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APPROXIMATION OF OPERATOR SEMIGROUPS OF OHARU'S CLASS $(C_{(k)})$

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In [13], Takahashi and Oharu have proved a convergence theorem for sequences of semigroups of Oharu's classes $(C_{(k)})$ which extends the "continuous" Trotter theorem [15]. We establish a discrete version of this result (Theorem 1), i.e., with the approximating sequence of semigroups replaced by a sequence of powers of bounded linear operators. This is applied to two particular sets of $(C_{(1)})$-semigroups, one originating in the theory of approximation with exponential orders [7], the other coming from a discretization of a certain Cauchy problem [12], [13; Ex. 5.4]. Instead of a semi-discrete version of the latter problem we now consider its full discretization and show, e.g., that the weakened stability condition of Theorem 1 applies in this case. Moreover it will be shown that the two examples are not of class (A), in general, and a summary of further approximation properties will be given.

1. A discrete convergence theorem for semigroups of class $(C_{(k)})$. The classes $(C_{(k)})$ of semigroups have been introduced by Oharu [11; p. 250] in connection with abstract Cauchy problems. They have been investigated futher in [13]. To define them we need the following notations.

Let X be a Banach space, $[X]$ the space of bounded linear operators from X into itself, and $\{T(t),\ t>0\}$ a semigroup of operators in $[X]$. Let $\omega_0 = \lim_{t\to\infty} t^{-1}\log\|T(t)\|$ be the type of the semigroup, and $\Sigma = \{f\in X;\ \lim_{t\to 0+}\|T(t)f-f\|=0\}$ its continuity set. Supposing $\{T(t),\ t>0\}$ to be strongly continuous on $(0,\infty)$, we denote by $R_0(\lambda)$ the Laplace transform

$$R_0(\lambda)f = \int_0^\infty e^{-\lambda t}T(t)f\,dt \quad (\lambda\in C) \tag{1.1}$$

whenever the right hand side exists as a Bochner integral. Let A_0 be the infinitesimal operator of the semigroup and set $X_0 = \bigcup_{t>0} T(t)(X)$.

Following Oharu [11] one says that the semigroup belongs to class $(C_{(k)})$ if it satisfies the following conditions:

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(C.1) $\bar{X}_0 = X$,

(C.2) there exists $\omega_1 > \omega_0$ such that for each $\lambda \in \boldsymbol{C}$ with $\mathrm{Re}(\lambda) > \omega_1$ there is an operator $R(\lambda) \in [X]$ with $R(\lambda)|_{X_0} = R_0(\lambda)|_{X_0}$,

(C.3) if $R(\lambda)f = 0$ for $\lambda > \omega_1$, then $f = 0$,

(C.4) there exists $k \in \boldsymbol{P}$ such that $R(\lambda_0)^k(X) \subset \Sigma$ for some $\lambda_0 > \omega_1$.

Here $\boldsymbol{P}$ denotes the set of non-negative integers. A semigroup satisfying (C.1)-(C.3) has a closable and densely defined infinitesimal operator A_0. Its closure A, the infinitesimal generator, satisfies

$$\{\lambda; \mathrm{Re}(\lambda) > \omega_1\} \subset \rho(A)\,, \quad R(\lambda, A) = R(\lambda) \quad \text{for} \quad \mathrm{Re}(\lambda) > \omega_1\,.$$

Here $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A)$ the resolvent of A at $\lambda \in \rho(A)$. Thus condition (C.4) can be replaced by

(C.4)′ there exists $k \in \boldsymbol{P}$ such that $D(A^k) \subset \Sigma$.

For $k \in \boldsymbol{P}$, $k \geqq 1$, the classes $(C_{(k)})$ properly contain the familiar classes [9; p. 320] of semigroups; in particular, $(1, A) \subset (0, A) \subset (C_{(1)})$ and $(A) \subset (C_{(2)})$, whereas $(C_0) = (C_{(0)})$, cf. [11; pp. 256-259].

A convergence theorem for $(C_{(k)})$-semigroups has been established by Takahashi and Oharu [13; Cor. 3.8.], i.e., a theorem giving sufficient conditions upon a sequence $\{T_n(t)\}_{n \in N}$ of $(C_{(k)})$-semigroups in order that there exists a limiting semigroup $T(t)$ of class $(C_{(k)})$ such that

$$\lim_{n\to\infty} \| T_n(t)f - T(t)f \| = 0 \quad \text{for each} \quad t > 0\,.$$

For $k = 0$, this result contains Trotter's well-known theorem [15; Thm. 5.2]. Discrete versions of Trotter's theorem, dealing with the sequence of powers of an operator $Q_n \in [X]$ instead of $\{T_n(t),\, t > 0\}$, have been given, e.g., in [15; Thm. 5.3], [10; Thm. (2.13)], [1; Satz 2.4]. There the limiting semigroup always belongs to (C_0), and the stability hypothesis implies in particular that the sequence $\{Q_n\}_{n \in N}$ has to be uniformly bounded in norm. The following extension of the discrete Trotter type theorem admits limiting semigroups which are of class $(C_{(k)})$ only, and sequences $\{Q_n\}_{n \in N}$ which are not necessarily approximation processes. To formulate this theorem we need the following notations.

