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A UNIFORMITY OF DISTRIBUTION OF G_Q IN G_A

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Introduction. 0-1. Let G be a connected and semisimple linear algebraic group defined over $\boldsymbol{Q}$. Denote by $G_{\boldsymbol{Q}}$ and G_A the group of $\boldsymbol{Q}$-rational points of G and the adele group of G, respectively. We identify $G_{\boldsymbol{Q}}$ with a subgroup of G_A in a natural manner. Then $G_{\boldsymbol{Q}}$ is discrete in G_A and the quotient $G_{\boldsymbol{Q}}\backslash G_A$ has a finite volume for a G_A-invariant measure.

In his paper [6], Kuga proposed the following problem:

How the set of points of $G_{\boldsymbol{Q}}$ is distributed in G_A?

He gave an answer for the case that G is a $\boldsymbol{Q}$-form of SL(2) of $\boldsymbol{Q}$-rank 0. With the help of Kuga's basic idea ("Kuga's criterion", see Proposition 1) and a deep result of representation theory due to Howe and Moore [5], the present author [9] showed a uniformity of distribution of $G_{\boldsymbol{Q}}$ in G_A with respect to a Haar measure dg on G_A when G is simply connected, absolutely almost simple and furthermore has $\boldsymbol{Q}$-rank zero. Roughly speaking, we showed that, for a relatively compact open subset X of G_A, the main term of the number $|X \cap G_{\boldsymbol{Q}}|$ is equal to $\int_X dg$, if $\int_X dg$ is sufficiently large. Here we normalize the Haar measure dg on G_A by $\int_{G_{\boldsymbol{Q}}\backslash G_A} dg = 1$. (In fact, we must impose some additional conditions on X. For detail, see Theorem.)

The object of the present paper is to show that the above result is also available even if G has $\boldsymbol{Q}$-rank greater than zero.

0-2. To explain our result more precisely, denote by G_f (resp. G_∞) the finite (resp. infinite) part of G_A; $G_f = \prod'_p G_{\boldsymbol{Q}_p}$ (the restricted direct product), $G_\infty = G_{\boldsymbol{R}}$. Then we have $G_A = G_f \cdot G_\infty$. For a finite set $\mathscr{S}$ of finite places of $\boldsymbol{Q}$, put $G_f(\mathscr{S}) = \prod_{p \in \mathscr{S}} G_{\boldsymbol{Q}_p} \times \prod_{p \notin \mathscr{S}} G_{\boldsymbol{Z}_p}$, which is an open subgroup of G_f.

Consider a sequence $\{X_j\}_{j=1}^{\infty}$ of relatively compact open subsets of G_A. A sequence $\{X_j\}_{j=1}^{\infty}$ is said to be *of Hecke type* if the following two conditions (0.1)-(0.2) are satisfied:

(0.1) Each X_j has the form $S(j) \times U$, where $S(j)$ is an open compact subset of $G_f(\mathscr{S})$ for a fixed finite set $\mathscr{S}$ of finite places of $\boldsymbol{Q}$ and U is a fixed relatively compact domain in G_∞.

(0.2) There exists an open compact subgroup K of G_f such that $K\cdot S(j)\cdot K = S(j)$ $(j = 1, 2, \cdots)$. Let dg be the Haar measure on G_A normalized by $\int_{G_Q\backslash G_A} dg = 1$. Then our main result is stated as follows.

THEOREM. *Let G be a connected, simply connected and absolutely almost simple linear algebraic group defined over Q, and let $\{X_j\}_{j=1}^{\infty}$ be a sequence of Hecke type. Assume that*

$$\lim_{j\to\infty} \int_{X_j} dg = \infty \,. \tag{0.3}$$

Then we have

$$\lim_{j\to\infty} |G_Q \cap X_j| \Big/ \int_{X_j} dg = 1 \,. \tag{0.4}$$

0–3. We present an implication of our theorem. Fix an embedding $G_Q \hookrightarrow \mathrm{GL}(N, Q)$ for some positive integer N. Set $\Gamma = G_Q \cap \mathrm{GL}(N, Z)$. Then Γ is a discrete subgroup of G_R. Let dg_R be the Haar measure on G_R normalized by $\int_{\Gamma\backslash G_R} dg_R = 1$. For a positive integer n, set $T(n) = \{g \in G_Q \,|\, n\cdot g \in \mathrm{M}(N, Z)\}$. It is easy to see that $\Gamma T(n)\Gamma = T(n)$ and that $\Gamma\backslash T(n)$ is a finite set. Put $\deg T(n) = |\Gamma\backslash T(n)|$. Let $\{n_j\}$ be a sequence of positive integers such that $\lim_{j\to\infty} \deg T(n_j) = \infty$ and that the primes which divide some n_j form a finite set. Then our theorem implies the following; for any relatively compact domain U in G_R, we have

$$\lim_{j\to\infty} |T(n_j) \cap U| / \deg T(n_j) = \int_U dg_R \,.$$

0–4. The present article consists of five sections. In the first section, we recall a result of Murase [9], which we call "Kuga's criterion". This criterion suggests a deep relation between "a uniformity of distribution of G_Q in G_A" and "an estimation of eigenvalues of Hecke's operators on the space of automorphic forms on G_A". The next two sections are of expository nature. In §2, we recall, after Arthur's exposition [1], the theory of the spectral decomposition of $L^2(G_Q\backslash G_A)$, which is due to Langlands [7]. In §3, we summarize some results of the theory of spherical functions on p-adic linear algebraic groups (after Satake [10] and Macdonald [8]). Especially, the explicit formula of zonal spherical functions is crucial in §4, where we study the behavior at infinity of the zonal spherical functions associated with parabolic subgroups of G. We prove our theorem in the last §5.

