$. Let $N = (n_{jk}) \in GL(2; Z)$ be a matrix with $\det N = -1$ having real eigenvalues α, $-1/\alpha$ such that $\alpha > 1$. Choose real eigenvectors (a_1, a_2), (b_1, b_2) of N corresponding to α and $-1/\alpha$, respectively, and we fix integers p, q, r $(r \neq 0)$. Define (c_1, c_2) to be the solution of

$$-(c_1, c_2) = (c_1, c_2)\,{}^tN + (e_1, e_2)(1/r)(b_1a_2 - b_2a_1)(p, q)\ ,$$

where

$$e_i = (1/2)n_{i1}(n_{i1} - 1)a_1b_1 + (1/2)n_{i2}(n_{i2} - 1)a_2b_2 + n_{i1}n_{i2}b_1a_2\ .$$

Let $G = G^{(-)}_{N,p,q,r}$ be the group of holomorphic transformations of $H \times C$ generated by

$$\sigma_0\colon (z, w) \mapsto (\alpha z, -w)\ ,$$
$$\sigma_i\colon (z, w) \mapsto (z + a_i, w + b_iz + c_i)\ ,$$
$$\sigma_3\colon (z, w) \mapsto (z, w + (1/r)((b_1a_2 - b_2a_1))\ .$$

Define $M = S^{(-)}_{N,p,q,r} = (H \times C)/G$. Since $S^{(-)}_{N,p,q,r}$ has $S^{(+)}_{N^2,p_1,q_1,r;0}$ with suitable p_1, q_1 as its unramified double covering [6] and since the latter has no $CO(2; C)$-structures, it follows that the former admits no holomorphic $CO(2; C)$-structures.

9. Ruled surfaces. Since we are interested only in surfaces free from exceptional curves of the first kind, by a ruled surface of genus g, we mean a holomorphic fibre bundle over a non-singular algebraic curve Δ of genus g with fibre P_1C and structure group $PGL(1; C)$. Then

$$\text{(9.1)} \qquad q = g\ , \qquad p_g = 0\ , \qquad c_2 = 4(1-g)\ , \qquad c_1^2 = 8(1-g)\ .$$

LEMMA (9.2). *Let M be a ruled surface over a curve Δ of genus g. If $TM = L' \oplus L''$ is a splitting such that L' is in the fibre direction, then M comes from a representation ρ of $\pi_1(\Delta)$ into $PGL(1; C)$, i.e.,*

$$M = \tilde{\Delta} \times_\rho P_1C\ ,$$

where $\tilde{\Delta}$ is the universal covering space of Δ, and L'' is the horizontal subspace of the natural flat connection in the bundle M.

PROOF. Consider L'' as the horizontal subspace for a generalized connection in the bundle M; since L'' is transversal to fibres everywhere, we can define the notion of parallel displacement of a fibre along a curve on the base Δ. Since L'' is an integrable distribution, the parallel displacement depends only on the homotopy class of the curve and maps the initial fibre holomorphically onto the terminal fibre. Hence, we obtain the holonomy representation $\rho\colon \pi_1(\Delta) \to PGL(1; C)$. The remainder of the proof is obvious. q.e.d.

LEMMA (9.3). *Let D be a small disk in C and $p\colon D \times P_1C \to D$ be the canonical projection. Then for every splitting $TN = L' \oplus L''$, either L' or L'' is in the fibre direction of p, where we set $N = D \times P_1C$.*

PROOF. Let z be the natural coordinate in D so that $\alpha = dz$ is a holomorphic 1-form on N. For each tangent vector V of N, write $V = V' + V''$, where $V' \in L'$ and $V'' \in L''$. Define a new holomorphic 1-form α' on N by setting $\alpha'(V) = \alpha(V')$. Assume neither L' nor L'' is vertical at some point $w \in N$. Let V be a nonzero vertical vector at w. Then $\alpha'(V) = \alpha(V') = dz(p_* V') \neq 0$ since $p_* V'$ is nonzero. Hence the restriction of α' onto the fibre $p^{-1}(p(z)) = P_1 C$ is a nonzero holomorphic 1-form. This is a contradiction. q.e.d.

The ruled surfaces of genus 0 can be classified as follows. Let H and 1 denote, respectively, the hyperplane line bundle and the trivial line bundle over $P_1 C$. For each nonnegative integer n, let $F_n = P(H^n \oplus 1)$ be the ruled surface associated to the vector bundle $H^n \oplus 1$ of rank 2.