Given a positive null sequence $\{h_n\}_{n \in N}$, a sequence $\{Q_n\}_{n \in N}$ of operators in $[X]$ will be said to satisfy property

(D1) if there exists an $\omega \in R$ such that

$$\sup_{n \in N} \| R(\lambda, A_n) \|_{[X]} < \infty$$

for each $\lambda > \omega$, where $A_n = h_n^{-1}(Q_n - I)$, with I the identity operator,
$(D2)_k$ if, for some $k \in P$, there exist constants M, K, n_0 such that

$$\|Q_n^j f\| \leqq M e^{Kh_n j}\|f\|_{k,n}$$

for all $f \in X$, $j, n \in N$, $n \geqq n_0$, where $\|f\|_{k,n} = \sum_{i=0}^{k} \|A_n^i f\|_X$,
(D3) if

$$\sup_{n \in N} \|\exp\{tA_n\}\|_{[X]} < \infty$$

uniformly in t on each compact interval $[a, b] \subset (0, \infty)$,
(D4) if

$$\sup_{n \in N} \|Q_n^{[t/h_n]}\|_{[X]} < \infty$$

uniformly in t on each compact interval $[a, b] \subset (0, \infty)$. Here $[a]$ denotes the integral part of $a \in R$.

Given a closed linear operator A with domain $D(A) \subset X$ and range $R(A) \subset X$, a set $D \subset D(A)$ is called a *core of* A if the closure of $A|_D$ is A. We then say that $\{Q_n\}_{n \in N}$ and A satisfy property

$(D5)_k$ if, for some $k \in P$, there is a core D of A with $D \subset D(A^{k+1})$ such that, for the ω of condition (D1),

$$\rho(A) \cap \{\lambda; \lambda > \omega\} \neq \varnothing\ , \quad D(A) \text{ is dense in } X\ , \quad \text{and}$$
$$\lim_{n\to\infty} \|A_n^i f - A^i f\| = 0 \quad \text{for each } f \in D \text{ and each } i = 1, 2, \cdots, k+1\ .$$

Conditions (D1)-(D3) are our substitutes for conditions (I), $(\mathrm{II}_{\exp})$, (IV)′ of [13; Cor. 3.8], respectively, and condition $(D5)_k$ corresponds to their consistency condition (III)′. Conditions $(D2)_k$ and (D4) together are the substitutes for the stability type condition of the classical Trotter theorem.

THEOREM 1. *Given a sequence* $\{Q_n\}_{n \in N} \subset [X]$ *and a closed linear operator with* $D(A) \subset X$, $R(A) \subset X$, *suppose that there exists a positive null sequence* $\{h_n\}_{n \in N}$ *such that, for some* $k \in P$, *conditions* (D1), $(D2)_k$, (D3), (D4), *and* $(D5)_k$ *are satisfied. Then*

(i) *A generates a* $(C_{(k)})$*-semigroup* $\{T(t); t > 0\}$,

(ii) $\lim_{n\to\infty} \|Q_n^{[t/h_n]}f - T(t)f\| = 0$

for each $f \in X$, *uniformly in t on each compact iuterval* $[a, b] \subset (0, \infty)$.

For the proof the auxiliary semigroup operators

$$T_n(t) = \exp(tA_n) \quad (t > 0,\ n \in N) \tag{1.2}$$

will be used. Proceeding as in [10; p. 363] and using $(D2)_k$ one readily obtains:

LEMMA 1. *Given $[a, b] \subset (0, \infty)$ and $\varepsilon \in (0, 1)$, there exists $n_1 \in N$ such that, under the hypotheses of Theorem 1,*

$$\| T_n(t)f - Q_n^{[t/h_n]}f \|_X \leqq 2Me^{b(K+\omega_1)}\{h_n \varepsilon^{-2} a^{-1} \| f \|_{k,n} + (3b\varepsilon + h_n) \| A_n f \|_{k,n}\}$$

holds for each $f \in X$, $n \geqq n_1$, $t \in [a, b]$.

PROOF OF THEOREM 1. For each $n \in N$, $\{T_n(t);\ t > 0\}$ forms a (C_0)-semigroup and hence belongs to $(C_{(k)})$ for each $k \in P$. The convergence theorem of Takahashi and Oharu [13; Cor. 3.8] can be applied to these semigroups. Indeed, the hypotheses (I), (III)′, and (IV) of that theorem are obviously satisfied in view of (D1), (D3), and $(D5)_k$, so that only condition (II_{exp}) remains to be verified, i.e., constants $\omega_1 \geqq \omega$ and M_1 are to be found such that for each $f \in D(A_n^k)$ and each $t > 0$

$$\| T_n(t)f \| \leqq M_1 e^{\omega_1 t} \| f \|_{k,n} . \tag{1.3}$$

Using $(D2)_k$ and observing that $D(A_n) = X = D(A_n^k)$, one has for each $n \geqq n_0$, $t > 0$, $f \in X$

$$\begin{aligned} \| T_n(t)f \| &= e^{-t/h_n} \Big\| \sum_{j=0}^{\infty} (t/h_n)^j (j!)^{-1} Q_n^j f \Big\| \leqq e^{-t/h_n} \sum_{j=0}^{\infty} (t/h_n)^j (j!)^{-1} M e^{Kh_n j} \| f \|_{k,n} \\ &= M \exp\{(t/h_n)(e^{Kh_n} - 1)\} \| f \|_{k,n} . \end{aligned}$$

