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EXISTENCE OF SOLUTIONS AND GALERKIN APPROXIMATIONS FOR NONLINEAR FUNCTIONAL EVOLUTION EQUATIONS

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1. Introduction—Preliminaries. In this paper we are concerned with existence and approximation results for nonlinear functional evolution equations in Banach spaces. Let X be a Banach space with norm $\|\cdot\|$, and let $C = C([-r, 0], X)$ be the Banach space of continuous functions mapping the interval $[-r, 0]$, for some $r > 0$, into X with norm $\|\psi\|_C = \sup_{\theta\in[-r,0]}\|\psi(\theta)\|$. Let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. In [9] we examined the existence of a unique strong solution of the abstract initial value problem

$$\text{(FDE)} \qquad x'(t) + A(t)x(t) = G(t, x_t)\,, \qquad t \in [0, T]\,, \qquad x_0 = \phi\,,$$

where $A(t)\colon D(A(t)) = D \subset X \to X$, G satisfies a global Lipschitz condition with respect to both variables, and $\phi \in C$ is such that $\phi' \in C$ and $\phi(0) \in D$. Furthermore, we required that X^*, the dual of X, be uniformly convex and for each $t \in [0, T]$, $A(t)$ be m-accretive (see definition below) and satisfy a Kato time-dependence condition of the form

$$(*) \qquad \|A(t)x - A(s)x\| \leqq |t - s|L(\|x\|)(1 + \|A(s)x\|)$$

for all $t, s \in [0, T]$ and $x \in D$, where $L\colon R_+ = [0, \infty) \to R_+$ is a given increasing function.

By a "strong solution" of (FDE) on $[0, T]$ we mean an absolutely continuous X-valued function which, for almost all $t \in [0, T]$, is strongly differentiable and satisfies (FDE). The unique strong solution $x(t)$ of (FDE), whose existence was known from previous results, was shown in [9] to be the uniform limit of strongly continuously differentiable solutions of approximating equations for (FDE) involving the Yosida approximants of $A(t)$. In [10] a method of lines for the approximation of the solution $x(t)$ of (FDE) was developed.

Our purpose in this paper is two-fold. We first establish a local existence result for a more general nonlinear abstract functional problem of the type:

$$\text{(DE)} \qquad x'(t) + A(t, x_t)x(t) = 0\,, \qquad t \in [0, T)\,, \qquad x_0 = \phi\,,$$

where $A(t, \phi)v$ is m-accretive in v for every $(t, \phi) \in [0, T) \times C_o$, C_o a certain closed subset of C, and satisfies a local Lipschitz-type condition in t and ϕ. As an important example of our result, we obtain the local existence of a unique strong solution of (FDE) under the given conditions, but with G satisfying now a *local* Lipschitz condition. This result is still new if the Lipschitz condition is global, and an application of it is given in Section 4.

Our second goal is to establish a Galerkin method for the approximation of the solutions of (FDE) for the case of a Hilbert space X, under the additional assumptions that $A(t)$ be defined on the whole of X and map bounded subsets of X into bounded sets. Our result, Theorem 2, is an improvement of the corresponding result of Kartsatos [8], and is illustrated in Section 4 by an example involving nonlinear partial elliptic operators of order $2m$.

For $x \in X$, $x^* \in X^*$, let $\langle x, x^* \rangle$ denote the number $x^*(x)$. We define the "duality mapping" $J: X \to 2^{X^*}$ as follows:

$$Jx = \{x^* \in X^*; \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} .$$

The set Jx is nonempty by the Hahn Banach theorem. However, if X^* is uniformly convex, then the duality mapping J is single valued and is uniformly continuous on bounded subsets of X. An operator $B: D(B) \subset X \to X$ is called "accretive" if

$$\mathrm{Re}\, \langle Bu - Bv, J(u - v) \rangle \geqq 0$$

for every $u, v \in D(B)$. An accretive operator B is "m-accretive" if $R(I + \lambda B) = X$ for some (equivalently, all) $\lambda > 0$. For further properties of m-accretive operators the reader is referred to Kato [11]. We denote by $\bar{D}$ the strong closure of the set $D \subset X$.

2. Existence. In this section we give a local existence result for the initial value problem

$$\text{(DE)} \qquad x'(t) + A(t, x_t)x(t) = 0 , \qquad t \in [0, T) , \qquad x_0 = \phi$$

under the following assumptions:

(A.1) X^* is uniformly convex.

(A.2) The domain of $A_1(\cdot, \cdot, \cdot)$ with $A_1(t, \psi, v) = A(t, \psi)v$ is the set $[0, T) \times C_o \times D$, where D is a subset of X and C_o consists of all $f \in C$ with $f(t) \in \bar{D} \cup M$, $t \in [-r, 0]$. Here $M = \{\phi(t); t \in [-r, 0]\}$.

(A.3) For every $(t, \psi) \in [0, T) \times C_o$, $A(t, \psi)v$ is m-accretive in v.

(A.4) For every $t, s \in [0, T)$, $\psi_1, \psi_2 \in C_o$, $v \in D$,

$$\|A(t, \psi_1)v - A(s, \psi_2)v\| \\ \leqq l(\|\psi_1\|_C, \|\psi_2\|_C, \|v\|)[|t - s|(1 + \|A(s, \psi_2)v\|) + \|\psi_1 - \psi_2\|_C] ,$$

where $l\colon R_+^3 \to R_+$ is increasing in all three variables.
(A.5) $\phi \in C_0$ is a given function with $\phi(0) \in D$ satisfying a Lipschitz condition on $[-r, 0]$ with Lipschitz constant K.

