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# DEFINING IDEALS OF THE CLOSURES OF THE CONJUGACY CLASSES AND REPRESENTATIONS OF THE WEYL GROUPS

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**1. Introduction.** Let $G$ be a connected reductive algebraic group over the complex number field $\boldsymbol{C}$ and $T$ be its maximal torus. We denote the Lie algebras of $G$ and $T$ by $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Let $O_x$ be the $G$-orbit containing $x \in \mathfrak{g}$ under the adjoint action of $G$ on $\mathfrak{g}$. Then the Weyl group $W$ of $(G, T)$ naturally acts on the coordinate ring $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_x]$ of the scheme-theoretic intersection of $\mathfrak{t}$ and the Zariski closure $\bar{O}_x$ of $O_x$. We consider the following problem due to Kostant, Kraft, DeConcini and Procesi. (See [1] and [5].)

PROBLEM. *Describe* $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_x]$ *as a $W$-module for each nilpotent orbit* $O_x$ *in* $\mathfrak{g}$.

When $x$ is regular nilpotent, $\bar{O}_x$ is just the variety $N$ consisting of all the nilpotent elements in $\mathfrak{g}$, and $\boldsymbol{C}[\mathfrak{t} \cap N]$ is isomorphic to the regular representation of $W$ (Cf. Kostant [4].).

DeConcini and Procesi [1] have shown that for $G = GL(n, \boldsymbol{C})$, $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_x]$ is isomorphic to the representation induced from the trivial representation of a certain subgroup of parabolic type. They also naturally identified $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_x]$ with a certain representation of $W$ constructed by Springer [11], [12] (Cf. §2 and §3 below for precise statements.). In [1] they conjectured that certain explicitly constructed polynomials form a generator system of the defining ideal of the variety $\bar{O}_x$ and proved the above results using these polynomials.

In this note we first give another candidate for a generator system of the defining ideal of $\bar{O}_x$ and show that the proof of the results in [1] can be a little simplified by replacing their polynomials by ours (§2, §3). Though some of the statements and the arguments in §2 and §3 are similar to those in [1], we include them for convenience of the readers.

For a general reductive group $G$ the structure of $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_x]$ is not yet clear. We secondly show that for a nilpotent orbit of a certain type

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in $\mathfrak{g} = \mathfrak{sp}(2n, \boldsymbol{C})$ (the Lie algebra of $Sp(2n, \boldsymbol{C})$), $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_x]$ is also isomorphic to the representation induced from the trivial representation of a subgroup of parabolic type (§4).

The first version of this paper contained the explicit descriptions of $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_x]$ for $C_2$, $C_3$ and $G_2$ except for one nilpotent orbit in the case of $G_2$. We omit them because for $C_2$ and $C_3$ they are already contained in Kraft [5] and our result is incomlete for $G_2$.

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**2. Structure of $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_x]$ in the case of $GL(n, \boldsymbol{C})$.** In §2 and §3 we consider the case $G = GL(n, \boldsymbol{C})$ and $\mathfrak{g} = M(n, \boldsymbol{C})$. Then the set of nilpotent orbits in $\mathfrak{g}$ is parametrized by the set of partitions of $n$. For a partition $\sigma = (b_0 \geqq b_1 \geqq b_2 \geqq \cdots)$ of $n$ we denote by $O_\sigma$ the nilpotent orbit consisting of the nilpotent matrices so that the sizes of their Jordan blocks are given by the $b_i$'s. We set $p_\sigma(s) = b_{n-s} + b_{n-s+1} + \cdots$ for $s = 1, \cdots, n$. For $x \in M(n, \boldsymbol{C})$ and $s = 1, \cdots, n$ let $d_s^x(t)$ be the greatest common divisor of all the $s$-minors of the matrix $(tI - x) \in M(n, \boldsymbol{C}[t])$.

LEMMA 1. (i) *$x \in O_\sigma$ if and only if $d_s^x(t) = t^{p_\sigma(s)}$ for $s = 1, \cdots, n$.*
(ii) *$x \in \bar{O}_\sigma$ if and only if $t^{p_\sigma(s)} \mid d_s^x(t)$ for $s = 1, \cdots, n$.*

PROOF. (i) follows from the theory of elementary divisors. It is well known that for two partitions $\sigma = (b_0 \geqq b_1 \geqq \cdots)$ and $\tau = (b_0' \geqq b_1' \geqq \cdots)$ of $n$ we have $\bar{O}_\sigma \supset O_\tau$ if and only if $b_0 \geqq b_0'$, $b_0 + b_1 \geqq b_0' + b_1'$, $\cdots$. Thus (ii) follows from (i). q.e.d.