For $n < n_0$ one has $\| T_n(t) \| \leqq M_n \exp\{\omega_n t\}$ with constants ω_n and M_n. Choosing

$$\omega_1 = \max\{\sup_{n \geqq n_0} (e^{Kh_n} - 1)/h_n,\ \omega,\ \max_{n<n_0} \omega_n\} \qquad \text{and} \qquad M_1 = \max_{n<n_0}\{M, M_n\} ,$$

which are finite, inequality (1.3) follows. Hence the Takahashi-Oharu theorem implies the existence of a $(C_{(k)})$-semigroup $\{T(t); t > 0\}$ which is generated by the operator A and satisfies

$$\lim_{n \to \infty} \| T_n(t)f - T(t)f \| = 0 \quad (f \in X) , \tag{1.4}$$

uniformly in t on each compact interval $[a, b] \subset (0, \infty)$. This proves assertion (i). Using (1.4), assertion (ii) will follow by

$$\begin{aligned} \|Q_n^{[t/h_n]}f - T(t)f\| &\leqq \| T(t)f - T_n(t)f \| + \| T_n(t)f - Q_n^{[t/h_n]}f \| \\ &= o(1) + \| T_n(t)f - Q_n^{[t/h_n]}f \| , \quad n \to \infty , \end{aligned}$$

if we show that, for each $f \in X$,

$$\lim_{n \to \infty} \| T_n(t)f - Q_n^{[t/h_n]}f \| = 0 \tag{1.5}$$

uniformly in t on each compact subinterval of $(0, \infty)$.

First we consider functions $f \in D$. By $(D5)_k$ there exists a constant

n_2, independent of i, such that for all $n \geqq n_2$

$$\|A_n^i f\| \leqq 1 + \|A^i f\| \leqq 1 + \sum_{j=1}^{k+1} \|A^j f\|$$

if $1 \leqq i \leqq k+1$. Thus $h_n \|f\|_{k,n} = o(1)$, $n \to \infty$, as well as

$$h_n \|A_n f\|_{k,n} = h_n \sum_{i=0}^{k} \|A_n^{i+1} f\| = o(1) , \quad n \to \infty ,$$

for each $f \in D$. Therefore, given any compact interval $[a, b] \subset (0, \infty)$, Lemma 1 yields

$$\|T_n(t)f - Q_n^{[t/h_n]} f\| = o(1) + O(\varepsilon \|A_n f\|_{k,n}) = \varepsilon \cdot O(1) , \quad n \to \infty$$

uniformly in t on $[a, b]$. Since ε can be arbitrarily small, (1.5) follows.

Now let $f \in X$. By (D3), (D4), and (1.2) the family of operators $\{T_n(t) - Q_n^{[t/h_n]}; n \in N, t \in [a, b]\}$ is uniformly bounded on X. Since D is dense in X, the Banach Steinhaus theorem yields (1.5) on all of X, and the proof is complete.

We remark that the assumptions of Theorem 1 in case $k = 0$ are identical with those of the known discrete version of Trotter's theorem, but the conclusion of the latter is somewhat stronger since it admits to choose $a = 0$. For $k \in N$, however, there is in general no uniform convergence on intervals of the form $[0, b]$.

2. Approximation of particular semigroups of class $(C_{(k)})$.

2.1. *The semigroups* $\{T_\varphi(t); t > 0\}$. In this section we consider an application of Theorem 1 with $k = 1$ to a particular set of operators Q_n on the space $C_{2\pi}$ of continuous, 2π-periodic functions, and obtain an interesting set of $(C_{(1)})$-semigroups whose properties will then be studied in more detail. For the Q_n we choose the general typical means $R_{\varphi,n}$ of the Fourier series of an $f \in C_{2\pi}$, where

$$R_{\varphi,n}(f; x) = \sum_{|m| \leqq n} (1 - \varphi(|m|)/\varphi(n+1)) f^\wedge(m) e^{imx} \quad (n \in N,\ x \in \boldsymbol{R}) , \tag{2.1}$$

with $f^\wedge(m)$ denoting the Fourier coefficient of f of order $m \in \boldsymbol{Z} = \{0, \pm 1, \pm 2, \cdots\}$ and φ being any function of class Ω below. Denoting by $C^r(0, \infty)$ the space of functions with continuous r-th derivative on $(0, \infty)$, we define

$$\Omega_0 = \{\varphi; \varphi\colon [0, \infty) \to \boldsymbol{R}, \varphi(0) = 1, \varphi \in C^1(0, \infty), \varphi'(x) > 0 \ \forall x > 0 , \lim_{x\to\infty} \varphi(x) = +\infty\} ,$$

$$\Omega = \{\varphi \in \Omega_0; \varphi(x) = e^{g(x)}, g \in C^3(0, \infty), \exists x_0 > 0 \text{ with } g''(x) \leqq 0 , g'''(x) \geqq 0 \ \forall x \geqq x_0, \text{ and } \limsup_{x\to\infty} |g''(x)|(g'(x))^{-2} < 1\} . \tag{2.2}$$