Our method in proving the existence of $x(t)$ follows that of Kartsatos [7], where the equation $x'(t) + A(t, x(t))x(t) = 0$ was studied. We first ensure the existence of the solution $x_u(t)$ of the problem

$$(\mathrm{DE})_u \qquad x'(t) + A(t, u_t)x(t) = 0\,, \qquad x_0 = \phi$$

on an interval $[0, T_1]$, where u is taken from a suitable metric space S of continuous functions. We then show that, for T_1 sufficiently small, the operator $U\colon u \to x_u$ maps the space S into itself and is a strict contraction. The resulting fixed point of this operator is the desired unique strong solution of (DE).

THEOREM 1. *Assume that Conditions* (A.1)–(A.5) *are satisfied. Then there exists* $T_1 < T$ *such that the initial value problem* (DE) *has a unique strong solution* $x(t)$, $t \in [0, T_1]$, *which is also Lipschitz continuous on* $[0, T_1]$.

PROOF. Let $N = 1 + \|A(0, \phi)\phi(0)\|$ and L be a positive constant with $L/N < T$. Let T_1 be such that $0 < T_1 \leqq L/N$. Consider the set

$$S = \{u\colon [-r, T_1] \to \bar{D} \cup M;\ u(t) \text{ is continuous, } u(t) = \phi(t) \text{ for } t \in [-r, 0] \text{ and } \|u(t_1) - u(t_2)\| \leqq N|t_1 - t_2| \text{ for } t_1, t_2 \in [0, T_1]\}\,.$$

$S \neq \emptyset$ because the function $u(t)$ such that $u(t) = \phi(t)$ for $t \in [-r, 0]$ and $u(t) \equiv \phi(0)$ for $t \in [0, T_1]$ belongs to S. Now, let $u \in S$ be given and consider the problem $(\mathrm{DE})_u$ on the interval $[0, T_1]$. Let $N_1 = \max\{N, K\}$. The operator $B_u(t)v \equiv A(t, u_t)v$ is m-accretive in v by Condition (A.3). Also, by Condition (A.4),

$$(1) \qquad \|B_u(t)v - B_u(s)v\| \leqq l(\|u_t\|_C, \|u_s\|_C, \|v\|)[|t - s|(1 + \|A(s, u_s)v\|) + \|u_t - u_s\|_C]$$

for every $t \in [0, T_1]$. Now, in order to show that B_u satisfies a condition like $(*)$ (in the introduction), we first observe that

$$\|u_t - u_s\|_C = \sup_{\theta \in [-r, 0]} \|u(t + \theta) - u(s + \theta)\|$$

for every $t, s \in [0, T_1]$. Suppose that $t, s \geqq r$. Then, for each $\theta \in [-r, 0]$, $\|u(t + \theta) - u(s + \theta)\| \leqq N|t - s| \leqq N_1|t - s|$. Suppose that $t, s < r$. Without loss of generality, assume that $t > s$. If $\theta \in [-r, -t]$, then $t + \theta \in [t - r, 0]$ and $s + \theta \in [s - r, s - t]$. For such θ, $\|u(t + \theta) - u(s + \theta)\| = \|\phi(t + \theta) - \phi(s + \theta)\| \leqq K|t - s| \leqq N_1|t - s|$. If $\theta \in [-t, -s]$, then $t + \theta \in [0, t - s]$

and $s+\theta\in[s-t, 0]$. For such θ, $\|u(t+\theta)-u(s+\theta)\| \leqq \|u(t+\theta)-u(0)\| + \|u(0)-u(s+\theta)\| \leqq N|t+\theta| + K|s+\theta| \leqq N_1|t-s|$. If $\theta\in[-s, 0]$, then $t+\theta\in[t-s, t]$ and $s+\theta\in[0, s]$, which implies again that the above inequality is true. Hence, for all $t, s<r$ and $\theta\in[-r, 0]$, we have $\sup_{\theta\in[-r,0]}\|u(t+\theta)-u(s+\theta)\| \leqq N_1|t-s|$. The same inequality holds if we assume that $t\geqq r$ and $s\leqq r$. The proof of this fact is similar to the above. It is therefore omitted.

In order to obtain a bound for u_t, we observe that since $u\in S$, we have $\|u(t+\theta)-u(0)\| \leqq Nt \leqq L$ for every $t\in[0, T_1]$ and every $\theta\in[-r, 0]$ such that $t+\theta\geqq 0$. Thus, for such t and θ, $\|u(t+\theta)\| \leqq \|\phi(0)\| + L \leqq \|\phi\|_C + L$. For t and θ such that $t+\theta<0$, $\|u(t+\theta)-\phi(0)\| \leqq \|\phi(t+\theta)\| + \|\phi(0)\| \leqq 2\|\phi\|_C$. It follows that for all $t\in[0, T_1]$, $\theta\in[-r, 0]$ we have the bound:

$$\|u_t\|_C = \sup_{\theta\in[-r,0]}\|u(t+\theta)\| \leqq 2\|\phi\|_C + L\,.$$

Using these estimates and (1), we obtain

$$(2)\qquad \|B_u(t)v - B_u(s)v\| \leqq l_1(\|v\|)|t-s|(1+\|B_u(s)v\|)\,,$$

where $l_1(\|v\|) = (1+N_1)l(2\|\phi\|_C+L, 2\|\phi\|_C+L, \|v\|)$. Consequently, the conditions of Theorems 1 and 2 of Kato [11] are satisfied. Thus, the problem $(\mathrm{DE})_u$ has a unique strong solution $x_u(t)$ on $[0, T_1]$. The function $x_u(t)$ is also weakly continuously differentiable on $[0, T_1]$ and such that $A(t, u_t)x_u(t)$ is weakly continuous in t. Furthermore, $x_u(t)$ satisfies $(\mathrm{DE})_u$ everywhere on $[0, T_1]$ if $x'(t)$ denotes now the weak derivative of $x(t)$.