We define a family of polynomials $\{g_i^\sigma\}$ in the variables $x_{ij}$ $(1 \leqq i, j \leqq n)$ to be the set of the coefficients of $t^m$ in $s$-minors of $(tI - (x_{ij}))$ with $s = 1, \cdots, n$ and $m \leqq p_\sigma(s) - 1$.

COROLLARY. *$x \in \bar{O}_\sigma$ if and only if $g_i^\sigma(x) = 0$ for all $i$.*

Let $T$ be a maximal torus of $G$ consisting of diagonal matrices which belong to $G$. Then its Lie algebra $\mathfrak{t}$ is given by

$$\mathfrak{t} = \left\{ \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix} \middle| \, x_i \in \boldsymbol{C} \right\}.$$

We define the dual partition $\check{\sigma} = (c_0 \geqq c_1 \geqq \cdots)$ of $\sigma = (b_0 \geqq b_1 \geqq \cdots)$ by $c_i = \#\{j \mid b_j \geqq i + 1\}$. Let $W_\sigma^\vee$ be the subgroup of the Weyl group $W = S_n$ defined by $W_\sigma^\vee = S_{c_0} \times S_{c_1} \times \cdots \subset S_n$. We prove the following theorem using $\{g_i^\sigma\}$.

THEOREM 1 (DeConcini-Proceci [1]). *$C[\mathfrak{t} \cap \bar{O}_\sigma]$ is isomorphic to $\mathrm{Ind}_{W_{\check{\sigma}}}^{W}(1_{W_{\check{\sigma}}})$ as a W-module.*

Let $P_{\check{\sigma}}$ be the parabolic subgroup of $G$ given by

$$P_{\check{\sigma}} = \left\{ \begin{bmatrix} A_0 & & * \\ & A_1 & \\ 0 & & \ddots \end{bmatrix} \middle| A_i \in GL(c_i, C) \right\}.$$

The Richardson orbit corresponding to $P_{\check{\sigma}}$ is $O_\sigma$ and the subgroup of $W$ corresponding to $P_{\check{\sigma}}$ is $W_{\check{\sigma}}$. Since $G^x$ is connected for $x \in O_\sigma$ and since $\bar{O}_\sigma$ is normal by Kraft-Procesi [6], $C[\mathfrak{t} \cap \bar{O}_\sigma]$ contains $\mathrm{Ind}_{W_{\check{\sigma}}}^{W}(1_{W_{\check{\sigma}}})$ by Kraft [5; Proposition 4].

Set $\bar{A}_\sigma^n = C[x_{ij}]/((g_i^\sigma) + (x_{ij} \mid i \neq j))$. Then we have only to prove the following (♯).

$$(\sharp) \qquad \dim \bar{A}_\sigma^n \leqq \binom{n}{\check{\sigma}} := n!/(c_0!\, c_1! \cdots).$$

We set $A^n = C[x_1, \cdots, x_n] = C[\mathfrak{t}]$. Let $\tilde{g}_i^\sigma \in A^n$ be the polynomial obtained by specializing $g_i^\sigma$ by $x_{rr} \mapsto x_r$ and $x_{rs} \mapsto 0$ $(r \neq s)$. If we put $K_\sigma = (\tilde{g}_i^\sigma)$, then $\bar{A}_\sigma^n = A^n/K_\sigma$ and $K_\sigma$ is generated by the coefficients of $t^m$ in $(t - x_{i_1}) \cdots (t - x_{i_s}) \in A^n[t]$ with $s$, $m$ and $i_1, \cdots, i_s$ running through the integers satisfying $1 \leqq s \leqq n$, $0 \leqq m \leqq p_\sigma(s) - 1$ and $1 \leqq i_1 < \cdots < i_s \leqq n$. In other words $K_\sigma$ is generated by the elementary symmetric functions in variables $x_{i_1}, \cdots, x_{i_s}$ with degree $\geqq s + 1 - p_\sigma(s)$, where $s$ and $i_1, \cdots, i_s$ are the integers satisfying $1 \leqq s \leqq n$ and $1 \leqq i_1 < \cdots < i_s \leqq n$.

We prove (♯) by induction on $n$. As the case $n = 1$ is trivial, we assume that $n \geqq 2$ and (♯) holds for $n - 1$ in the following.

DEFINITION. For a partition $\sigma = (b_0 \geqq b_1 \geqq \cdots)$ of $n$ with $b_0 > i \geqq 0$, we define a partition $\sigma_i = (b'_0 \geqq b'_1 \geqq \cdots)$ of $n - 1$ as follows. If we set $t_0 = \max\{t \geqq 0 \mid b_t > i\}$, then $b'_{t_0} = b_{t_0} - 1$ and $b'_j = b_j$ $(j \neq t_0)$.

Let $\Phi\colon A^n \to A^{n-1}$ be the algebra homomorphism defined by $\Phi(x_j) = x_j$ $(j \neq n)$ and $\Phi(x_n) = 0$.