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Notation. As usual, we denote by $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{Z}_p, \boldsymbol{Q}_p, \boldsymbol{Q}_A$ and $\boldsymbol{Q}_A^\times$ the ring of rational integers, the rational number field, the real number field, the complex number field, the ring of p-adic integers, the p-adic number field, the adele ring of $\boldsymbol{Q}$ and the idele group of $\boldsymbol{Q}$, respectively. We define the modulus $|a|_A$ of an idele $a \in \boldsymbol{Q}_A^\times$ by $d(ax) = |a|_A \cdot dx$, where dx is a Haar measure on $\boldsymbol{Q}_A$. We denote by $\boldsymbol{R}_+^\times$ the group of positive real numbers. For a linear algebraic group $\boldsymbol{H}$ defined over $\boldsymbol{Q}$, we denote by $\boldsymbol{H}_A$, H_f and H_∞ the adele group of $\boldsymbol{H}$, the finite part and the infinite part of $\boldsymbol{H}_A$, respectively. We denote by $C_c^\infty(H_f)$ the space of locally constant functions on H_f with compact support. For a continuous function f on a locally compact group X and a subgroup K of X, we say that f is right K-finite if the set $\{R_k f \mid k \in K\}$ spans a finite dimensional subspace of the space $C^0(X)$ of continuous functions on X. Here we put $R_k f(x) = f(xk)$ $(x \in X, k \in K)$.

1. 1–1. In this section, we suppose that $\boldsymbol{G}$ is a connected semisimple linear algebraic group defined over $\boldsymbol{Q}$. Let $\{S(j)\}_{j=1}^\infty$ be a sequence of compact subsets of G_f satisfying $K \cdot S(j) \cdot K = S(j)$ $(j = 1, 2, \cdots)$ with some open compact subgroup K of G_f. Denote by $L^2(\boldsymbol{G}_Q \backslash \boldsymbol{G}_A)$ and $L^2(\boldsymbol{G}_Q \backslash \boldsymbol{G}_A/K)$ the Hilbert space of square integrable functions on $\boldsymbol{G}_Q \backslash \boldsymbol{G}_A$ and its closed subspace consisting of right K-invariant functions. For $f \in L^2(\boldsymbol{G}_Q \backslash \boldsymbol{G}_A)$ and $\varphi \in C_c^\infty(G_f)$, set $f * \varphi(g) = \int_{G_f} f(gx_f^{-1})\varphi(x_f)dx_f$ $(g \in \boldsymbol{G}_A)$. Here dx_f is the Haar measure on G_f normarized by $\int_K dx_f = 1$. Let ξ_j be the characteristic function of $S(j)$ on G_f. Then ξ_j belongs to $C_c^\infty(G_f)$. Put $\deg \xi_j = \int_{G_f} \xi_j(x_f)dx_f = \int_{S(j)} dx_f$. It is easy to see that the mapping $f \mapsto f * \xi_j$ $(f \in L^2(\boldsymbol{G}_Q \backslash \boldsymbol{G}_A/K))$ defines a linear bounded operator on $L^2(\boldsymbol{G}_Q \backslash \boldsymbol{G}_A/K)$ and that $\|f * \xi_j\| \leqq \deg \xi_j \cdot \|f\|$ for any $f \in L^2(\boldsymbol{G}_Q \backslash \boldsymbol{G}_A/K)$. Note that $\mathbf{1} * \xi_j = \deg \xi_j \cdot \mathbf{1}$ where $\mathbf{1}$ denotes the constant function on $\boldsymbol{G}_Q \backslash \boldsymbol{Q}_A$ taking the value one. The following fact, which we shall call "Kuga's criterion", plays a basic role in the proof of our theorem.

PROPOSITION 1. *Let the notation be the same as above. Assume that, for any* $f \in L^2(\boldsymbol{G}_Q \backslash \boldsymbol{G}_A/K)$, *we have*

$$\lim_{j \to \infty} \| f * \xi_j / \deg \xi_j - (f, \mathbf{1}) \cdot \mathbf{1} \| = 0 . \tag{1.1}$$

Then, for any relatively compact domain U *in* G_∞, *the following equality holds*:

$$\lim_{j \to \infty} |(S(j) \times U) \cap \boldsymbol{G}_Q| \Big/ \int_{S(j) \times U} dg = 1 . \tag{1.2}$$

For the proof, see Kuga [6, Theorem] and Murase [9, Proposition 1]. Note that linear operators $T_j: f \to f * \xi_j$ can be seen as Hecke's operators on the space of automorphic forms on G_A.