We are planning to show that the operator $U: u\to x_u$ is a strict contraction on S if T_1 is chosen sufficiently small. To this end, fix $u\in S$ and consider the approximating equations

$$(\mathrm{E})_u \qquad x_n'(t) + A_n(t)x_n(t) = 0\,, \qquad x_{n_0} = \phi\,,$$

where $A_n(t) = A_n(t, u_t) = A(t, u_t)[I+(1/n)A(t, u_t)]^{-1}$, $n = 1, 2, \cdots$, are the Yosida approximants of $A(t, u_t)$. The operators $A_n(t)$ are defined and Lipschitz continuous on X with Lipschitz constants $\leqq 2n$. Moreover, the operators $J_n(t) = [I+(1/n)A(t, u_t)]^{-1}: X\to D$ are also Lipschitz continuous on X with Lipschitz constants $\leqq 1$. Since $B_u(t)$ is m-accretive for each $t\in[0, T_1]$, so are the operators $A_n(t)$ [11, Lemma 2.3]. Also, as in Lemma 4.1 of the same reference, we obtain

$$\|A_n(t)v - A_n(s)v\| = l_1(\|J_n(s)v\|)|t-s|(1+\|A_n(s)v\|)\,.$$

Since, by (2),

$$\begin{aligned}(1/n)\|A_n(s)v\| &\leqq (1/n)\|A_n(s)\phi(0)\| + 2\|v - \phi(0)\| \\ &\leqq \|A_n(s, u_s)\phi(0)\| + 2(\|v\| + \|\phi(0)\|) \\ &\leqq \|A(0, \phi)\phi(0)\| + l_1(\|\phi(0)\|)(L/N)(1 + \|A(0, \phi)\phi(0)\|) \\ &\quad + 2(\|v\| + \|\phi(0)\|) \\ &= K_1 + 2\|v\|\end{aligned}$$

and $\|J_n(s)v\| \leqq \|v\| + (1/n)\|A_n(s)v\|$, we finally arrive at

$$\begin{aligned}(3)\qquad \|A_n(t, u_t)v - A_n(s, u_s)v\| &= \|A_n(t)v - A_n(s)v\| \\ &\leqq l_2(\|v\|)|t - s|(1 + \|A_n(s)v\|)\,,\end{aligned}$$

where $l_2(\|v\|) = l_1(3\|v\| + K_1)$. Hence each of the equations $(\mathrm{E})_n$ has a unique strongly continuously differentiable solution $x_n(t)$ defined on $[0, T_1]$ and such that $\lim_{n\to\infty} x_n(t) = x_u(t)$ strongly and uniformly on $[0, T_1]$ (cf. Kato [11]).

We shall show that the sequence $\{x_n(t)\}$, $n = 1, 2, \cdots$, is uniformly bounded and uniformly Lipschitz continuous on $[0, T_1]$ independently of $u \in S$. To this end, using [11, Lemma 1.3], the accretiveness of $A_n(t)$ and (3), we get

$$\begin{aligned}2\|x_n(t) - \phi(0)\|(d/dt)\|x_n(t) - \phi(0)\| &= (d/dt)\|x_n(t) - \phi(0)\|^2 \\ &= 2\,\mathrm{Re}\,\langle x_n'(t), J(x_n(t) - \phi(0))\rangle \\ &= -2\,\mathrm{Re}\,\langle A_n(t)x_n(t) - A_n(t)\phi(0), J(x_n(t) - \phi(0))\rangle \\ &\quad - 2\,\mathrm{Re}\,\langle A_n(t)\phi(0), J(x_n(t) - \phi(0))\rangle \\ &\leqq 2\|A_n(t)\phi(0)\|\,\|x_n(t) - \phi(0)\| \\ &\leqq 2[\|A_n(t, u_t)\phi(0) - A_n(0, \phi)\phi(0)\| + \|A_n(0, \phi)\phi(0)\|]\|x_n(t) - \phi(0)\| \\ &\leqq 2[\|A_n(0, \phi)\phi(0)\| + l_2(\|\phi(0)\|)T_1(1 + \|A_n(0, \phi)\phi(0)\|)]\|x_n(t) - \phi(0)\| \\ &\leqq 2[\|A(0, \phi)\phi(0)\| + l_2(\|\phi(0)\|)(L/N)(1 + \|A(0, \phi)\phi(0)\|)]\|x_n(t) - \phi(0)\|\,.\end{aligned}$$

This inequality holds a.e. in $[0, T_1]$. Dividing by $2\|x_n(t) - \phi(0)\|$ and integrating from 0 to $t \leqq T_1$, we obtain

$$(4)\qquad \|x_n(t) - \phi(0)\| \leqq K_2 T_1\,,$$

where $K_2 = \|A(0, \phi)\phi(0)\| + l_2(\|\phi(0)\|)(L/N)(1 + \|A(0, \phi)\phi(0)\|)$ is independent of T_1, n and $u \in S$. In order to find a uniform upper bound for the derivatives $x_n'(t)$, we consider the function $z_n(t) \equiv x_n(t + h) - x_n(t)$, $0 \leqq t$, $t + h < T_1$. Using again Lemma 1.3 of Kato [11], the accretiveness of $A_n(t + h)$, the uniform boundedness of $\{x_n(t)\}$ from (4), and the appraisal (3), we get

$$\begin{aligned}(1/2)(d/dt)\|z_n(t)\|^2 &= \mathrm{Re}\,\langle z_n'(t), J(z_n(t))\rangle \\ &= -\mathrm{Re}\,\langle A_n(t + h)x_n(t + h) - A_n(t)x_n(t), J(z_n(t))\rangle\end{aligned}$$

$$
\begin{aligned}
&= -\mathrm{Re}\,\langle A_n(t+h)x_n(t+h) - A_n(t+h)x_n(t),\, J(z_n(t))\rangle \\
&\quad - \mathrm{Re}\,\langle A_n(t+h)x_n(t) - A_n(t)x_n(t),\, J(z_n(t))\rangle \\
&\leqq \|A_n(t+h,\, u_{t+h})x_n(t) - A_n(t,\, u_t)x_n(t)\|\,\|z_n(t)\| \\
&\leqq |h|\,l_2(\|x_n(t)\|)(1 + \|A_n(t)x_n(t)\|)\,\|z_n(t)\| \\
&\leqq |h|\,l_2(\|\phi(0)\| + K_2T_1)(1 + \|x_n'(t)\|)\,\|z_n(t)\|\,.
\end{aligned}
$$