LEMMA 2. $\Phi(K_\sigma) \subset K_{\sigma_i}$.

PROOF. We first remark that $p_{\sigma_i}(s)$ is given by

$$p_{\sigma_i}(s) = \begin{cases} p_\sigma(s+1) & \text{if } c_i \leqq n - 1 - s \\ p_\sigma(s+1) - 1 & \text{if } c_i > n - 1 - s\,. \end{cases}$$

Set $(t - x_{i_1}) \cdots (t - x_{i_s}) = t^s + a_{s-1}t^{s-1} + \cdots + a_0$ for $1 \leqq i_1 < \cdots < i_s \leqq n$.

(i) In the case $i_s < n$ we have $t^s + \Phi(a_{s-1})t^{s-1} + \cdots + \Phi(a_0) = (t - x_{i_1}) \cdots (t - x_{i_s})$. If $m \leqq p_\sigma(s) - 1$, then $m \leqq p_\sigma(s) - 1 = p_{\sigma_i}(s-1) - 1 \leqq p_{\sigma_i}(s) - 1$ for $c_i \leqq n - s$, and $m \leqq p_\sigma(s) - 1 = p_{\sigma_i}(s-1) = p_{\sigma_i}(s) - b'_{n-1-s} \leqq p_{\sigma_i}(s) - i - 1 \leqq p_{\sigma_i}(s) - 1$ for $c_i > n - s$.

(ii) In the case $i_s = n$ we have $\Phi(a_0) = 0$ and $t^{s-1} + \Phi(a_{s-1})t^{s-2} + \cdots + \Phi(a_1) = (t - x_{i_1}) \cdots (t - x_{i_{s-1}})$. If $m \leqq p_\sigma(s) - 1$, then $m - 1 \leqq p_\sigma(s) - 2 \leqq p_{\sigma_i}(s-1) - 1$. q.e.d.

Thus $\Phi$ induces a surjective homomorphism $\Phi_i \colon \bar{A}^n_\sigma \to \bar{A}^{n-1}_{\sigma_i}$.

LEMMA 3. *$(\operatorname{Ker} \Phi_i) \cdot x^i_n \subset (x^{i+1}_n)$ in $\bar{A}^n_\sigma$.*

PROOF. It is easy to see that $\operatorname{Ker} \Phi_i$ is generated by $x_n$ and the coefficients of $t^m$ in $(t - x_{i_1}) \cdots (t - x_{i_s})$ with $s$, $m$ and $i_1, \cdots, i_s$ running through the integers satisfying $1 \leqq s \leqq n$, $0 \leqq m \leqq p_{\sigma_i}(s) - 1$ and $1 \leqq i_1 < \cdots < i_s \leqq n - 1$. Set $(t - x_{i_1}) \cdots (t - x_{i_s}) = t^s + a_{s-1}t^{s-1} + \cdots + a_0 \in \bar{A}^n_\sigma[t]$ for $1 \leqq i_1 < \cdots < i_s \leqq n - 1$. Then it is sufficient to prove that $a_m x^i_n \in (x^{i+1}_n)$ in $\bar{A}^n_\sigma$ for $m \leqq p_{\sigma_i}(s) - 1$. Since the coefficient of $t^m$ in $(t - x_{i_1}) \cdots (t - x_{i_s})(t - x_n)$ vanishes for $m \leqq p_\sigma(s+1) - 1$, we see that $-a_0 x_n = 0$, $a_0 - a_1 x_n = 0, \cdots, a_{p_\sigma(s+1)-2} - a_{p_\sigma(s+1)-1} x_n = 0$. Thus $a_m x^i_n = a_{m+1} x^{i+1}_n$ for $m \leqq p_\sigma(s+1) - 2$. We may thus assume that $s \leqq n - c_i - 1$ and $m = p_\sigma(s+1) - 1 = p_{\sigma_i}(s) - 1$. Since $p_\sigma(s+1) - p_\sigma(s) \leqq i$ and $a_{p_\sigma(s)-1} = 0$ in $\bar{A}^n_\sigma$, we have $a_{p_\sigma(s+1)-1} x^i_n = a_{p_\sigma(s+1)-2} x^{i-1}_n = \cdots = a_{p_\sigma(s)-1} x^{i-(p_\sigma(s+1)-p_\sigma(s))}_n = 0$ and we are done. q.e.d.

PROOF OF THEOREM 1. Let $J_i$ be the principal ideal of $\bar{A}^n_\sigma$ generated by $x^i_n$. Then since $J_i/J_{i+1}$ is a cyclic $\bar{A}^{n-1}_{\sigma_i}$-module by Lemma 3, we have

$$\dim (J_i/J_{i+1}) \leqq \dim \bar{A}^{n-1}_{\sigma_i} \leqq \binom{n-1}{\sigma_i} .$$

Thus

$$\dim \bar{A}^n_\sigma = \sum_{i \geqq 0} \dim (J_i/J_{i+1}) \leqq \sum_{i \geqq 0} \binom{n-1}{\sigma_i} = \binom{n}{\sigma} .$$

This proves (♯) for $n$, and so the proof of Theorem 1 is complete.

q.e.d.