LEMMA (9.4). *$F_0 = P_1 C \times P_1 C$ is the only ruled surface of genus 0 admitting a holomorphic $CO(2; C)$-structure.*

PROOF. We represent a point of F_n by a pair (u_0, u_1), where $u_0 \in H^n$ and $u_1 \in 1$. The bundle F_n has two natural sections s_0 and s_∞ given by

$$s_0 = \{u_1 = 0\} \qquad \text{and} \qquad s_\infty = \{u_0 = 0\} .$$

Let the group $C^* = C - \{0\}$ act on F_n by $\lambda: (u_0, u_1) \mapsto (\lambda u_0, u_1)$ for $\lambda \in C^*$. Let V be the holomorphic vertical vector field induced by this action of C^*. Since C^* leaves the section s_∞ fixed, V vanishes at s_∞.

Let $TF_n = L' \oplus L''$ be a splitting. (Remark F_n is simply connected.) Assume that neither L' nor L'' is in the fibre direction at some point of F_n. Decompose $V = V' + V''$, where $V' \in L'$ and $V'' \in L''$. Since V vanishes at s_∞, so do V' and V''. On the other hand, as we have seen in the proof of (6.11), every holomorphic vector field on F_n projects to a holomorphic vector field on the base space. In particular, V' and V'' project to holomorphic vector fields on the base space. Since they vanish at the section s_∞, their projections must be zero. In other words, V' and V'' are vertical vector fields. This is a contradiction. Hence, either L' or L'' is vertical. Now our assertion follows from (9.2). q.e.d.

THEOREM (9.5). *A ruled surface M over a curve Δ of genus $g \geqq 1$ admits a holomorphic $CO(2; C)$-structure if and only if $M = \tilde{\Delta} \times_\rho P_1 C$, where $\tilde{\Delta}$ is the universal covering space of Δ and $\rho: \pi_1(\Delta) \to PGL(1; C)$ is a representation, and the $CO(2; C)$-structure is the natural one arising from the natural quadric structure on $\tilde{\Delta} \times P_1 C$. The quadric $P_1 C \times P_1 C$ is the only ruled surface of genus 0 admitting a holomorphic $CO(2; C)$-structure.*

PROOF. Let $p: M \to \Delta$ be the fibration. Take a sufficiently fine covering $\Delta = \bigcup U_\alpha$ by small disks U_α so that $p^{-1}(U_\alpha) = D_\alpha \times P_1C$. By restricting the $CO(2; C)$-structure onto $P^{-1}(U_\alpha)$, we have the splitting $T(M) \mid P^{-1}(U_\alpha) = L' \oplus L''$. From Lemma (9.3) we may assume L' is in the fibre direction. From this we see the $CO(2; C)$-structure on M gives rise to the splitting $TM = L' \oplus L''$. Then our assertion follows from Lemma (9.2) and Lemma (9.4). q.e.d.

10. Surfaces with holomorphic $CO(2; C)$-structures and quadric structures. Let M be an algebraic surface and Φ_{mK} the pluri-canonical map associated with the pluri-canonical system $|mK|$; it is a rational map of M into P_NC, where $N = \dim |mK|$. The Kodaira dimension $\kappa(M)$ of M is the maximum dimension of the image $\Phi_{mK}(M)$ for $m \geqq 1$. If $|mK| = \varnothing$, we set $\dim \Phi_{mK}(M) = -\infty$. Then the classification theorem of Enriques may be stated as follows:

THEOREM (10.1). (1) *A minimal algebraic surface M with $\kappa(M) = -\infty$ is either the projective plane P_2C or a ruled surface;*

(2) *A minimal algebraic surface M with $\kappa(M) = 0$ satisfies $4K = 0$ or $6K = 0$, and it is either a* K3 *surface (if $q = 0$ and $p_g = 1$), an Enriques surface (if $q = 0$ and $p_g = 0$), a bielliptic (or hyperelliptic) surface (if $q = 1$), or an Abelian surface (if $q = 2$);*

(3) *A minimal algebraic surface M with $\kappa(M) = 1$ satisfies $c_1^2 = 0$ and is elliptic.*

If $\kappa(M) = 2$, then M is called a surface of general type.