Dividing through by $|h|\,\|z_n(t)\|$, integrating and then passing to the limit as $h \to 0$, we get

$$\|x_n'(t)\| = \|x_n'(0)\| + \int_0^t l_2(\|\phi(0)\| + K_2T_1)\,\|x_n'(s)\|\,ds + l_2(\|\phi(0)\| + K_2T_1)T_1\,.$$

Applying Gronwall's inequality, we find

$$\|x_n'(t)\| \leqq (K_3T_1 + K_4)e^{K_3T_1}\,, \tag{5}$$

where $K_3 = l_2(\|\phi(0)\| + K_2(L/N))$ is independent of T_1, n and $u \in S$, and $K_4 = \|A(0,\, \phi)\phi(0)\|$. From (4) and (5) we conclude that

$$\|x_u(t)\| \leqq \|\phi(0)\| + K_2T_1\,, \qquad \|x_u(t_1) - x_u(t_2)\| \leqq K_5|t_1 - t_2|$$

for every $t_1,\, t_2 \in [0,\, T_1]$, where K_5 is the right hand side of (5).

Now, let $u_1,\, u_2 \in S$ be given and let $x_1,\, x_2$ be the corresponding solutions of $(\mathrm{DE})_{u_i}$, $i = 1, 2$. Then we have

$$
\begin{aligned}
(6)\qquad &(1/2)(d/dt)\|x_1(t) - x_2(t)\|^2 \\
&= -\mathrm{Re}\,\langle A(t,\, u_{1_t})x_1(t) - A(t,\, u_{2_t})x_2(t)\,,\, J(x_1(t) - x_2(t))\rangle \\
&\leqq -\mathrm{Re}\,\langle A(t,\, u_{1_t})x_2(t) - A(t,\, u_{2_t})x_2(t),\, J(x_1(t) - x_2(t))\rangle \\
&\leqq l(\|u_{1_t}\|_C,\, \|u_{2_t}\|_C,\, \|x_2(t)\|)\,\|u_{1_t} - u_{2_t}\|_C\,\|x_1(t) - x_2(t)\|
\end{aligned}
$$

from which, dividing by $\|x_1(t) - x_2(t)\|$ and then integrating, we arrive at

$$\|x_1(t) - x_2(t)\| \leqq K_6 \sup_{t \in [0, T_1]} \|u_{1_t} - u_{2_t}\|_C\,,$$

where $K_6 = T_1 l(2\|\phi\|_C + L,\; 2\|\phi\|_C + L,\; \|\phi(0)\| + K_2(L/N))$. Since $\sup_{t \in [0,T_1]} \|u_{1_t} - u_{2_t}\|_C = \sup_{t \in [0,T_1]} \|u_1(t) - u_2(t)\|$, we conclude that

$$\sup_{t \in [0,T_1]} \|x_1(t) - x_2(t)\| \leqq K_6 \sup_{t \in [0,T_1]} \|u_1(t) - u_2(t)\|\,.$$

Now, we choose T_1 so small that $K_5 \leqq N$ and $K_6 < 1$. Then the operator $U\colon u \to x_u$ is a strict contraction on a complete metric space. Let $x(t)$, $t \in [0,\, T_1]$ be the unique fixed point of U. Then $x(t)$ is the desired solution of the problem (DE). Its uniqueness follows from (6) by replacing $u_1,\, u_2$ by $x_1,\, x_2$, respectively.

The above result can be extended to include the infinite (unbounded) delay version of (DE). In fact, in that case we let C equal the space of all bounded and uniformly continuous functions $f: (-\infty, 0] \to X$ with the sup-norm. Moreover, we let C_o be now the space of all $f \in C$ such that $f(t) \in \bar{D} \cup M$, $t \in (-\infty, 0]$, where $M = \{\phi(t); t \in (-\infty, 0]\}$. The proof of this result follows as above and is therefore omitted.

3. Galerkin approximations. In this section we consider a Galerkin approximation scheme for the solution of the abstract initial value problem

(FDE) $$x'(t) + A(t)x(t) = G(t, x_t)\ , \qquad t \in [0, T]$$

with $X = H$, a real Hilbert space, $\phi \in C$ such that $\phi' \in C$ and the operators $A(t)$ and G satisfying the following conditions:

(C.1) For each $t \in [0, T]$, $A(t): H \to H$ is m-accretive.

(C.2) There exists a nondecreasing function $L_1: R_+ \to R_+$ such that

$$\|A(t)x - A(s)x\| \leqq |t - s| L_1(\|x\|)(1 + \|A(s)x\|)\ .$$

(C.3) $A(0)$ maps bounded sets into bounded sets.

(C.4) There exists a constant $b > 0$ such that for every $\phi, \psi \in C$, $t \in [0, T]$, $\|G(t, \phi) - G(t, \psi)\| \leqq b\|\phi - \psi\|_C$.

(C.5) There exists $L_2: R_+ \to R_+$, nondecreasing and such that for every $s, t \in [0, T]$, $\phi \in C$, $\|G(t, \phi) - G(s, \phi)\| \leqq L_2(\|\phi\|_C)|t - s|$.