1–2. We consider the simplest case. Assume that there exists an orthonormal basis $\{f_k\}_{k=0}^{\infty}$ of $L^2(G_Q \backslash G_A/K)$ consisting of common eigenfunctions of linear operators $T_j (j = 1, 2, \cdots)$. Let $\lambda_k(j)$ be the eigenvalues of T_j with respect to f_k; $T_j f_k = \lambda_k(j) f_k$. We may assume that $f_0 = 1$ (note that $f_0 * \xi_j/\deg \xi_j - (f_0, 1)1 = 0$). Then, in this special case, the assumption (1.1) of Proposition 1 is satisfied if and only if

$$\lim_{j \to \infty} \lambda_k(j)/\deg \xi_j = 0$$

for any $k = 1, 2, \cdots$. This suggests a relation between our problem and an estimation of eigenvalues $\lambda_k(j)$ of Hecke's operators T_j. In fact, Kuga [6] proved the so-called "Kuga's lemma", which gives an estimation of eigenvalues of Hecke's operators on the space of automorphic forms on the upper half plane with respect to an arithmetic Fuchsian group. (For a representation theoretical version of Kuga's lemma, see Murase [9, Lemma 3].) However, we cannot always find a basis of $L^2(G_Q \backslash G_A/K)$ consisting of common eigenfunctions of T_j. Especially, $L^2(G_Q \backslash G_A/K)$ has continuous spectrum if $\mathrm{rank}_Q G > 0$. Thus we are led to the study of the spectral decomposition of $L^2(G_Q \backslash G_A)$.

2. Let G be a reductive linear algebraic group defined over Q. Fix once for all a minimal parabolic subgroup P_0 of G defined over Q and a Levi subgroup L_0 of P_0.

2.1. A Q-parabolic subgroup P is said to be *standard* if P_0 is a subgroup of P. We assume throughout this section that a parabolic subgroup P of G is defined over Q and standard. We denote by L_P the unique Levi subgroup of P that contains L_0, and by U_P the unipotent radical of P. Let A_0 (resp. A_P) be the split component of the center of L_0 (resp. L_P). Then A_P is a subgroup of A_0.

Fix a maximal compact subgroup M_∞ of $G_\infty = G_R$ whose Lie algebra is orthogonal to the Lie algebra of $(A_0)_R$ under the Killing form. For almost all finite places p, G_{Z_p} is a maximal compact subgroup of G_{Q_p}. For such p, we set $M_p = G_{Z_p}$. For other finite places p, we choose and fix any maximal compact subgroups M_p of G_{Q_p}. Set $M_f = \prod_{p<\infty} M_p$ and $M = M_f M_\infty$. Then we have $G_A = (U_P)_A \cdot (L_P)_A \cdot M$ for any parabolic subgroup P of G.

Let $X(L_P)_Q$ be the Z-module of Q-rational characters of L_P. Put $\mathfrak{a}_P = \mathrm{Hom}(X(L_P)_Q, R)$, the group of all homomorphisms from $X(L_P)_Q$ to R. Then $\mathfrak{a}_P$ is a vector space over R whose dimension is equal to $\dim A_P$.

For $g \in G_A$, we define a vector $H_P(g) \in \mathfrak{a}_P$ by $\exp(H_P(g)(\chi)) = |\chi(l_g)|_A$ for any $\chi \in X(\boldsymbol{L}_P)_Q$. Here we choose $l_g \in (\boldsymbol{L}_P)_A$ so that $g \in (\boldsymbol{U}_P)_A \cdot l_g \cdot M$. It is easy to see that $H_P(g)$ does not depend on the choice of $l_g \in (\boldsymbol{L}_P)_A$. Define a homomorphism Δ_{P_A} from $(\boldsymbol{L}_P)_A$ to $\boldsymbol{R}_+^\times$ by setting $\Delta_{P_A}(l) = d(lu_P l^{-1})/du_P$ for $l \in (\boldsymbol{L}_P)_A$. Here du_P is a Haar measure on $(\boldsymbol{U}_P)_A$.

Let W be the restricted Weyl group of $(\boldsymbol{G}, \boldsymbol{A}_0)$. Then W acts on the $\boldsymbol{R}$-vector space $X(\boldsymbol{L}_0)_Q \otimes_Z \boldsymbol{R}$ and hence acts on its dual $\mathfrak{a}_{P_0}$. Fix a positive definite W-invariant bilinear form $\langle\, , \rangle$ on $\mathfrak{a}_{P_0}$, by which we identify $\mathfrak{a}_{P_0}$ with its dual. This allows us to embed each $\mathfrak{a}_P$ in $\mathfrak{a}_{P_0}$. For a pair of parabolic subgroups $(\boldsymbol{P}, \boldsymbol{P}')$, let $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ be the set of distinct isomorphisms from $\mathfrak{a}_P$ onto $\mathfrak{a}_{P'}$ which are induced on $\mathfrak{a}_P$ by the action of elements of W. If $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ is not empty, $\boldsymbol{P}$ and $\boldsymbol{P}'$ are said to be *associated.*

2–2. In this subsection, we fix a parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}$. For simplicity, we often omit P as subscripts; namely we write $\mathfrak{a}$, $\boldsymbol{U}$, $\boldsymbol{L}$ and $\boldsymbol{A}$ for $\mathfrak{a}_P$, $\boldsymbol{U}_P$, $\boldsymbol{L}_P$ and $\boldsymbol{A}_P$, respectively. Put $\mathfrak{a}_C = \mathfrak{a} \otimes_R \boldsymbol{C}$. We shall now construct a series of representations $I_P(\Lambda)$ parametrized by $\Lambda \in \mathfrak{a}_C$. Let $\mathscr{H}_P^0$ be the space of functions Φ: $\boldsymbol{U}_A \boldsymbol{L}_Q \boldsymbol{A}_R^0 \setminus \boldsymbol{G}_A \to \boldsymbol{C}$ satisfying the following conditions (2.1)–(2.3):

(2.1) For any $x \in \boldsymbol{G}_A$, the function $l \mapsto \Phi(lx)$ on $\boldsymbol{L}_A$ is $\mathscr{Z}_R$-finite.