By (3.21), the projective plane P_2C admits no holomorphic $CO(2; C)$-structures. From (9.5) we conclude:

THEOREM (10.2). *An algebraic surface M with $\kappa(M) = -\infty$ admits a holomorphic $CO(2; C)$-structure if and only if it is one of the following:*

(1) *A ruled surface over a curve Δ of genus $\geqq 1$ such that $M = \tilde{\Delta} \times_\rho P_1C$, where $\tilde{\Delta}$ is the universal covering space of Δ and $\rho: \pi_1(\Delta) \to PGL(1; C)$ is a representation. (In this case, the $CO(2; C)$-structure is the natural one coming from the natural quadric structure on $\tilde{\Delta} \times P_1C$).*

(2) *The quadric $P_1C \times P_1C$.*

THEOREM (10.3). *An algebraic surface M with $\kappa(M) = 0$ admits a holomorphic $CO(2; C)$-structure if and only if it is one of the following:*

(1) *A bielliptic (or hyperelliptic) surface.*

(2) *An Abelian surface.*

In both cases, it admits a quadric structure.

PROOF. In this case, $c_1 = 0$ in $H^2(M; \boldsymbol{R})$. By (3.21) a necessary condition for the existence of a holomorphic $CO(2; \boldsymbol{C})$-structure is $c_2 = 0$. This eliminates the $K3$ surfaces and the Enriques surfaces (which are doubly covered by $K3$ surfaces).

A complex torus $\boldsymbol{C}^2/\Gamma$ admits a quadric structure coming from the natural quadric structure on $\boldsymbol{C}^2$ invariant under the translation.

It is known (see, for example, [19]) that a bielliptic surface can be expressed as the quotient of an Abelian surface A by the group generated by an automorphism g of A of the following form: $g(z^1, z^2) = (z^1 + 1/m, \zeta z^2)$, where ζ is an m-th root of 1 and $m = 2, 3, 4$, or 6. It is clear that the natural quadric structure on A induces a quadric structure on the quotient bielliptic surface. q.e.d.

THEOREM (10.4). *An algebraic surface M with $\kappa(M) = 1$ admits a holomorphic $CO(2; \boldsymbol{C})$-structure if and only if $c_1^2 = 0$ (which is equivalent to minimality for an elliptic surface) and $c_2 = 0$. In this case, it admits a quadric structure.*

PROOF. The first part follows from (3.21), (5.11) and (6.8). The second half follows from (6.15). q.e.d.

THEOREM (10.5). *An algebraic surface M of general type admits a holomorphic $CO(2; \boldsymbol{C})$-structure if and only if its universal covering space is biholomorphic to the bidisk $D \times D$. In this case, it admits a quadric structure.*

PROOF. According to Kodaira [16], an algebraic surface of general type M has an ample canonical bundle if and only if it contains no non-singular rational curve C with self-intersection $C \cdot C = -1$ or -2. Our assertion now follows from (4.5) and (5.14). q.e.d.

Kodaira [15] classified the compact complex surfaces without exceptional curves of the first kind into seven classes I_0 to VII_0. We shall now examine his classification table to determine the surfaces which admit holomorphic $CO(2; \boldsymbol{C})$-structures and quadric structures.

Class I_0. This is the class of minimal algebraic surfaces with $p_g = 0$. The algebraic case was dealt with in (10.2)-(10.5).

Class II_0. This is the class $K3$ surfaces. Since $c_1^2 = 0$ and $c_2 = 24$ for a $K3$ surface, there is no holomorphic $CO(2; \boldsymbol{C})$-structure on a $K3$ surface by (3.21).

Class III_0. This is the class of complex tori. Clearly, every complex torus admits a natural quadric structure.

Class IV_0. This is the class of minimal elliptic surfaces with even

Betti number, $p_g > 0$ and $c_1^2 = 0$ (but $c_1 \neq 0$ in $H^2(M; \boldsymbol{Z})$). By (6.10), a surface in this class admits a holomorphic $CO(2; \boldsymbol{C})$-structure. By (6.15) it actually admits a quadric structure.

Class V_0. This is the class of minimal algebraic surfaces with $p_g > 0$ and $c_1^2 > 0$. The algebraic case was dealt with in (10.2)-(10.5).

Class VI_0. This is the class of minimal elliptic surfaces with odd first Betti number, $p_g > 0$ and $c_1^2 = 0$. By (6.13) an elliptic surface with odd first Betti number, fibred over a curve of positive genus, admits no holomorphic $CO(2; \boldsymbol{C})$-structures. By (6.14), an elliptic surface over $P_1\boldsymbol{C}$ with odd first Betti number cannot admit a holomorphic $CO(2; \boldsymbol{C})$-structure unless it is a Hopf surface (which is in Class VII_0). Hence, no surface of Class VI_0 admits a holomorphic $CO(2; \boldsymbol{C})$-structure.