Under the assumptions (C.1), (C.2), (C.4) and (C.5) we have the existence of a unique strong solution of (FDE) on $[0, T]$, for example, by [9, Theorem 2.1]. In what follows, the space H is separable. Let $e_1, e_2, \cdots$ be a basis of H and let H_n be the subspace of H spanned by the vectors $e_1, e_2, \cdots, e_n$. Let $P_n: H \to H_n$ be the projection on H_n. We consider the (finite dimensional) approximating problems

(FDE)$_n$ $$x_n'(t) + P_nA(t)x_n(t) = P_nG(t, x_{n_t})\ , \qquad t \in [0, T]\ ,$$
$$x_n(t) = P_n\phi(t)\ , \qquad t \in [-r, 0]\ .$$

The Galerkin method has been already used by other authors to obtain the existence and/or approximation of solutions of nonlinear evolution equations. We should mention here the paper of Browder [3], where the Galerkin method was used to obtain the existence of the unique solution of the problem

$$x'(t) + A(t)x(t) = 0\ , \qquad t \in R_+\ ,$$
$$x(0) = x_o\ .$$

Here, $A(t)$ is a continuous m-accretive (thus maximal monotone) operator defined on the whole of H and mapping bounded sets into bounded sets.

Gajewski and Zacharias established in [5] the convergence of the Galerkin approximants for the unique strong solution of the perturbed evolution equation

$$x'(t) + A(t)x(t) = G(t, x(t)) , \qquad t \in [0, T] ,$$
$$x(0) = x_0 .$$

Their results where extended by Kartsatos [8] to operators $A(t)$ defined on a proper subset of H. Abstract semigroup theory has been the setting for applying the Galerkin method in Banks [1], [2], Kappel and Schappacher [6] and Webb [13]. These authors have considered equations that fall into the type:

$$x'(t) = f(t, x_t) + g(t) , \qquad t \in [0, T] .$$

Their approach in these papers is to consider an abstract equation in the space of initial functions involving an operator which generates a nonlinear semigroup on that space. The Galerkin approximations are then given for that equation.

We note that, in our case, since $A(t)$ is defined on the whole space, it is demicontinuous, i.e., it is continuous from the strong topology of H to the weak topology of H [12, p. 107].

In what follows the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product of H. We should also remark that $P_n y(t) \to y(t)$ strongly and uniformly as $n \to \infty$ for any continuous function $y: [a, b] \subset [-r, T] \to H$.

THEOREM 2. *Assume that Conditions* (C.1)-(C.5) *are satisfied. Then the sequence* $\{x_n(t)\}$ *of the Galerkin approximants satisfying* $(\text{FDE})_n$ *exists and converges strongly and uniformly to the unique solution* $x(t)$ *of* (FDE).

PROOF. As we mentioned above, the unique strong solution $x(t)$ of (FDE) exists by [9, Theorem 2.1]. We note that in $(\text{FDE})_n$ the operator $P_n A(t)$ is accretive on H_n. Since it is also demicontinuous on H, it is continuous on H_n. Thus, by a well known result, $P_n A(t)$ is m-accretive on H_n. It is also easy to see that $P_n A(t)$ is Lipschitz continuous in t, satisfying a condition similar, but not identical, to (C.3). Since the projection P_n has norm 1, the function $P_n G(t, \phi)$ satisfies the Lipschitz conditions (C.4) and (C.5). With these facts established, the existence of the unique strong solution of $(\text{FDE})_n$ is guaranteed by the following argument. Consider the equations

$$(\text{FDE})_{mn} \qquad u'_{mn}(t) + P_n A_m(t) u_{mn}(t) = P_n G(t, u_{mn_t}) , \qquad t \in [0, T] ,$$
$$u_{mn_0} = x_{n_0} ,$$

where $x_{n_0}(\theta) = x_n(\theta) = P_n\phi(\theta)$, $\theta \in [-r, 0]$, and $A_m(t)$ are the Yoshida approximants of $A(t)$. Following the proof of Lemma 2.3 of [9], we can show that, for a fixed n, the (unique) solutions $u_{mn}(t)$ ($u_{mn}(t) \in H_n$, $m = 1, 2, \cdots$, $t \in [0, T]$) of the problems $(\mathrm{FDE})_{mn}$ are uniformly bounded. On the other hand, since $P_nA(t)x$ is continuous on the set $[0, T] \times H_n$, it maps bounded subsets of it into bounded subsets of H_n. Using this fact, we can easily see that there exists a constant $K_n > 0$ such that $\|P_nA_m(t)u_{mn}(t)\| \leqq K_n$ for every $m = 1, 2, \cdots$ and every $t \in [0, T]$. This in turn implies that there exists a constant $L_n > 0$ such that the functions $u'_{mn}(t)$, given by $(\mathrm{FDE})_{mn}$, satisfy: $\|u'_{mn}(t)\| \leqq L_n$ for every $m = 1, 2, \cdots$ and every $t \in [0, T]$. The uniform convergence of $u_{mn}(t)$ and $u'_{mn}(t)$ to $x_n(t)$ and $x'_n(t)$, for $m \to \infty$, respectively, follows now almost exactly as in [9]. Its proof is therefore omitted.

In order to show that the sequence $\{x_n(t)\}$ is uniformly bounded, we start with the inequality

$$\begin{aligned}
(1/2)(d/dt)\|x_n(t) - P_n\phi(0)\|^2 &= \langle x'_n(t), x_n(t) - P_n\phi(0)\rangle \\
&= -\langle P_nA(t)x_n(t), x_n(t) - P_n\phi(0)\rangle + \langle P_nG(t, x_{n_t}), x_n(t) - P_n\phi(0)\rangle \\
&= -\langle A(t)x_n(t), x_n(t) - P_n\phi(0)\rangle + \langle G(t, x_{n_t}), x_n(t) - P_n\phi(0)\rangle \\
&= -\langle A(t)x_n(t) - A(t)P_n\phi(0), x_n(t) - P_n\phi(0)\rangle \\
&\quad - \langle A(t)P_n\phi(0), x_n(t) - P_n\phi(0)\rangle + \langle G(t, x_{n_t}), x_n(t) - P_n\phi(0)\rangle \\
&\leqq \|A(t)P_n\phi(0)\|\,\|x_n(t) - P_n\phi(0)\| + \|G(t, x_{n_t}) - G(t, x_{n_0})\|\,\|x_n(t) - P_n\phi(0)\| \\
&\quad + \|G(t, x_{n_0})\|\,\|x_n(t) - P_n\phi(0)\| \\
&\leqq \|A(0)P_n\phi(0)\|\,\|x_n(t) - P_n\phi(0)\| \\
&\quad + TL_1(\|P_n\phi(0)\|)(1 + \|A(0)P_n\phi(0)\|)\|x_n(t) - P_n\phi(0)\| \\
&\quad + b\|x_{n_t} - x_{n_0}\|_C\|x_n(t) - P_n\phi(0)\| + \|G(t, x_{n_0})\|\,\|x_n(t) - P_n\phi(0)\|
\end{aligned}$$