These classes have been introduced in connection with exponential rates of approximation (see, e.g., [6], [7]), and the approximation behavior of typical means with $\varphi \in \Omega$ has been studied in [4, II]. One of the general results there implies that, if φ grows faster than any polynomial, i.e., $\varphi \in \Omega_1$, where

$$\Omega_1 = \{\varphi \in \Omega; \lim_{x\to\infty} |g''(x)|(g'(x))^{-2} = 0\} , \tag{2.3}$$

then the corresponding operators $R_{\varphi,n}$ are no longer uniformly bounded in n on $C_{2\pi}$. In fact, for each $\varphi \in \Omega$ one has ([4, II; Thm. 4.1])

$$\| R_{\varphi,n} \|_{[C_{2\pi}]} \sim (4/\pi^2) \log \{ng'(n)\} , \quad n \to \infty , \tag{2.4}$$

where $a_n \sim b_n$, $n \to \infty$ denotes two sequences with $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n \to \infty$. Nevertheless, on a suitable set of "smooth" functions the $R_{\varphi,n}$ yield the same rate of approximation as the trigonometric polynomials of best approximation of degree n.

On the other hand, if $\varphi \in \Omega \backslash \Omega_1$, then $g'(n) = O(1/n)$, $n \to \infty$ ([4, I; Lemma 4.1]) and thus $\| R_{\varphi,n} \|_{[C_{2\pi}]} = O(1)$, $n \to \infty$, by (2.4). So there is an essential distinction between the two cases $\varphi \in \Omega_1$ and $\varphi \in \Omega \backslash \Omega_1$ which will also be called "exponential" and "classical" orders, respectively. Such a distinction will also be important for the properties of the limiting semigroup $\{T_\varphi(t); t > 0\}$ obtained from the $R_{\varphi,n}$ via Theorem 1.

We apply Theorem 1 with $X = C_{2\pi}$, equipped with the maximum-norm, $Q_n = R_{\varphi,n}$ for some $\varphi \in \Omega$, and $A = A_\varphi$ where A_φ is defined by

$$(A_\varphi f)^\wedge(m) = -\varphi(|m|)f^\wedge(m) \quad (m \in \mathbf{Z}) \tag{2.5}$$

$$D(A_\varphi) = \{f \in C_{2\pi};\ \exists\, h \in C_{2\pi} \text{ with } \varphi(|m|)f^\wedge(m) = h^\wedge(m)\ \forall\, m \in \mathbf{Z}\} . \tag{2.6}$$

Obviously A_φ is a closed operator. Moreover we choose

$$h_n = 1/\varphi(n+1) , \qquad A_{\varphi,n} = \varphi(n+1)(R_{\varphi,n} - I) .$$

As is easily checked, conditions $(D2)_k$ and $(D5)_k$ are satisfied with $k = 1$ for each $\varphi \in \Omega$. In particular, the set $P_{2\pi}$ of all trigonometric polynomials is a core of A_φ, and ω may be taken to be 0. Condition (D4) is satisfied since for each $t \in [a, b] \subset (0, \infty)$ and each $f \in C_{2\pi}$

$$\begin{aligned} \| R_{\varphi,n}^{[t\varphi(n+1)]} f \|_{C_{2\pi}} &= \Big\| \sum_{|m| \le n} (1 - \varphi(|m|)/\varphi(n+1))^{[t\varphi(n+1)]} f^\wedge(m) e^{imx} \Big\|_{C_{2\pi}} \\ &\le 2\|f\|_{C_{2\pi}} \Big(C_1 + \sum_{m=n_0}^{n} (1 - \varphi(m)/\varphi(n+1))^{a\varphi(n+1)-1} \Big) \\ &\le 2\|f\|_{C_{2\pi}} \Big(C_1 + \sum_{m=n_0}^{n} \exp\{-\varphi(m)(a\varphi(n+1) - 1)/\varphi(n+1)\} \Big) \end{aligned}$$

where n_0 is chosen such that $a\varphi(n+1) > 1\ \ \forall\, n \ge n_0$, and the latter sum is bounded as $n \to \infty$ since $\limsup_{n\to\infty} (\log x)/g(x) < 1$ for each $\varphi \in \Omega$ and therefore $\varphi(k) \geqq k$ for k large enough. Similarly one obtains that $\|\exp\{tA_{\varphi,n}\}f\|_{C_{2\pi}} \leqq C_2\|f\|$ with a constant C_2 independent of f and t, for each $t \in [a, b] \subset (0, \infty)$, whence (D3) is satisfied. Using (D3) and the fact that $\{\exp\{tA_{\varphi,n}\}; t > 0\}$ is a (C_0)-semigroup on $C_{2\pi}$ for each $n \in N$, one has, for each $\lambda > 0$, $n \in N$

$$\|R(\lambda, A_{\varphi,n})f\|_{C_{2\pi}} = \left\|\int_0^\infty e^{-\lambda t} \exp\{tA_{\varphi,n}\}f\,dt\right\|_{C_{2\pi}} \leqq \|f\|_{C_{2\pi}}\left(C_3 + C_2\int_0^\infty e^{-\lambda t}dt\right)$$

with a constant C_3, thus (D1) holds with $\omega = 0$, and Theorem 1 now yields:

COROLLARY 1. *For each $\varphi \in \Omega$, the operator A_φ is the infinitesimal generator of a $(C_{(1)})$-semigroup $\{T_\varphi(t); t > 0\}$ on $C_{2\pi}$. Moreover,*

$$\lim_{n\to\infty} \|R_{\varphi,n}^{[t\varphi(n+1)]}f - T_\varphi(t)f\|_{C_{2\pi}} = 0 \tag{2.7}$$

for each $f \in X$, uniformly in t on compact subintervals of $(0, \infty)$.