**3. Relations with Springer's representation.** We first review the cohomology algebra of the flag variety. Set $G = GL(n, \boldsymbol{C}) = GL(V)$ ($V = \boldsymbol{C}^n$). We denote the projective variety consisting of all the complete flags of $V$ by $\mathscr{F}$, that is,

$$\mathscr{F} = \{(0 = V_0 \subset V_1 \subset \cdots \subset V_n = V) \mid \dim V_i = i \quad \text{for all } i\} .$$

Then the cohomology algebra $H^*(\mathscr{F}) = H^*(\mathscr{F}, \boldsymbol{C})$ can be described as follows. Let $\tilde{V}_j$ be the subbundle of the trivial vector bundle $\mathscr{F} \times V$ over $\mathscr{F}$ whose fiber at $(V_i) \in \mathscr{F}$ is just $V_j$. We denote the first Chern class of the line bundle $\tilde{V}_i/\tilde{V}_{i-1}$ by $\bar{x}_i \in H^2(\mathscr{F})$.

PROPOSITION 1 (cf. Kleiman [3]). (i) *$H^*(\mathscr{F})$ is generated by $\bar{x}_1, \cdots, \bar{x}_n$ as an algebra.*

(ii) *Define the algebra homomorphism $\pi$ from the polynomial ring $\boldsymbol{C}[\mathrm{t}] = \boldsymbol{C}[x_1, \cdots, x_n]$ onto $H^*(\mathscr{F})$ by $\pi(x_i) = \bar{x}_i$. Then* $\operatorname{Ker}\pi$ *is generated as an ideal by the elementary symmetric functions $f_1, \cdots, f_n$.*

Thus we obtain an algebra isomorphism

$$\bar{\pi}\colon \boldsymbol{C}[\mathrm{t} \cap N] = \boldsymbol{C}[x_1, \cdots, x_n]/(f_1, \cdots, f_n) \to H^*(\mathscr{F}) .$$

On the other hand the Weyl group $W = S_n$ acts on $\mathscr{F}$ as follows. For any $(V_i) \in \mathscr{F}$, there exists $g \in U(n)$ so that $V_i = \bigoplus_{j=1}^{i} \boldsymbol{C}g(e_j)$, where $\{e_1, \cdots\cdot, e_n\}$ is the canonical basis of $V = \boldsymbol{C}^n$. Then the action of $w \in W = S_n$ on $\mathscr{F}$ can be defined by

$$(V_i)\cdot w = (V_i') \quad \text{with} \quad V_i' = \bigoplus_{j=1}^{i} \boldsymbol{C}g(e_{w^{-1}(j)}) .$$

Thus $W$ acts on $H^*(\mathscr{F})$. Then the algebra isomorphism $\bar{\pi}$ is also an isomorphism as $W$-modules.

Now for a partition $\eta$ of $n$ we fix an element $x_0 \in O_\eta$ and define a subvariety $\mathscr{F}_\eta$ of $\mathscr{F}$ by

$$\mathscr{F}_\eta = \{(V_i) \in \mathscr{F} \mid x_0(V_i) \subset V_{i-1} \quad \text{for all} \quad i\} .$$

Springer [11], [12] defined a $W$-module structure on the cohomology algebra $H^*(\mathscr{F}_\eta)$. Furthermore for $\eta_0 = (1 \geqq 1 \geqq \cdots)$ the $W$-module structure on $H^*(\mathscr{F}) = H^*(\mathscr{F}_{\eta_0})$ defined by Springer coincides with the ordinary one described above. (We are considering here the $W$-module structure obtained by tensoring the one-dimensional sign representation of $W$ with the original one defined in [11], [12].) The natural algebra homomorphism $\rho_\eta\colon H^*(\mathscr{F}) \to H^*(\mathscr{F}_\eta)$ induced by the inclusion $\mathscr{F}_\eta \hookrightarrow \mathscr{F}$ is known to be a homomorphism as $W$-modules (Cf. Hotta-Springer [2].).