which implies

$$\begin{aligned}
&(d/dt)\|x_n(t) - P_n\phi(0)\| \\
&\quad \leqq \|A(0)P_n\phi(0)\| + TL_1(\|P_n\phi(0)\|)(1 + \|A(0)P_n\phi(0)\|) + \|G(t, x_{n_0})\| + b\|x_{n_t} - x_{n_0}\|_C \\
&\quad \leqq K + b\|x_{n_t} - x_{n_0}\|_C\,,
\end{aligned}$$

a.e. in $[0, T]$, where K is a positive constant. Here we have used the boundedness of $A(0)$ and $G(t, x_{n_0})$ on $[0, T]$. Integrating, we obtain

$$\|x_n(t) - P_n\phi(0)\| \leqq KT + b\int_0^t \|x_{n_s} - x_{n_0}\|_C ds\,.$$

Thus, for any $t_1 \in [0, t]$, we have

$$\|x_n(t_1) - P_n\phi(0)\| \leqq KT + b\int_0^{t_1} \|x_{n_s} - x_{n_0}\|_C ds$$

$$\leqq KT + b\int_0^t \|x_{n_s} - x_{n_0}\|_C ds .$$

If $t_1 \in [-r, 0]$, then $\|x_n(t_1) - P_n\phi(0)\| \leqq \|P_n\phi(t_1)\| + \|P_n\phi(0)\| \leqq 2\|\phi\|_C$. Hence

$$\sup_{\theta\in[-r,0]} \|x_n(t+\theta) - P_n\phi(0)\| \leqq KT + 2\|\phi\|_C + b\int_0^t \|x_{n_s} - x_{n_0}\|_C ds ,$$

which implies

$$\begin{aligned}\|x_{n_t} - x_{n_0}\|_C &= \sup_{\theta\in[-r,0]} \|x_n(t+\theta) - x_{n_0}(\theta)\| \\ &\leqq \sup_{\theta\in[-r,0]} \|x_n(t+\theta) - P_n\phi(0)\| + \sup_{\theta\in[-r,0]} \|P_n\phi(0) - x_{n_0}\| \\ &\leqq KT + 4\|\phi\|_C + b\int_0^t \|x_{n_s} - x_{n_0}\|_C ds .\end{aligned}$$

Applying Gronwall's inequality above, we obtain the boundedness of $\{x_{n_t} - x_{n_0}\}$, which implies the boundedness of $\{x_n(t)\}$. We are now ready to show the convergence of $x_n(t)$ to $x(t)$ uniformly on $[0, T]$. We first observe that

$$(7)\qquad \langle x_n'(t), x_n(t) - x(t)\rangle + \langle P_nA(t)x_n(t), x_n(t) - x(t)\rangle = \langle P_nG(t, x_{n_t}), x_n(t) - x(t)\rangle$$

$$(8)\qquad \langle x'(t), x_n(t) - x(t)\rangle + \langle A(t)x(t), x_n(t) - x(t)\rangle = \langle G(t, x_t), x_n(t) - x(t)\rangle .$$

Subtracting (8) from (7), we find

$$\begin{aligned}&\langle x_n'(t) - x'(t), x_n(t) - x(t)\rangle \\ &\quad = -\langle A(t)x_n(t), x_n(t) - P_nx(t)\rangle + \langle A(t)x(t), x_n(t) - x(t)\rangle \\ &\qquad + \langle G(t, x_{n_t}), x_n(t) - P_nx(t)\rangle - \langle G(t, x_t), x_n(t) - x(t)\rangle \\ &\quad = -\langle A(t)x_n(t) - A(t)x(t), x_n(t) - x(t)\rangle - \langle A(t)x_n(t), x(t) - P_nx(t)\rangle \\ &\qquad + \langle G(t, x_{n_t}) - G(t, x_t), x_n(t) - x(t)\rangle + \langle G(t, x_{n_t}), x(t) - P_nx(t)\rangle ,\end{aligned}$$

which implies

$$\begin{aligned}&(1/2)(d/dt)\|x_n(t) - x(t)\|^2 \\ &\quad \leqq \|A(t)x_n(t)\|\,\|x(t) - P_nx(t)\| + b\|x_{n_t} - x_t\|_C^2 + \|G(t, x_{n_t})\|\,\|x(t) - P_nx(t)\| .\end{aligned}$$

Integrating this inequality, we arrive at

$$(9)\qquad \|x_n(t) - x(t)\|^2 \leqq \|x_n(0) - x(0)\|^2 + \int_0^T h(t)dt + 2b\int_0^t \|x_{n_s} - x_s\|_C^2 ds ,$$

where $h(t) = 2\|A(t)x_n(t)\|\,\|x(t) - P_nx(t)\| + 2\|G(t, x_{n_t})\|\,\|x(t) - P_nx(t)\|$.