Using a uniqueness theorem of Oharu [11; p. 250], an explicit representation of the operators $T_\varphi(t)$ is obtained:

$$T_\varphi(t)(f; x) = \sum_{m\in Z} e^{-t\varphi(|m|)} f^\wedge(m)e^{imx} \quad (f \in C_{2\pi};\ t > 0)\ ; \tag{2.8}$$

they are also called *generalized Abel-Cartwright means*. In case $\varphi \in \Omega\backslash\Omega_1$ it can be shown that the $T_\varphi(t)$ belong to (C_0). But for $\varphi \in \Omega_1$ they are no longer in (C_0), as has been shown in [7; p. 170], already. Thus, relation (2.7) cannot be deduced from the discrete version of the classical Trotter theorem. Moreover one has:

PROPOSITION 1. *For each $\varphi \in \Omega_1$, the semigroup defined by* (2.8) *does not belong to the class* (A).

PROOF. For each $\varphi \in \Omega_1$ one has $(\log x)/g(x) = o(1)$, $x \to \infty$ ([7; Lemma 1]), thus there exists $x_1 > 0$ such that $\varphi(x) \geqq x^2$ for each $x \geqq x_1$. Using this one obtains

$$\|T_\varphi(t)\|_{[C_{2\pi}]} \leqq \sum_{m\in Z} e^{-t\varphi(|m|)} \leqq 2x_1 + 1 + \sqrt{\pi/t} \tag{2.9}$$

for each $t > 0$, and consequently the operator $R_\varphi(\lambda)$ defined by

$$R_\varphi(\lambda)f = \int_0^\infty e^{-\lambda t}T_\varphi(t)f\,dt \quad (f \in C_{2\pi}) \tag{2.10}$$

belongs to $[C_{2\pi}]$, if $\mathrm{Re}(\lambda) > 0$. In view of the definition of (A) [9; p. 322] and the uniform boundedness principle, the assertion follows if we can show that

$$\limsup_{\lambda\to\infty} \|\lambda R_\varphi(\lambda)\|_{[C_{2\pi}]} = +\infty \ . \tag{2.11}$$

For this purpose the familiar test functions

$$f_n(x) = 2\sin(n+1)x \sum_{m=1}^{n} m^{-1}\sin mx \quad (n \in \boldsymbol{N},\ x \in \boldsymbol{R}) \ , \tag{2.12}$$

can be used, which are uniformly bounded in x and n (cf. [18; p. 61]). Setting $n(\lambda) = [\varphi^{-1}(\lambda)]$ and $a(\lambda) = 1 + [1/g'(\varphi^{-1}(\lambda))]$ for $\lambda \geqq 1$ one has

$$\begin{aligned}
\|\lambda R_\varphi(\lambda) f_{n(\lambda)}\|_{C_{2\pi}} &\geqq \Big|\sum_{m\in \boldsymbol{Z}} f^{\wedge}_{n(\lambda)}(m)\lambda(\lambda+\varphi(|m|))^{-1}\Big| \\
&= \sum_{j=1}^{n(\lambda)} (\lambda/j)\{(\lambda+\varphi(n(\lambda)+1-j))^{-1} - (\lambda+\varphi(n(\lambda)+1+j))^{-1}\} \\
&\geqq \sum_{j=a(\lambda)+1}^{n(\lambda)} (\lambda/j)(\lambda+\varphi(n(\lambda)+1-j))^{-1} - \sum_{j=a(\lambda)+1}^{n(\lambda)} \lambda/(j\varphi(n(\lambda)+1+j)) \\
&= I_1 - I_2\ , \quad \text{say.}
\end{aligned}$$

By (2.2) there is a constant C_1 such that $g'(x) \leqq C_1$ for all $x \geqq 0$. Thus we have

$$\begin{aligned}
I_1 &\geqq \lambda(\lambda + \varphi(\varphi^{-1}(\lambda) - a(\lambda)))^{-1} \sum_{j=a(\lambda)+1}^{n(\lambda)} j^{-1} \geqq (1/2)\int_{a(\lambda)+1}^{\varphi^{-1}(\lambda)} x^{-1}\,dx \\
&\geqq (1/2)\log(\varphi^{-1}(\lambda)g'(\varphi^{-1}(\lambda))/(C_1+1))\ ,
\end{aligned}$$

and, since $\varphi \in \Omega_1$ implies that $\lim_{x\to\infty} xg'(x) = \infty$ ([7; Lemma 1]), it follows that

$$\limsup_{\lambda\to\infty} I_1 = +\infty \ . \tag{2.13}$$

For I_2 we use the convexity of φ for large arguments and obtain

$$\begin{aligned}
I_2 &\leqq \sum_{j=a(\lambda)+1}^{n(\lambda)} (\lambda/j)(\lambda + j\varphi'(\varphi^{-1}(\lambda)))^{-1} \leqq \sum_{j=a(\lambda)+1}^{n(\lambda)} (jg'(\varphi^{-1}(\lambda)))^{-1} \\
&\leqq (g'(\varphi^{-1}(\lambda)))^{-1}\int_{a(\lambda)}^{\varphi^{-1}(\lambda)} x^{-2}dx \leqq (g'(\varphi^{-1}(\lambda)))^{-1}\{g'(\varphi^{-1}(\lambda)) - 1/\varphi^{-1}(\lambda)\} \leqq 1\ ,
\end{aligned}$$

which, together with (2.13), proves (2.11).