(2.2) Φ is right M-finite.

$$\|\Phi\|^2 = \int_M \int_{A_R^0 \cdot L_Q \setminus L_A} |\Phi(lm)|^2 dl dm < \infty\ . \tag{2.3}$$

Here we denote by $\mathscr{Z}_R$ the center of the universal envelopping algebra of $\mathfrak{l}_C = \mathfrak{l} \otimes_R \boldsymbol{C}$, the complexification of the Lie algebra $\mathfrak{l}$ of $\boldsymbol{L}_R$, by $\boldsymbol{A}_R^0$ the identity component of $\boldsymbol{A}_R$ and by dl (resp. dm) the Haar measure on $\boldsymbol{L}_A$ (resp. M) normalized by $\int_{A_R^0 \cdot L_Q \setminus L_A} dl = 1 \left(\text{resp. } \int_M dm = 1\right)$. Let $\mathscr{H}_P$ be the Hilbert space obtained as the completion of $\mathscr{H}_P^0$.

For $\Lambda \in \mathfrak{a}_C$, $\Phi \in \mathscr{H}_P$ and $x, y \in \boldsymbol{G}_A$, set

$$I_P(\Lambda, y)\Phi(x) = \Phi(xy) \exp(\langle \Lambda + \rho_P, H_P(xy) - H_P(x) \rangle)\ . \tag{2.4}$$

Here ρ_P is the vector in $\mathfrak{a} = \mathfrak{a}_P$ defined by

$$\exp(\langle \rho_P, H_P(l) \rangle) = \Delta_{P_A}(l)^{1/2} \quad (l \in \boldsymbol{L}_A)\ . \tag{2.5}$$

Then $y \mapsto I_P(\Lambda, y) (y \in \boldsymbol{G}_A)$ defines a representation $I_P(\Lambda)$ of $\boldsymbol{G}_A$ on $\mathscr{H}_P$. If $\Lambda \in i\mathfrak{a}$, then $I_P(\Lambda)$ is a unitary representation. For $\varphi \in C_c^\infty(G_f)$ and $\Lambda \in \mathfrak{a}_C$, we define a bounded operator $I_P(\Lambda, \varphi)$ on $\mathscr{H}_P$ by

$$I_P(\Lambda, \varphi)\Phi = \int_{G_f} \varphi(x_f) I_P(\Lambda, x_f) \Phi dx_f$$

for $\Phi \in \mathscr{H}_P$.

2–3. Let $\mathscr{P}$ be an associated class of (standard) parabolic subgroups of $\boldsymbol{G}$. Let $\Xi_{\mathscr{P}}$ be the linear space of collections $F=\{F_P\}_{P\in\mathscr{P}}$ of measurable functions $F_P\colon i\mathfrak{a}_P\to\mathscr{H}_P$ satisfying the following conditions (2.6)–(2.7):

(2.6) If $P, P'\in\mathscr{P}$ and $w\in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$, then

$$F_{P'}(w\Lambda)=M(w, \Lambda)F_P(\Lambda)\quad(\Lambda\in i\mathfrak{a}_P)\ .$$

$$\|F\|^2=\sum_{P\in\mathscr{P}}(2\pi i)^{-\dim\mathfrak{a}_P}c_P^{-1}\int_{i\mathfrak{a}_P}\|F_P(\Lambda)\|^2 d\Lambda<\infty\ . \tag{2.7}$$

Here $M(w, \Lambda)$ is an intertwining operator from $\mathscr{H}_P$ onto $\mathscr{H}_{P'}$, and c_P is the number of chambers in $\mathfrak{a}_P$ (for detail, see Arthur [1, pp. 255–256]).

For $\varphi\in C_c^\infty(G_f)$ and $F=\{F_P\}_{P\in\mathscr{P}}\in\Xi_{\mathscr{P}}$, set

$$(F_P*\varphi)(\Lambda)=I_P(\Lambda, \varphi^\vee)F_P(\Lambda)\quad(\Lambda\in i\mathfrak{a}_P)\ , \tag{2.8}$$

$$F*\varphi=\{F_P*\varphi\}_{P\in\mathscr{P}}\ , \tag{2.9}$$

where we put $\varphi^\vee(x)=\varphi(x^{-1})(x\in G_f)$. It is easy to see that $F*\varphi\in\Xi_{\mathscr{P}}$.

2.4. One of the main results of the theory of the spectral decomposition of $L^2(\boldsymbol{G}_Q\backslash\boldsymbol{G}_A)$ is stated as follows.