THEOREM 2. (DeConcini-Procesi [1]). *There exists a unique isomorphism $j_\eta$ as algebras and $W$-modules which makes the following diagram commutative;*

$$\begin{array}{ccc} \boldsymbol{C}[\mathrm{t} \cap N] & \xrightarrow{\bar{\pi}} & H^*(\mathscr{F}) \\ {\scriptstyle p_\eta^\vee}\big\downarrow & & \big\downarrow{\scriptstyle \rho_\eta} \\ \boldsymbol{C}[\mathrm{t} \cap \bar{O}_\eta^\vee] & \xrightarrow[j_\eta]{} & H^*(\mathscr{F}_\eta) . \end{array}$$

*Here $p_{\check{\eta}}$ is the natural algebra homomorphism.*

From the cellular decomposition of $\mathscr{F}_\eta$ given by Spaltenstein [9] (cf. also Hotta-Springer [2]), we have $\dim H^*(\mathscr{F}_\eta) = \binom{n}{\eta}$. Thus $\dim \boldsymbol{C}[t \cap \bar{O}_{\check{\eta}}] = \dim H^*(\mathscr{F}_\eta)$ by Theorem 1. Since $p_{\check{\eta}}$ and $\rho_\eta$ are surjective homomorphism, it is sufficient to prove that the images under $\rho_\eta \circ \bar{\pi}$ of the elements in the generator system of $\operatorname{Ker} p_{\check{\eta}}$ vanish in $H^*(\mathscr{F}_\eta)$.

In order to prove Theorem 2 we need some basic facts about the Grassmann and Scubert varieties. For $1 \leqq s \leqq n$ we denote by $Gr_s(V)$ the Grassmann variety consisting of all the $s$-dimensional subspaces of $V = \boldsymbol{C}^n$. We fix a complete flag $(0 = U_0 \subset U_1 \subset \cdots \subset U_n = V)$ obtained by refining the flag $(\cdots \subset x_0^2(V) \subset x_0(V) \subset V)$ for a fixed $x_0 \in O_\eta$. For a sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_s)$ of integers with $0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_s \leqq n - s$, let $Y_\lambda$ be the subvariety of $Gr_s(V)$ given by

$$Y_\lambda = \{T \in Gr_s(V) \mid \dim(T \cap U_{\lambda_i + i}) \geqq i \quad (i = 1, \cdots, s)\} .$$

Then $Y_\lambda$ is called the Schubert variety corresponding to $\lambda$. Let $\geqq$ be the ordering on $\{\lambda\}$ given by $\lambda \geqq \mu$ iff $\lambda_i \geqq \mu_i$ $(i = 1, \cdots, s)$.

PROPOSITION 2 (cf. Kleiman [3]). (i) *$Y_\lambda \supset Y_\mu$ if and only if $\lambda \geqq \mu$.*

(ii) *If we set $\mathring{Y}_\lambda = Y_\lambda - \bigcup_{\mu \lneqq \lambda} Y_\mu$, then $Gr_s(V) = \coprod_\lambda \mathring{Y}_\lambda$, which gives a cellular decomposition of $Gr_s(V)$.*

PROPOSITION 3. *Let $p: \mathscr{F} \to Gr_s(V)$ be the natural projection given by $p((V_i)) = V_s$. Then we have $p(\mathscr{F}_\eta) \subset Y_{\lambda_0}$, where $\lambda_0 = (0, \cdots, 0, n - s, \cdots, n - s)$ with 0 repeated $p_{\check{\eta}}(s)$-times and $n - s$ repeated $(s - p_{\check{\eta}}(s))$-times.*

PROOF. From the definition of $\mathscr{F}_\eta$, we see that $V_s \supset x_0^{n-s}(V)$ for $(V_i) \in \mathscr{F}_\eta$. On the other hand $\dim x_0^{n-s}(V) = \operatorname{rank} x_0^{n-s} = p_{\check{\eta}}(s)$. Thus $x_0^{n-s}(V) = U_{p_{\check{\eta}}(s)}$. Hence $\dim(V_s \cap U_i) = \dim U_i = i$ for $i \leqq p_{\check{\eta}}(s)$ and $\dim(V_s \cap U_{(n-s)+i}) \geqq i$ for $i > p_{\check{\eta}}(s)$, and we are done. q.e.d.

DEFINITION. For a sequence of integers $\lambda = (\lambda_1, \cdots, \lambda_s)$ with $0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_s$, we set

$$[\lambda_1, \cdots, \lambda_s] = \det(x_i^{\lambda_j + j - 1})_{1 \leqq i, j \leqq s} ,$$

and $S_\lambda(x_1, \cdots, x_s) = [\lambda_1, \cdots, \lambda_s]/[0, \cdots, 0]$. ($S_\lambda(x_1, \cdots, x_s)$ is a symmetric polynomial which is called the Schur function.)

REMARK. Let $h_{s,j}$ be the $j$-th elementary symmetric polynomial in the variables $x_1, \cdots, x_s$, that is,

$$(t - x_1) \cdots (t - x_s) = t^s - h_{s,1} t^{s-1} + \cdots + (-1)^s h_{s,s} .$$

Then we have $h_{s,j} = S_{\mu_{s,j}}$, where $\mu_{s,j} = (0, \cdots, 0, 1, \cdots, 1)$ with 0 repeated

$(s-j)$-times and 1 repeated $j$-times.