Since the above inequality holds for any $t_1 \in [0, t]$, and since, for $t_1 \in [-r, 0]$,

$$\| x_n(t_1) - x(t_1) \| = \| P_n\phi(t_1) - \phi(t_1) \| \leqq \sup_{\theta \in [-r,0]} \| P_n\phi(\theta) - \phi(\theta) \| ,$$

we actually have

$$\| x_n(t) - x(t) \|^2 \leqq \| x_n(0) - x(0) \|^2 + \sup_{\theta \in [-r,0]} \| P_n\phi(\theta) - \phi(\theta) \|^2 + \int_0^T h(t)dt + 2b \int_0^t \| x_{n_s} - x_s \|_C^2 ds , \qquad t \in [-r, T] .$$

Consequently, by Gronwall's inequality, we get

$$\sup_{\theta \in [-r,0]} \| x_{n_t}(\theta) - x_t(\theta) \| \leqq \left[\| x_n(0) - x(0) \|^2 + \sup_{\theta \in [-r,0]} \| P_n\phi(\theta) - \phi(\theta) \|^2 + \int_0^T h(t)dt \right] e^{2bt} .$$

Now, $x_n(0) - x(0) = P_n\phi(0) - \phi(0) \to 0$ as $n \to \infty$ and $P_n\phi(\theta) - \phi(\theta) \to 0$ uniformly on $[-r, 0]$. In addition, for the three normed expressions in $h(t)$ we have the following properties. From the boundedness of $A(0)$ and the inequality $\|A(t)x_n(t)\| \leqq \|A(0)x_n(t)\| + TL_1(\|x_n(t)\|)(1 + \|A(0)x_n(t)\|)$ we obtain the uniform boundedness of $\{A(t)x_n(t)\}$. The uniform boundedness of $\{x_{n_t}\}$ implies the same property for $\{G(t, x_{n_t})\}$. Finally, $P_nx(t) \to x(t)$ uniformly on $[0, T]$. Thus, an application of Lebesgue's bounded convergence theorem shows that

$$\sup_{\theta \in [-r,0]} \| x_{n_t}(\theta) - x_t(\theta) \| \to 0 \quad \text{as} \quad n \to \infty ,$$

which in turn shows that $x_n(t) \to x(t)$ uniformly on $[0, T]$.

It should be noted here that we do not assume that $A(t)P_nx \to A(t)x$ for every $x \in H$. This assumption is actually included in the result of Kartsatos [8] if the domain of $A(t)$ there is the whole of H. Also, the constant b in (C.5) can be replaced by a Lebesgue integrable function $b: [0, T] \to R_+$.

4. Applications. As an example to which we can apply our result of Section 3, we cite the nonlinear initial-boundary value problem:

$$\text{(E)} \quad \begin{aligned} &(\partial/\partial t)u(x, t) + A(t, x, u(x, t)) = f(t, x, u(x, t - r)) , \quad t \in (0, T) , \quad x \in \Omega , \\ &u(x, \theta) = \phi(x, \theta) , \qquad x \in \Omega , \qquad \theta \in [-r, 0] , \\ &D^\alpha u(x, t) = 0 , \qquad x \in \partial\Omega , \qquad t \in (0, T] , \qquad |\alpha| < m , \end{aligned}$$

where u is a real valued function, r is a positive constant, Ω is a bounded open subset of R^n ($R = (-\infty, \infty)$, $n \geqq 2$) with sufficiently smooth boundary, $A(t, x, u)$, $f(t, x, u)$ are nonlinear elliptic partial differential operators in divergence form:

$$A(t, x, u) = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} b_\alpha(t) D^\alpha A_\alpha(x, \xi(u)) ,$$
$$f(t, x, u) = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha f_\alpha(t, x, \xi(u)) ,$$

and $\phi: \Omega \times [-r, 0] \to R$ is a given function. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers we adopt the notation

$$|\alpha| = \alpha_1 + \cdots + \alpha_n , \qquad D_i = (\partial/\partial x_i) , \qquad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} .$$

By R^{n_m} we denote the space of all real vectors of the form $\xi = \{\xi_\alpha; |\alpha| \leqq m\}$. Thus, $\xi(u) = \{D^\alpha u; |\alpha| \leqq m\}$.

For the results concerning such partial differential operators the reader is referred, for example, to Browder [4] and Pascali and Sburlan [12].

Now, let $W^{m,2}(\Omega)$ be the Sobolev space of all real valued functions u such that $D^\alpha u \in L^2(\Omega)$ for every α with $|\alpha| \leqq m$. $W^{m,2}(\Omega)$ is a separable Hilbert space with inner product

$$\langle u, v \rangle_m = \sum_{|\alpha| \leqq m} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)} .$$

Let $C_C^\infty(\Omega)$ be the space of all $f \in C^\infty(\Omega)$ with compact support. We denote by $W_o^{m,2}(\Omega)$ the closure of the space $C_C^\infty(\Omega)$ in $W^{m,2}(\Omega)$. The space $W_o^{m,2}(\Omega)$ is thus another separable Hilbert space. We let H denote this space and we make the following additional assumptions:

(i) for each α, $A_\alpha: \Omega \times R^{n_m} \to R$ satisfies the Caratheodory conditions and there exists a function $g \in L^2(\Omega)$ and a constant $c > 0$ such that

$$|A_\alpha(x, \xi)| \leqq c|\xi| + g(x) , \qquad (x, \xi) \in \Omega \times R^{n_m} ,$$

where $|\xi| = (\sum_{|\alpha| \leqq m} \xi_\alpha^2)^{1/2}$.

(ii) For $x \in \Omega$ and $\xi, \xi' \in R^{n_m}$ we have

$$\sum_{|\alpha| \leqq m} [A_\alpha(x, \xi) - A_\alpha(x, \xi')](\xi_\alpha - \xi'_\alpha) \geqq 0 .$$

(iii) Each $b_\alpha: [0, T] \to R_+$ is Lipschitz continuous on $[0, T]$, $\phi(\cdot, \theta) \in W_o^{m,2}(\Omega)$ for every $\theta \in [-r, 0]$, $\phi(x, \theta)$ is continuous and satisfies a Lipschitz condition with respect to θ uniformly in $x \in \Omega$.