Class VII_0. This is the class of minimal surfaces with $p_g = 0$ and $b_1 = 1$. In § 6, § 7 and § 8, we have shown that a surface of Class VII_0 admitting a holomorphric $CO(2; \boldsymbol{C})$-structure is either an Inoue surface S_U in the notation of § 8 or a Hopf surface $(\boldsymbol{C}^2 - \{0\})/\Gamma$, where Γ contains only elements of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

and that such a surface actually admits a quadric structure.

BIBLIOGRAPHY

[1] T. AUBIN, Equations du type Monge-Ampère sur les variétés Kähleriénnes compactes, C.R. Acad. Sci. Paris 283 (1976), 119-121.

[2] A. BEAUVILLE, Surfaces algébriques complexes, Astériques 54 (1978).

[3] R. BOTT, A residue formula for holomorphic vector fields, J. Differential Geometry 1, (1967), 311-330.

[4] R. GUNNING, On uniformization of complex manifolds; the role of connections, Math. Notes No. 22, Princeton University Press 1978.

[5] F. HIRZEBRUCH, Automorphe Formen und der Satz von Riemann-Roch, Symp. Intern. Top. Alg. 1956, 129-144, Univ. de Mexico, 1958.

[6] M. INOUE, On surfaces of Class VII_0, Inventiones Math. 24 (1974), 269-310.

[7] M. INOUE, S. KOBAYASHI AND T. OCHIAI, Holomorphic affine connections on compact complex surfaces, J. Fac. Sci. Univ. Tokyo 27 (1980), 247-264.

[8] S. KOBAYASHI, On compact Kähler manifolds with positive definite Ricci tensor, Ann. Math. 74 (1961), 570-574.

[9] S. KOBAYASHI, The first Chern class and holomorphic symmetric tensor fields, J. Math. Soc. Japan, 32 (1980), 325-329.

[10] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, Vol. II, Intersci. Tracts No. 15, New York, Wiley, 1969.

[11] S. KOBAYASHI AND T. OCHIAI, Characterizations of complex projective spaces and hyperquadrics, J. Math. Kyoto Univ. 13 (1972), 31-47.

[12] S. KOBAYASHI AND T. OCHIAI, Holomorphic projective structures on compact complex

surfaces, Math. Ann. 249 (1980), 75–94.

[13] S. KOBAYASHI AND T. OCHIAI, Holomorphic projective structures on compact complex surfaces II, Math. Ann. 255 (1981), 519–522.

[14] K. KODAIRA, On compact complex analytic surfaces I, Ann. Math. 71 (1960), 111–152; II 77 (1963), 563–626; III 78 (1963), 1–40.

[15] K. KODAIRA, On the structure of compact complex analytic surfaces I, Amer. J. Math. 86 (1964), 751–798; II 88 (1966), 682–721; III 90 (1968), 55–83.

[16] K. KODAIRA, Pluricanonical systems on algebraic surfaces of general type, J. Math. Soc. Japan 20 (1968), 170–192.

[17] A. LASCOUX AND M. BERGER, Variétés Kähleriennes compactes, Lecture Notes in Math. No. 154, Springer-Verlag, 1970.

[18] K. MAEHARA, On elliptic surfaces whose first Betti numbers are odd, Proc. Intl. Symp. Algebraic Geometry, Kyoto 1977, 565–574.

[19] T. SUWA, On hyperelliptic surfaces, J. Fac. Sci. Univ. Tokyo, Sec. 1A, Math. 16 (1970), 469–476.

[20] S. T. YAU, On the Ricci curvature of a compact Kaehler manifolds and the complex Monge-Ampère equation I, Comm. Pure Appl. Math. 31 (1978), 339–411.

[21] Y. MIYAOKA, Kähler metrics on elliptic surfaces, Proc. Japan Acad. 50 (1974), 533–536.

[22] S. BUNGAARD AND J. NIELSEN, On normal subgroups with finite index in F-groups, Mat. Tidsskr. B (1951), 56–58.

[23] W. FENCHEL, Remarks on finite groups of mapping classes, Mat. Tidsskr. B (1950), 90–95.

[24] R. FOX, On Fenchel's conjecture about F-groups, Mat. Tidsskr. B (1952), 61–65.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720
U.S.A.

AND

INSTITUTE OF MATHEMATICS
UNIVERSITY OF TOKYO
TOKYO 113
JAPAN