PROPOSITION 4 (cf. Kleiman [3]). (i) *Let* $p^*: H^*(Gr_s(V)) \to H^*(\mathscr{F})$ *be the homomorphism induced by* $p: \mathscr{F} \to Gr_s(V)$. *Then* $p^*$ *is injective and its image is the set of all the symmetric polynomials in the Chern classes* $\bar{x}_1, \cdots, \bar{x}_s$. (*Thus we identify* $H^*(Gr_s(V))$ *with a subalgebra of* $H^*(\mathscr{F})$ *in the following.*)

(ii) $S_\lambda(\bar{x}_1, \cdots, \bar{x}_s)$ *is not zero if and only if* $\lambda_s \leqq n-s$.

(iii) $\{S_\lambda(\bar{x}_1, \cdots, \bar{x}_s) \mid \lambda_s \leqq n-s\}$ *is the dual basis of the basis of the homology group* $H_*(Gr_s(V))$ *given by the cells* $\mathring{Y}_\lambda$, *that is*, $(S_\lambda(\bar{x}_1, \cdots, \bar{x}_s), \mathring{Y}_\mu) = \delta_{\lambda\mu}$.

PROOF OF THEOREM 2. By the proof of Theorem 1, it is sufficient to prove that $\rho_\eta(h_{s,j}(\bar{x}_{i_1}, \cdots, \bar{x}_{i_s})) = 0$ for $1 \leqq i_1 < \cdots < i_s \leqq n$ and $j \geqq s - (p_\eta^\vee(s) - 1)$. Since $\rho_\eta$ is a homomorphism of $W$-modules, we may assume that $i_1 = 1, \cdots, i_s = s$. Then by the remark above we have $h_{s,j}(\bar{x}_1, \cdots, \bar{x}_s) = S_{\mu_{s,j}}(\bar{x}_1, \cdots, \bar{x}_s)$. Since $p(\mathscr{F}_\eta) \subset Y_{\lambda_0}$ by Proposition 3, we have a commutative diagram;

$$\begin{array}{ccc} H^*(\mathscr{F}) & \xleftarrow{p^*} & H^*(Gr_s(V)) \\ \rho_\eta \downarrow & & \downarrow i^* \\ H^*(\mathscr{F}_\eta) & \xleftarrow[k^*]{} & H^*(Y_{\lambda_0}) \end{array} .$$

If $j \geqq s - (p_\eta^\vee(s) - 1)$, then $\lambda_0 \ngeqq \mu_{s,j}$. Thus $i^*(S_{\mu_{s,j}}(\bar{x}_1, \cdots, \bar{x}_s)) = 0$. Hence $\rho_n(S_{\mu_{s,j}}(\bar{x}_1, \cdots, \bar{x}_s)) = k^* \circ i^*(S_{\mu_{s,j}}(\bar{x}_1, \cdots, \bar{x}_s)) = 0$, and we are done.

**4. Structure of $C[\mathfrak{t} \cap \bar{O}_x]$ for some $\bar{O}_x$ in the case of $Sp(2n, C)$.** In this section we consider the case

$$G = Sp(2n, C) = \{g \in GL(2n, C) \mid {}^t gJg = J\} \quad \text{and}$$
$$\mathfrak{g} = \mathfrak{sp}(2n, C) = \{x \in M(2n, C) \mid {}^t xJ + Jx = 0\} ,$$

where

$$J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} .$$

Then

$$T = \left\{ \begin{bmatrix} h & 0 \\ 0 & h^{-1} \end{bmatrix} \middle| \ h\text{: nonsingular diagonal matrix} \right\}$$

is a maximal torus of $G$ whose Lie algebra is

$$\mathfrak{t} = \left\{ \left[ \begin{array}{ccc|ccc} x_1 & & & & & \\ & \ddots & & & 0 & \\ & & x_n & & & \\ \hline & & & -x_1 & & \\ & 0 & & & \ddots & \\ & & & & & -x_n \end{array} \right] \middle| \; x_i \in \boldsymbol{C} \right\},$$

and the Weyl group $W$ of $(G, T)$ is isomorphic to the semi-direct product of $S_n$ and $(\boldsymbol{Z}/2\boldsymbol{Z})^n$, as is well known.

The nilpotent orbits in $\mathfrak{g}$ are parametrized as follows. For a partition $\sigma = (b_0 \geqq b_1 \geqq \cdots)$ of $2n$ with the following condition $(\nabla)$ let $O_\sigma$ be the set of matrices in $\mathfrak{sp}(2n, \boldsymbol{C})$ with the Jordan type $\sigma$.

$$(\nabla) \qquad \#\{i \mid b_i = 2r-1\} \equiv 0 (\mathrm{mod}\, 2) \quad \text{for each} \quad r \in \boldsymbol{N}.$$

Then the following is well known.