Without proof we state that, in the present case, beyond the mere convergence property (2.7), there is a rather high rate of approximation available, namely:

PROPOSITION 2. *For each $\varphi \in \Omega$ and $a > 0$,*

$$\|R_{\varphi,n}^{[t\varphi(n+1)]} - T_\varphi(t)\|_{[C_{2\pi}]} = O(1/\varphi(n+1))\ , \quad n\to\infty\ ,$$

uniformly in t for $a \leqq t < \infty$.

2.2 *The semigroups* $\{T_q(t); t > 0\}$. To introduce another typical example of a $(C_{(k)})$-semigroup which is encountered as a limit in Theorem 1, we consider the following initial value problem. For the sake of consistency with the usual notations in the theory of difference methods for initial value problems we will use in this subsection the letters h and k as step parameters of difference operators though they have another meaning in the rest of the paper.

Let $\boldsymbol{L}^2(\boldsymbol{R}) = L^2(\boldsymbol{R}) \times L^2(\boldsymbol{R})$ with elements $f = (f_1, f_2)$ and norm $\|f\| = (\|f_1\|_2^2 + \|f_2\|_2^2)^{1/2}$ where $\|f_1\|_2$ denotes the usual norm

$$\|f_1\|_2 = \left((2\pi)^{-1/2}\int |f_1(x)|^2\,dx\right)^{1/2},$$

and let $q \in \boldsymbol{P}$, $0 \leqq q \leqq 4$. The problem is to determine a vector valued function $u(x, t) = (u_1(x, t), u_2(x, t))$ of the variables $x \in \boldsymbol{R}$, $t > 0$ such that $u(\cdot, t) \in \boldsymbol{L}^2(\boldsymbol{R})$ for each $t > 0$, and u solves the Cauchy problem

$$\partial u_1/\partial t = \partial^2 u_1/\partial x^2 + i^{-q}\,\partial^q u_2/\partial x^q\,, \qquad \partial u_2/\partial t = \partial^2 u_2/\partial x^2 \tag{2.14}$$

with initial condition $u(x, 0) = f(x)$ for a given $f \in \boldsymbol{L}^2(\boldsymbol{R})$. This is a slightly modified version of an example by Sunouchi [12; p. 403, 405], who considered a discretization of this problem with respect to the space variable x in connection with his generalization of the continuous Trotter theorem. We now consider the discretization of this problem both in the space and time variables in connection with Theorem 1.

Denoting by $P_{q,0}$ the differential operator defined by the right hand side of (2.14), $P_{q,0}$ is not necessarily closed (in fact, it is closed for $q = 0, 1, 2$, not for $q = 3, 4$) but it has a closure which will be denoted by P_q (cf. [17; p. 78]). So we consider the abstract Cauchy problem (cf. [12; p. 403, (A.2)])

$$\partial u/\partial t = P_q u\,, \qquad u(x, 0) = f(x)\,. \tag{2.15}$$

To set up the corresponding discrete problem, P_q will now be replaced by an operator $A_{q,h}$ which is obtained by replacing the derivatives $\partial u(x, t)/\partial x$ on the right hand side of (2.14) by central difference quotients $(2h)^{-1}(u(x + h, t) - u(x - h, t))$ with increment h, and $\partial u/\partial t$ will be replaced by the forward difference quotient $k^{-1}(u(x, t + k) - u(x, t))$. The increments h and k will always be related by

$$k/h^M = \mu \tag{2.16}$$

where $\mu > 0$ is assumed to be fixed number, $\mu \leqq 2$, and $M = \max\{2, q\}$. So the discrete solution $u_k(x, t)$ satisfies

$$k^{-1}(u_k(x, t + k) - u_k(x, t)) = (A_{q,h}u_k)(x, t)\,, \qquad u_k(x, 0) = f(x)\,.$$

To be explicit, we reformulate this as

$$u_k(x, t + k) = (E_{q,k} u_k)(x, t) , \qquad u_k(x, 0) = f(x) , \tag{2.17}$$

where

$$E_{q,k} = kA_{q,h} + I \tag{2.18}$$

and apply the Fourier transform with respect to x. Then the matrix $E_{q,k}$ turns into

$$E_{q,k}^{\wedge} = \begin{pmatrix} 1 - k(h^{-1}\sin hv)^2 & k(h^{-1}\sin hv)^q \\ 0 & 1 - k(h^{-1}\sin hv)^2 \end{pmatrix} \tag{2.19}$$

where $v \in \boldsymbol{R}$, and the transformed problem consists in determining $u_k^{\wedge}(v, t) \in \boldsymbol{L}^2(\boldsymbol{R})$ such that

$$u_k^{\wedge}(v, t + k) = E_{q,k}^{\wedge} u_k^{\wedge}(v, t) , \qquad u_k^{\wedge}(v, 0) = f^{\wedge}(v) .$$

Using Theorem 1, the solution of (2.15) can now be obtained from the $E_{q,k}$ when k tends to zero.