PROPOSITION 2. (1) *For any associated class $\mathscr{P}$ of parabolic subgroups of $\boldsymbol{G}$, there exists a linear operator $F\mapsto\Theta_{\mathscr{P}}(F)$ from $\Xi_{\mathscr{P}}$ onto a closed $\boldsymbol{G}_A$-invariant subspace $L^2_{\mathscr{P}}(\boldsymbol{G}_Q\backslash\boldsymbol{G}_A)$ of $L^2(\boldsymbol{G}_Q\backslash\boldsymbol{G}_A)$ which satisfies the following conditions* (2.10)–(2.11):

$$\|\Theta_{\mathscr{P}}(F)\|=\|F\|\ , \tag{2.10}$$

(2.11) *for $\varphi\in C_c^\infty(G_f)$, we have*

$$\Theta_{\mathscr{P}}(F)*\varphi=\Theta_{\mathscr{P}}(F*\varphi)\ .$$

(2) *We have the orthogonal decomposition;*

$$L^2(\boldsymbol{G}_Q\backslash\boldsymbol{G}_A)=\sum_{\mathscr{P}}L^2_{\mathscr{P}}(\boldsymbol{G}_Q\backslash\boldsymbol{G}_A)\ ,$$

where $\mathscr{P}$ runs over all associated classes of parabolic subgroups of $\boldsymbol{G}$.

(3) *The space $L^2_{\{G\}}(\boldsymbol{G}_Q\backslash\boldsymbol{G}_A)$ decomposes into a direct sum of countably infinite $\boldsymbol{G}_A$-invariant irreducible closed subspaces.*

(For detail, see Arthur [1, p. 256].)

3. We shall suppose throughout §3 and §4 that $\boldsymbol{G}$ is simply connected and F-almost simple linear algebraic group defined over a non-archimedian local field F. We put $G=\boldsymbol{G}_F$, the group of F-rational points of $\boldsymbol{G}$.

3-1. We fix a maximal F-split torus $\boldsymbol{S}$ of $\boldsymbol{G}$ and a minimal F-parabolic subgroup $\boldsymbol{B}$ of $\boldsymbol{G}$ which contains $\boldsymbol{S}$. Let $\boldsymbol{N}$ (resp. $\boldsymbol{Z}$) be the normalizer (resp. the centralizer) of $\boldsymbol{S}$ in $\boldsymbol{G}$. We write S, B, N and Z

for the groups of F-rational points of $\boldsymbol{S}, \boldsymbol{B}, \boldsymbol{N}$ and $\boldsymbol{Z}$, respectively. Due to Bruhat and Tits [3, 4], there exists a reduced and irreducible root system Σ_0, subgroups U_α of G ($\alpha \in \Sigma$, the affine root system associated with Σ_0) and a surjective homomorphism $\nu: N \to W$ (W is the Weyl group of Σ) such that the triple $(N, \nu, (U_\alpha)_{\alpha \in \Sigma})$ satisfies several axioms stated in Macdonald [8, p. 27, p. 35, (I)–(IX)]. Note that every root in Σ_0 is proportional to a root of $\boldsymbol{G}$ relative to $\boldsymbol{S}$ and conversely.

We assume that Σ_0 is a finite subset of the dual of a finite dimensional vector space V over $\boldsymbol{R}$ with a positive definite scalar product (,). For $a \in \Sigma_0$, let $\boldsymbol{U}_a$ be the root subgroup of $\boldsymbol{G}$ correspoding to a. Put $\Sigma_0^+ = \{a \in \Sigma_0 | \boldsymbol{U}_a \subset \boldsymbol{B}\}$ and $C_0 = \{x \in V | a(x) > 0$ for any $a \in \Sigma_0^+\}$. Then C_0 is a Weyl chamber of Σ_0 in V and Σ_0^+ is the set of positive roots relative to C_0.

Let T be the translation subgroup of the Weyl group W of Σ. Set $T^{++} = \{t \in T \mid t(0) \in \bar{C}_0\}$ and $Z^{++} = \nu^{-1}(T^{++})$, where $\bar{C}_0$ is the topological closure of C_0 in V. Then Z^{++} is a subsemigroup of Z. Let K be the subgroup of G generated by U_{a+k} with $a \in \Sigma_0$ and non-negative integers k. The group K is a maximal compact open subgroup of G. Let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{B}$, and put $U = \boldsymbol{U}_F$. Then we have

$$G = U \cdot Z \cdot K \quad \text{(Iwasawa decomposition)}\,, \tag{3.1}$$

$$G = K \cdot Z^{++} \cdot K \quad \text{(Cartan decomposition)}\,. \tag{3.2}$$

For $a \in \Sigma_0$, the subgroup of G generated by U_{a+k} with $k \in \boldsymbol{Z}$, which we denote by $U_{(a)}$, coincides with the group of F-rational points of $\boldsymbol{U}_a$. We have

$$U = \prod_{a \in \Sigma_0^+} U_{(a)} \quad \text{(in any order)}\,. \tag{3.3}$$

Let du_a (resp. du) be a Haar measure on $U_{(a)}$ (resp. U). For $z \in Z$, put

$$\Delta_a(z) = d(zu_a z^{-1})/du_a\,, \quad \Delta(z) = d(zuz^{-1})/du\,.$$

It follows from (3.3) that $\Delta(z) = \prod_{a \in \Sigma_0^+} \Delta_a(z)$. (Note that $\Delta_a(z)$ and $\Delta(z)$ do not depend on the choice of Haar measures du_a and du.) Since the functions $\Delta_a(z)$ and $\Delta(z)$ depend only on $t = \nu(z)$, we can define homomorphisms $\delta_a, \delta: T \to \boldsymbol{R}_+^\times$ in such a way that $\delta_a(\nu(z)) = \Delta_a(z)$ and $\delta(\nu(z)) = \Delta(z)$ for any $z \in Z$. For $a \in \Sigma_0$, put $\ell_a = |U_{a-1}/U_{a+1}|^{1/2}$ (note that $\ell_a > 1$ and that $\ell_a = (q_a q_{a+1})^{1/2}$ in the notation of Macdonald [8, p. 38]). The following statement is proved in Macdonald [8, Prop. (3.2.4)].