LEMMA 4. *If $\sigma$ satisfies $(\nabla)$, then $O_\sigma \neq \emptyset$. Any nilpotent orbit in $\mathfrak{g}$ coincides with some $O_\sigma$ for a $\sigma$ satisfying $(\nabla)$.*

We determine the $W$-module structure of $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_\sigma]$ for a special $\sigma$ satisfying the following condition $(\nabla\nabla)$.

$$(\nabla\nabla) \qquad \#\{i \mid b_i = 2r-1\} = 0 \quad \text{for each} \quad r \in \boldsymbol{N}.$$

THEOREM 3. *Let $\sigma = (b_0 \geqq b_1 \geqq \cdots)$ be a partition of $2n$ which satisfies $(\nabla\nabla)$. We denote the dual partition of $(\sigma/2) = ((b_0/2) \geqq (b_1/2) \geqq \cdots)$ by $\tau = (d_0 \geqq d_1 \geqq \cdots)$. Then $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_\sigma]$ is isomorphic to $\mathrm{Ind}_{W_\tau}^W(1_{W_\tau})$ as a $W$-module, where $W_\tau = S_{d_0} \times S_{d_1} \times \cdots \subset S_n \subset W$.*

We prove this theorem in exactly the same manner as Theorem 1.

Let $P_\tau$ be the parabolic subgroup given by

$$P_\tau = \left\{ \begin{bmatrix} x & y \\ 0 & {}^t x^{-1} \end{bmatrix} \in G \;\middle|\; x = \begin{bmatrix} A_0 & & * \\ & A_1 & \\ 0 & & \ddots \end{bmatrix}, \quad A_i \in GL(d_i, \boldsymbol{C}) \right\}.$$

Then the Richardson orbit corresponding to $P_\tau$ is $O_\sigma$ and the subgroup of $W$ corresponding to $P_\tau$ is $W_\tau$. Since $\bar{O}_\sigma$ is normal by Kraft-Procesi [6] and $G^x = P^x$ for $x \in O_\sigma$ by Springer-Steinberg [10; III, 4.16], $\boldsymbol{C}[\mathfrak{t} \cap \bar{O}_\sigma]$ contains $\mathrm{Ind}_{W_\tau}^W(1_{W_\tau})$ by Kraft [5; Proposition 4].

Let $h_i^\sigma \in \boldsymbol{C}[\mathfrak{g}]$ be the restriction of $g_i^\sigma \in \boldsymbol{C}[M(2n, \boldsymbol{C})]$ (cf. §3) to $\mathfrak{g} = \mathfrak{sp}(2n, \boldsymbol{C})$. Then the following is obvious.

LEMMA 5. *$x \in \bar{O}_\sigma$ if and only if $h_i^\sigma(x) = 0$ for all $i$.*

Set $B^n = C[t] = C[x_1, \cdots, x_n]$. We denote the restriction of $h_i^q$ to $t$ by $\tilde{h}_i^q \in C[x_1, \cdots, x_n]$. For $L_\sigma = (\tilde{h}_i^q)$ and $\bar{B}_\sigma^n = B^n/L_\sigma$ we have only to prove the following (##).

$$(\#\#) \qquad \dim \bar{B}_\sigma^n \leqq 2^n \binom{n}{\tau}.$$

We note that $L_\sigma$ is generated by the coefficients of $t^m$ in $\prod_{p=1}^k (t^2 - x_{i_p}^2) \prod_{q=1}^r (t - \varepsilon_q x_{j_q})$ with $k, r, m, \varepsilon_q, i_1, \cdots, i_k, j_1, \cdots, j_r$ running through the integers satisfying $0 \leqq k \leqq n$, $0 \leqq r \leqq n$, $0 \leqq m \leqq p_\sigma(2k+r) - 1$, $\varepsilon_q = \pm 1$, $1 \leqq i_1 < \cdots < i_k \leqq n$, $1 \leqq j_1 < \cdots < j_r \leqq n$, $i_p \neq j_q$ $(1 \leqq p \leqq k, 1 \leqq q \leqq r)$.

We prove (##) by induction on $n$. The case $n = 1$ being trivial, we assume that $n \geqq 2$ and (##) holds for $n - 1$.

DEFINITION. For a partition $\sigma = (b_0 \geqq b_1 \geqq \cdots)$ of $2n$ with $(\nabla\nabla)$ and an integer $i$ with $b_0 > i \geqq 0$, we define a partition $\sigma_{(i)} = (b_0' \geqq b_1' \geqq \cdots)$ of $2(n-1)$ which also satisfies $(\nabla\nabla)$ as follows. $b_{t_0}' = b_{t_0} - 2$ for $t_0 = \max\{t \mid b_t > i\}$, and $b_j' = b_j$ for $j \neq t_0$.