COROLLARY 2. *For each $q \in \boldsymbol{P}$, $q \leqq 4$, there exists a $(C_{(1)})$-semigroup $\{T_q(t); t > 0\}$ such that $u(x, t) = T_q(t)f(x)$ is a solution of (2.15). Moreover,*

$$\lim_{k \to 0+} \| E_{q,k}^{[t/k]} f - T_q(t) f \| = 0 \tag{2.20}$$

for each $f \in \boldsymbol{L}^2(R)$, uniformly on compact subintervals of $(0, \infty)$.

PROOF. Let $\{k_n\}$ be a fixed null sequence of positive numbers and define h_n by $k_n = \mu h_n^M$ (cf. (2.16)). In Theorem 1 we take $X = \boldsymbol{L}^2(\boldsymbol{R})$, $Q_n = E_{q,k_n}$, $A = P_q$ with $D(P_q) = \{u \in \boldsymbol{L}^2(\boldsymbol{R}); P_q u \in \boldsymbol{L}^2(\boldsymbol{R})\}$, and $k = 1$. Moreover we replace the sequence $\{h_n\}$ in Theorem 1 by the sequence $\{k_n\}$. Condition (D1) of the theorem is then satisfied with $\omega = 0$ since $\sup_{x \geqq 0} x^q(\lambda + x^2)^{-2} < \infty$ for each $\lambda > 0$. To verify condition $(\mathrm{D2})_1$ it has to be shown that

$$\| E_{q,k_n}^j u \| \leqq M e^{K k_n j} \{ \| u \| + \| A_{q,h_n} u \| \} \tag{2.21}$$

for each $u \in \boldsymbol{L}^2(\boldsymbol{R})$, $j, n \in \boldsymbol{N}$, $n \geqq n_0$. This can be proved by observing that

$$(E_{q,k}^j)^{\wedge} = \begin{pmatrix} (1 - k(h^{-1}\sin hv)^2)^j & jk(h^{-1}\sin hv)^q(1 - k(h^{-1}\sin hv)^2)^{j-1} \\ 0 & (1 - k(h^{-1}\sin hv)^2)^j \end{pmatrix}$$

and therefore

$$\begin{aligned} \| E_{q,k}^j u \| &= \| (E_{q,k}^j)^{\wedge} (I - A_{q,h}^{\wedge})^{-1} (I - A_{q,h}^{\wedge}) u^{\wedge} \| \\ &\leqq \sup_{v \in \boldsymbol{R}} | (E_{q,k}^j)^{\wedge} (I - A_{q,h}^{\wedge})^{-1} | \{ \| u \| + \| A_{q,h} u \| \} , \end{aligned}$$

which implies (2.21). Conditions (D3) and (D4) are satisfied since $\sup_{x\geq 0} x^{q/2}e^{-x}<\infty$. In order to verify $(D5)_1$ we use as a core of P_q the set D of functions $u\in \boldsymbol{L}^2(\boldsymbol{R})$ for which $u^\wedge$ has compact support. In view of

$$P_q^\wedge = \begin{pmatrix} -v^2 & v^q \\ 0 & -v^2 \end{pmatrix}$$

one obviously has $D\subset D(P_q^2)$. For $\lambda>0$ the resolvent $R(\lambda, P_q)$ is given by

$$R(\lambda, P_q)^\wedge = \begin{pmatrix} (\lambda+v^2)^{-1} & -v^q(\lambda+v^2)^{-2} \\ 0 & (\lambda+v^2)^{-1} \end{pmatrix}, \tag{2.22}$$

it is bounded on $\boldsymbol{L}^2(\boldsymbol{R})$, and thus $\{\lambda; \lambda>0\}\subset\rho(P_q)$. It remains to prove that, for each $u\in D$, $j=1, 2$,

$$\lim_{n\to\infty} \|A^j_{q,h_n}u - P^j_q u\| = 0 . \tag{2.23}$$

If $u\in D$ and v_0 is chosen such that $u^\wedge(v)=0 \ \forall |v|>v_0$, one has $|(h^{-1}\sin hv)^r - v^r| \leqq h^2 r v_0^{r+2}/6$ for each $|v|\leqq v_0$. The assertion (2.23) for $j=1$ now follows by

$$\begin{aligned}\|A_{q,h}u - P_q u\| &\leqq \sup_{|v|\leqq v_0}\left|\begin{pmatrix} v^2-(h^{-1}\sin hv)^2 & (h^{-1}\sin hv)^q - v^q \\ 0 & v^2-(h^{-1}\sin hv)^2\end{pmatrix}\right|\|u\| \\ &\leqq \max(h^2v_0^4/3,\ h^2v_0^{r+2}q/6)\|u\|\end{aligned}$$

which tends to zero as $h\to 0$, and similarly for $j=2$. Hence all the hypotheses of Theorem 1 are satisfied, and, since $\{k_n\}$ is an arbitrary null sequence, the proof is complete.