(iv) The functions $f_\alpha: [0, T] \times \Omega \times R^{n_m} \to R$ are continuous and such that: there exists a nonnegative function $h \in L^2(\Omega)$ and a constant $L > 0$ with

$$|f_\alpha(t, x, \xi) - f_\alpha(t', x, \xi')| \leqq h(x)|t - t'| + L|\xi - \xi'|$$

for every $t, t' \in [0, T]$, $x \in \Omega$, and $\xi, \xi' \in R^{n_m}$.

If for each $t \in [0, T]$, $u, v \in W_o^{m,2}(\Omega)$ we let

$$a^t(u, v) = \sum_{|\alpha| \leqq m} b_\alpha(t) \int_\Omega A_\alpha(x, \xi(u(x))) D^\alpha v(x) dx ,$$

then $a^t(u, v)$ is a bounded linear functional in v. By the Riesz representation theorem, there exists a nonlinear operator $T(t): H \to H$ such that

$$\langle T(t)u, v \rangle_m = a^t(u, v) , \qquad (u, v) \in H \times H .$$

The operator $T(t)$ is continuous, m-accretive, and maps bounded subsets of H into bounded sets for each $t \in [0, T]$. The proof of this fact follows as in [12, p. 275]. It is also easy to see that T satisfies the condition (C.2). Similarly, we can obtain an operator $F(t): H \to H$ such that

$$\langle F(t)u, v \rangle_m = \sum_{|\alpha| \leqq m} \int_\Omega f_\alpha(t, x, \xi(u(x))) D^\alpha v(x) dx$$

for every $t \in [0, T]$, $u, v \in W_o^{m,2}(\Omega)$. In order to show that $F(t)u$ satisfies a global Lipschitz condition on $[0, T] \times W_o^{m,2}(\Omega)$, we observe that

$$\begin{aligned}
&\left| \int_\Omega [f_\alpha(t, x, \xi(u(x))) - f_\alpha(t', x, \xi(v(x)))] D^\alpha v(x) dx \right| \\
&\leqq \left(\int_\Omega [f_\alpha(t, x, \xi(u(x))) - f_\alpha(t', x, \xi(v(x)))]^2 dx \right)^{1/2} \cdot \| v \|_{m,2} \\
&\leqq \left(\int_\Omega [h(x) | t - t' | + L | \xi(u(x)) - \xi(v(x)) |]^2 dx \right)^{1/2} \cdot \| v \|_{m,2} \\
&\leqq \left(\left(\int_\Omega h^2(x) dx \right)^{1/2} | t - t' | + L \| u - v \|_{m,2} \right) \| v \|_{m,2} \\
&= (K | t - t' | + L \| u - v \|_{m,2}) \| v \|_{m,2} ,
\end{aligned}$$

where $\| \cdot \|_{m,2}$ is the norm of $W_o^{m,2}(\Omega)$ and K is an obvious constant. Adding these inequalities, we obtain our assertion.

Now, we consider the abstract problem

$$\text{(AE)} \qquad \begin{aligned} &u'(t) + T(t)u(t) = G(t, u_t) , \qquad t \in [0, T] , \\ &u_0 = \phi , \end{aligned}$$

where $G(t, \psi) = F(t)\psi(-r)$, for any $\psi \in C$, and $u'(t)$ denotes the weak derivative of $u(t)$. Since the conditions (C.1)–(C.5) are satisfied, the unique strong solution of (AE) can be approximated by the Galerkin method.

As an application of Theorem 1, we consider the initial-boundary value problem consisting of the equation

$$(\partial/\partial)u(x, t) + A(t, x, u(x, t - r), u(x, t)) = 0 , \qquad t \in (0, T) , \qquad x \in \Omega$$

and the initial and boundary conditions in (E). We assume that the initial and boundary conditions satisfy the hypotheses made above, and we let the elliptic differential operator A have the form

$$A(t, x, u, v) = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^{\alpha} b_{\alpha}(t, x, \xi(u)) A_{\alpha}(x, \xi(v)) .$$

We assume, further, that the following conditions hold:

(1) Each A_{α} satisfies (i) and (ii) above with $g(x)$ constant and $c = 0$.

(2) Each b_{α} is defined and continuous on $[0, T) \times \Omega \times R^{nm}$, it has values in R_{+} and, for some constants $K > 0$, $L > 0$,

$$|b_{\alpha}(t, x, \xi) - b_{\alpha}(t', x, \xi')| \leqq K|t - t'| + L|\xi - \xi'|$$

for every $t, t' \in [0, T)$, $x \in \Omega$, $\xi, \xi' \in R^{nm}$.

Now, let $T(t, u)v$ be defined on $W_{o}^{m,2}(\Omega)$ from the equation

$$\langle T(t, u)v, w\rangle_{m} = \sum_{|\alpha| \leqq m} \int_{\Omega} b_{\alpha}(t, x, \xi(u(x))) A_{\alpha}(x, \xi(v(x))) D^{\alpha} w(x) dx .$$

It is easy to see that $T(t, u)v$ is continuous, monotone and bounded in v and satisfies the following Lipschitz condition:

$$\| T(t, u)v - T(t', u')v \|_{m,2} \leqq K_1 |t - t'| + L_1 \| u - u' \|_{m,2}$$

for all $t, t' \in [0, T)$, $u, u', v \in W_{o}^{m,2}$, where K_1, L_1 are positive constants. Setting $T(t, \phi)v = T(t, \phi(-r))v$ for $(t, \phi, v) \in [0, T) \times C \times W_{o}^{m,2}$, we see that all the conditions of Theorem 1 are satisfied for the abstract problem

$$u'(t) + T(t, u_t)u(t) = 0 , \qquad t \in [0, T) ,$$
$$u_0 = \phi .$$

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