LEMMA 1. *If $t \in T$ and $x = t(0) \in V$, we have*

$$\delta_a(t) = \ell_a^{a(x)}\,, \quad \delta(t) = \prod_{a \in \Sigma_0^+} \ell_a^{a(x)}\,.$$

3-2. We summarize several results of the theory of spherical functions on $G = G_F$ due to Satake [10] and Macdonald [8] in a form convenient for later applications. Let the notation be the same as in 3-1.

A function ω on G is called a zonal spherical function relative to K if the following conditions (3.4)-(3.6) are satisfied:

(3.4) ω is continuous on G.

(3.5) ω is not identically zero.

$$\omega(g_1)\cdot\omega(g_2) = \int_K \omega(g_1 k g_2) dx \quad (g_1, g_1 \in G)\ , \tag{3.6}$$

where the Haar measure dk on K is normalized by $\int_K dk = 1$.

Let $\mathrm{Hom}(T, \boldsymbol{C}^\times)$ be the group of all homomorphisms from T to $\boldsymbol{C}^\times$. For $s \in \mathrm{Hom}(T, \boldsymbol{C}^\times)$ and $t \in T$, we write (s, t) for $s(t)$. For $s \in \mathrm{Hom}(T, \boldsymbol{C}^\times)$, we define a function ϕ_s on G by $\phi_s(uzk) = (s^{-1}\cdot\delta^{1/2}, \nu(z)) = (s, t)^{-1}\cdot(\delta, t)^{1/2}$ $(t = \nu(z))$ for $u \in U$, $z \in Z$ and $k \in K$ (recall that $G = U\cdot Z\cdot K$). It is easy to see that ϕ_s is well-defined. Set

$$\omega_s(g) = \int_K \phi_s(kg) dk \quad (g \in G)\ . \tag{3.7}$$

LEMMA 2 (Satake; see Macdonald [8, Theorem (3.3.12)]). *For any $s \in \mathrm{Hom}(T, \boldsymbol{C}^\times)$, the function ω_s on G is a zonal spherical function relative to K. Conversely, every zonal spherical function is equal to ω_s for some $s \in \mathrm{Hom}(T, \boldsymbol{C}^\times)$.*

The Weyl group W_0 of Σ_0 acts on T by inner automorphisms and hence acts on $\mathrm{Hom}(T, \boldsymbol{C}^\times)$; we define ws by $(ws, t) = (s, w^{-1}tw)$ for $s \in \mathrm{Hom}(T, \boldsymbol{C}^\times)$, $t \in T$ and $w \in W_0$.

LEMMA 3 (Satake; see Macdonald [8, Prop. (3.3.3)]). *For $w \in W_0$ and $s \in \mathrm{Hom}\ (T, \boldsymbol{C}^\times)$, we have $\omega_{ws} = \omega_s$.*

The following explicit formula for the zonal spherical function ω_s plays a basic role in the next section.

LEMMA 4 (Macdonald [8, Prop. (4.6.2)]). *For $s \in \mathrm{Hom}(T, \boldsymbol{C}^\times)$ and $z \in Z^{++}$, we have*

$$\omega_s(z) = \sum_{s' \in W_0 s} \Phi_{s'}(x)(s'\cdot\delta^{1/2}, t^{-1})\ .$$

Here $t = \nu(z)$, $x = t(0)$, $W_0 s$ is the W_0-orbit of s in $\mathrm{Hom}(T, \boldsymbol{C}^\times)$ and $\Phi_{s'}$ is a polynomial function on V depending on $s' \in W_0 s$.

4. Let the notation be the same as in §3. In this section, we define zonal spherical functions ω_{P_F} on $G = G_F$ associated with F-parabolic

subgroups $\boldsymbol{P}$ of $\boldsymbol{G}$ and study their behavior at infinity.