To give an explicit representation of $T_q(t)$, let

$$\chi_t(x) = (2t)^{-1/2}e^{-x^2/4t} \quad (t>0,\ x\in\boldsymbol{R})$$

denote the Weierstraß kernel, and let

$$\chi_{q,t}(x) = ti^{-q}\chi_t^{(q)}(x) ,$$

where (q) denotes the q-th derivative. With the usual notation $(f*g)(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(x-u)g(u)du$ for convolution, $T_q(t)f$ for an $f=(f_1, f_2)\in\boldsymbol{L}^2(\boldsymbol{R})$ is given by

$$T_q(t)f = \begin{pmatrix} (\chi_t * f_1) + (\chi_{q,t}*f_2) \\ \chi_t * f_2\end{pmatrix}. \tag{2.24}$$

For $q\leqq 2$ these semigroups belong to (C_0) since their norm in this case is

$$\|T_q(t)\| = \sup_v |\exp\{tP_q^\wedge\}| = \sup_v\left|\begin{pmatrix} e^{-tv^2} & tv^qe^{-tv^2} \\ 0 & e^{-tv^2}\end{pmatrix}\right| = O(1) ,$$

as $t \to 0+$, and the convergence $\|T_q(t)f - f\| \to 0$ as $t \to 0+$ is trivial for each $f \in D$. For $q = 3$ or 4, however, the resolvent can be estimated from below by

$$\|\lambda R(\lambda, P_q)\| \geqq \lambda \sup_v |v|^q/(\lambda + v^2)^2 = C(q)\lambda^{(q/2)-1}$$

with a constant $C(q)$ only depending on q. Therefore,

PROPOSITION 3. *For $q \in \{3, 4\}$ the semigroup $\{T_q(t), t > 0\}$ does not belong to class (A).*

3. Further properties of semigroups of class $(C_{(k)})$. In conclusion we summarize some further properties of the two sets of $(C_{(1)})$-semigroups, which will be proved in a forthcoming paper [8]. The rate of approximation of a semigroup $T(t)f$ to f as $t \to 0+$ can, generally, be derived via interpolation from its saturation properties, i.e., from a characterization of the set $S(\{T(t)\}) = \{f \in X; \|T(t)f - f\| = O(t), t \to 0+\}$, which is called the saturation class of $\{T(t)\}$. Therefore one is interested to know whether the classical characterization for (C_0)-semigroups remains valid, namely (see [4; p. 224], [2]), for a general Banach space X,

$$S(\{T(t)\}) = D(A)^{\sim X} \tag{3.1}$$

and, in case X is reflexive, also

$$S(\{T(t)\}) = D(A)\ . \tag{3.2}$$

Here $D(A)^{\sim X}$ denotes the relative completion of the Banach subspace $D(A)$ with respect to X, cf. [2; p. 14]. It can be shown that this result extends to $(0, C_1)$-semigroups. But for $(C_{(k)})$-semigroups, $k \geqq 1$, this is no longer true.

Indeed, for the semigroups $\{T_\varphi(t), t > 0\}$ with $\varphi \in \Omega_1$ and $X = C_{2\pi}$ one can show the following relations

$$\begin{array}{ccccc} & \nearrow\!\!\!\!\!\!\subsetneqq & S(\{T_\varphi(t)\}) & = & \{f \in C_{2\pi};\ \|R_{\varphi,n}f - f\| = O(1/\varphi(n+1)),\ n \to \infty\} \\ D(A_{\varphi,0}) & & \not\cap\ \not\cup & & \supsetneqq \\ & \searrow\!\!\!\!\!\!\subsetneqq & D(A_\varphi) & \subsetneqq & D(A_\varphi)^{\sim C_{2\pi}} \end{array} \tag{3.3}$$

where $A_{\varphi,0}$ denotes the infinitesimal operator, and $D(A_\varphi)^{\sim C_{2\pi}} = \{f \in C_{2\pi};\ \exists\, g \in L^\infty_{2\pi}$ with $\varphi(|m|)f^\wedge(m) = g^\wedge(m)\ \forall\, m \in \boldsymbol{Z}\}$. In particular, (3.1) no longer holds.

For our second example $\{T_q(t), t > 0\}$, $q = 3, 4$, we have, in the reflexive space $\boldsymbol{L}^2(\boldsymbol{R})$,

$$D(P_{q,0}) \subsetneqq S(\{T_q(t)\}) \subsetneqq D(P_q)^{\sim L^2} = D(P_q)\ ,$$

so that (3.2) is incorrect, too.

Moreover, it can be shown that, for a $(C_{(k)})$-semigroup $\{T(t), t > 0\}$ on a Banach space X, which is not in $(0, C_1)$ and whose operator norm satisfies $\| T(t) \|_{[X]} = O(t^{-\alpha})$, $t \to 0+$, for some $\alpha < 1$, one generally has

$$\begin{array}{ccccc} & \subset & S(\{T(t)\}) & \subsetneqq & \\ D(A_0) & & \not\subset & & D(A)^{\sim X}\,. \\ & \subsetneqq & D(A) & \subset & \end{array}$$

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