Let $\Psi: B^n \to B^{n-1}$ be the algebra homomorphism given by $\Psi(x_j) = x_j$ $(j \neq n)$ and $\Psi(x_n) = 0$. We fix a partition $\sigma = (b_0 \geqq b_1 \geqq \cdots)$ of $2n$ satisfying $(\nabla\nabla)$. Let $\check{\sigma} = (c_0 \geqq c_1 \geqq \cdots)$ be the dual partition of $\sigma$ and let $\tau = (d_0 \geqq d_1 \geqq \cdots)$ be the partition of $n$ as in the statements of Theorem 3. We can prove the following just in the same way as in the proof of Lemma 2. So we omit the proof.

LEMMA 6. $\Psi(L_\sigma) \subset L_{\sigma_{(i)}}$.

Thus $\Psi$ induces a surjective homomorphism $\Psi_i: \bar{B}_\sigma^n \to \bar{B}_{\sigma_{(i)}}^{n-1}$.

LEMMA 7. $(\operatorname{Ker} \Psi_i) x_n^i \subset (x_n^{i+1})$ *in* $\bar{B}_\sigma^n$.

PROOF. It is easily seen that $\operatorname{Ker} \Psi_i$ is generated as an ideal by $x_n$ and the coefficients of $t^m$ in $\prod_{p=1}^k (t^2 - x_{i_p}^2) \prod_{q=1}^r (t - \varepsilon_q x_{j_q})(t + x_n)$ with $k, r, m, \varepsilon_q, i_1, \cdots, i_k, j_1, \cdots, j_r$ running through the integers which satisfy $0 \leqq k \leqq n-1$, $0 \leqq r \leqq n-1$, $0 \leqq m \leqq p_{\sigma_{(i)}}(2k+r)$, $\varepsilon_q = \pm 1$, $1 \leqq i_1 < \cdots < i_k \leqq n-1$, $1 \leqq j_1 < \cdots < j_r \leqq n-1$ and $i_p \neq j_q$ for any $p$ and $q$. Set

$$\prod_{p=1}^k (t^2 - x_{i_p}^2) \prod_{q=1}^r (t - \varepsilon_q x_{j_q})(t + x_n) = \sum_{s=0}^{2k+r+1} a_s t^s \in \bar{B}_\sigma^n[t].$$

Then the coefficient of $t^m$ in $(\sum_{s=0}^{2k+r+1} a_s t^s)(t - x_n)$ vanishes for $m \leqq p_\sigma(2k + r + 2) - 1$. Thus arguments similar to those in the proof of Lemma 6 show that $a_m x_n^i$ is contained in the ideal $(x_n^{i+1})$ of $\bar{B}_\sigma^n$ for $m \leqq p_{\sigma_{(i)}}(2k + r)$, and we are done. q.e.d.

PROOF OF THEOREM 3. Let $J_i$ be the principal ideal of $\bar{B}^n_\sigma$ generated by $x^i_n$. Then since $J_i/J_{i+1}$ is a cyclic $\bar{B}^{n-1}_{\sigma(i)}$-module, we have $\dim(J_i/J_{i+1}) \leqq \dim \bar{B}^n_{\sigma(i)} \leqq 2^{n-1}\binom{n-1}{\tau(i)}$, where $\tau(i)$ is the dual partition of $(\sigma_{(i)})/2$. Thus

$$\dim \bar{B}^n_\sigma = \sum_i \dim(J_i/J_{i+1}) \leqq 2^{n-1}\sum_i \binom{n-1}{\tau(i)} = 2^n\binom{n}{\tau},$$

which proves (##) for $n$ and the proof of Theorem 3 is complete. q.e.d.

REMARK. $C[\mathfrak{t} \cap \bar{O}_x]$ is the direct sum of the subspaces $C[\mathfrak{t} \cap \bar{O}_x]_i$ of degree $i$ which are $W$-invariant. For a partition $\sigma = (b_0 \geqq b_1 \geqq \cdots)$ of $2n$ satisfying ($\nabla\nabla$) we set $d(\sigma) = (b_0/2)^2 + (b_1/2)^2 + \cdots$. Then it follows from the proof of Theorem 3 and Kraft [5; Proposition 2] that $C[\mathfrak{t} \cap \bar{O}_\sigma]_i = (0)$ for $i > d(\sigma)$ and $C[\mathfrak{t} \cap \bar{O}_\sigma]_{d(\sigma)}$ is the irreducible representation corresponding to $((0), \tau)$ where $\tau$ is a partition of $n$ as in the statement of Theorem 3. (An irreducible representation of the Weyl group of $\mathfrak{sp}(2n, C)$ is characterized by an ordered pair of two partitions $(\lambda, \mu)$ with $|\lambda| + |\mu| = n$. Cf. Mayer [8].)

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