4–1. Let $\boldsymbol{P}$ be a proper F-parabolic subgroup of $\boldsymbol{G}$ containing $\boldsymbol{B}$, the fixed minimal F-parabolic subgroup of $\boldsymbol{G}$. Denote by $\boldsymbol{U}_P$ the unipotent radical of $\boldsymbol{P}$. We write P and U_P for the groups of F-rational points of $\boldsymbol{P}$ and $\boldsymbol{U}_P$, respectively. For $p \in P$, put

$$\Delta_P(p) = d(pu_Pp^{-1})/du_P \tag{4.1}$$

where du_P is a Haar measure on U_P. Since $G = P \cdot K$, we can extend Δ_P to a function on G by $\Delta_P(pk) = \Delta_P(p)$ $(p \in P, k \in K)$. It is easy to see that Δ_P is well-defined as a function on G. Set

$$\omega_P(g) = \int_K \Delta_P(kg)^{1/2} dk \quad (g \in G) . \tag{4.2}$$

LEMMA 5 (Macdonald [8, Prop. (1.4.7)]). *The function ω_P is a positive definite zonal spherical function on G relative to K.*

It follows from Lemma 2 that ω_P is equal to ω_s for some $s \in \mathrm{Hom}(T, \boldsymbol{C}^\times)$. Put $\Sigma_0^+(P) = \{a \in \Sigma_0 \mid U_a \subset U_P\}$. Since $U_P \subset U = U_B$, $\Sigma_0^+(P)$ is contained in Σ_0^+. Define $s_P \in \mathrm{Hom}(T, \boldsymbol{C}^\times)$ by

$$\begin{aligned} s_P &= \prod_{a \in \Sigma_0^+ - \Sigma_0^+(P)} \delta_a(t)^{-1/2} \quad (t \in T) \\ &= \prod_{a \in \Sigma_0^+ - \Sigma_0^+(P)} \not\ell_a^{-a(x)/2} \quad (x = t(0)) . \end{aligned} \tag{4.3}$$

Then we have:

LEMMA 6. $\Delta_P(g)^{1/2} = \phi_{s_P^{-1}}(g)$ $(g \in G)$.

PROOF. Observing that U_P is generated by $U_{(a)}$ with $a \in \Sigma_0^+(P)$, we have, for any $z \in Z$,

$$\Delta_P(z)^{1/2} = \prod_{a \in \Sigma_0^+(P)} \Delta_a(z)^{1/2} = \prod_{a \in \Sigma_0^+(P)} \delta_a(\nu(z))^{1/2} = (s_P \cdot \delta^{1/2}, \nu(z)) = \phi_{s_P^{-1}}(z) .$$

This proves the lemma since Δ_P and $\phi_{s_P^{-1}}$ are, as functions on G, right K-invariant and left U-invariant. q.e.d.

In view of the above lemma and the definitions (3.7) and (4.2), we have $\omega_P = \omega_{s_P^{-1}}$. There exists an element $w_0 \in W_0$ such that $w_0 a = -a$ for $a \in \Sigma_0$. Then we have $w_0 s_P = s_P^{-1}$. Thus we have proved:

LEMMA 7. *Let s_P be the element of* $\mathrm{Hom}(T, \boldsymbol{C}^\times)$ *given by* (4.3). *Then we have $\omega_P = \omega_{s_P}$. Furthermore we have, for $z \in Z^{++}$,*

$$\omega_P(z) = \sum_{w \in W_0/W_0(s_P)} \Phi_{ws_P}(x)(ws_P \cdot \delta^{1/2}, t^{-1})$$

where $t = \nu(z)$, $x = t(0)$ and $W_0(s_P)$ is the stabilizer of s_P in W_0; $W_0(s_P) =$

$\{w \in W_0 \mid ws_P = s_P\}$.

4–2. A continuous function f on a locally compact space X is said to *vanish at infinity* on X if, for any $\varepsilon > 0$, there exists a compact subset C of X such that $\sup_{x \in X-C}|f(x)| < \varepsilon$.

We now show that ω_P vanishes at infinity on G, if $\boldsymbol{P} \neq \boldsymbol{G}$. Since $G = K \cdot Z^{++} \cdot K$, we have only to show that the function $z \mapsto \omega_P(z)$ vanishes at infinity on Z^{++}. Denote by L the lattice in V generated by $t(0)$ with $t \in T$. For $x \in \mathrm{L}$, let t_x be the element of T given by $t_x(0) = x$. By a straightforward calculation, we obtain

$$(ws_P \cdot \delta^{1/2}, t_x^{-1}) = \prod_{a \in w(\Sigma_0^+ - \Sigma_0^+(P))} q_{w^{-1}a}^{a(x)/2} \cdot \prod_{a \in \Sigma_0^+} q_a^{-a(x)/2} .$$

Set $\Sigma_0^+(w, P) = w(\Sigma_0^+ - \Sigma_0^+(P)) \cap \Sigma_0^+$ and $\Sigma_0^-(w, P) = w(\Sigma_0^+ - \Sigma_0^+(P)) \cap (-\Sigma_0^+)$. Observing that $q_{w^{-1}a} = q_a$ (see Macdonald [8, p. 39]), we obtain

$$(ws_P \cdot \delta^{1/2}, t_x^{-1}) = \prod_{a \in \Sigma_0^-(w,P)} q_a^{a(x)/2} \cdot \prod_{a \in \Sigma_0^+ - \Sigma_0^+(w,P)} q_a^{-a(x)/2} .$$

Since $a(x) \leqq 0$ for $a \in \Sigma_0^-(w, P)$ and $x \in \bar{C}_0$, and since $q_a > 1$, we have

$$(ws_P \cdot \delta^{1/2}, t_x^{-1}) \leqq \prod_{a \in \Sigma_0^+ - \Sigma_0^+(w,P)} q_a^{-a(x)/2}$$

for $x \in \bar{C}_0 \cap \mathrm{L}$. Put $q = \mathrm{Min}\{q_a \mid a \in \Sigma_0^+\}$. Since $a(x) \geqq 0$ for $a \in \Sigma_0^+$ and $x \in \bar{C}_0$, we have proved:

LEMMA 8. *For $x \in \bar{C}_0 \cap \mathrm{L}$, we have*

$$(ws_P \cdot \delta^{1/2}, t_x^{-1}) \leqq q^{-\Lambda_{w,P}(x)/2} .$$

Here $\Lambda_{w,P}$ is a linear form on V given by

$$\Lambda_{w,P}(x) = \sum_{a \in \Sigma_0^+ - \Sigma_0^+(w,P)} a(x) .$$

Combining Lemmas 7 and 8, we obtain:

LEMMA 9. *For $z \in Z^{++}$, we have*

$$|\omega_P(z)| \leqq \sum_{w \in W_0/W_0(s_P)} |\Phi_{ws_P}(x)| q^{-\Lambda_{w,P}(x)/2} \quad (x = \nu(z)(0)) .$$

Note that the kernel of the homomorphism from Z to L defined by $z \mapsto \nu(z)(0)$ is compact. Hence, in order to show the vanishing at infinity on Z^{++} of ω_P, it is sufficient to verify that $x \mapsto |\Phi_{ws_P}(x)| \cdot q^{-\Lambda_{w,P}(x)/2}$ vanishes at infinity on $\bar{C}_0$ for any $w \in W_0$.

4–3. Let $\{a_1, \cdots, a_r\}$ be the set of the simple roots of Σ_0 relative to the Weyl chamber C_0. Then $a_1, \cdots, a_r$ are linearly independent over $\boldsymbol{R}$ and $C_0 = \bigcap_{i=1}^r \{x \in V \mid a_i(x) > 0\}$. Recall that every positive root a can be written in the form $a = m_1 a_1 + \cdots + m_r a_r$ with non-negative integers m_i. Since $\Lambda_{w,P}$ is a sum of positive roots, we have $\Lambda_{w,P} = \sum_{i=1}^r m_i(w) \cdot a_i$ with non-negative integers $m_i(w)$.

We claim that $m_i(w) > 0$ $(1 \leqq i \leqq r)$ for any $w \in W_0$. Suppose the contrary. We may assume that $m_1(w) = 0$ for some $w \in W_0$. Denote by $\Sigma_0^+(a_1)$ (resp. $\Sigma_0(a_1)$) the set of all roots in Σ_0 whose coefficients at a_1 are positive (resp. non-negative). It is easy to see that $\Sigma_0^+(a_1) \subset \Sigma_0^+$. Then the assumption $m_1(w) = 0$ implies that $(\Sigma_0^+ - \Sigma_0^+(w, P)) \cap \Sigma_0^+(a_1)$ is empty and hence that $\Sigma_0^+(a_1)$ is a subset of $\Sigma_0^+(w, P)$. Thus we have

$$w(\Sigma_0^+ - \Sigma_0^+(P)) \supset \Sigma_0^+(a_1) \,. \tag{4.4}$$

Observe that the root subgroup $\boldsymbol{U}_a$ corresponding to $a \in w(\Sigma_0^+ - \Sigma_0^+(P))$ is contained in $n_w \boldsymbol{L}_P n_w^{-1}$, where $\boldsymbol{L}_P$ is a Levi subgroup of $\boldsymbol{P}$ and n_w is any element of N_F such that $\nu(n_w) = w$. On the other hand, let $\boldsymbol{P}(a_1)$ be the subgroup of $\boldsymbol{G}$ generated by $\boldsymbol{Z}$ and $\boldsymbol{U}_a$ with $a \in \Sigma_0(a_1)$. Then $\boldsymbol{P}(a_1)$ is a proper parabolic subgroup of $\boldsymbol{G}$ and the unipotent radical $U(a_1)$ of $\boldsymbol{P}(a_1)$ is generated by $\boldsymbol{U}_a$ with $a \in \Sigma_0^+(a_1)$. If the inclusion relation (4.4) were true, we would obtain the following:

$$n_w \boldsymbol{L}_P n_w^{-1} \supset \langle \boldsymbol{U}_a;\, a \in w(\Sigma_0^+ - \Sigma_0^+(P))\rangle \supset \prod_{a \in \Sigma_0^+(a_1)} \boldsymbol{U}_a = \boldsymbol{U}(a_1) \,. \tag{4.5}$$

Observe that $n_w \boldsymbol{L}_P n_w^{-1}$ is a Levi subgroup of a proper parabolic subgroup $n_w \boldsymbol{P} n_w^{-1}$. The following lemma shows that the inclusion relation (4.5) never occurs and hence that the assumption $m_1(w) = 0$ gives rise to a contradiction.

LEMMA 10. *Let $\boldsymbol{P}$ and $\boldsymbol{P}'$ be proper F-parabolic subgroups of $\boldsymbol{G}$. Then the unipotent radical $\boldsymbol{U}'$ of $\boldsymbol{P}'$ is not contained in any Levi subgroup $\boldsymbol{L}$ of $\boldsymbol{P}$.*

PROOP. Suppose that $\boldsymbol{U}' \subset \boldsymbol{L}$ for some Levi subgroup $\boldsymbol{L}$ of $\boldsymbol{P}$. Then $\boldsymbol{U}' \subset \boldsymbol{P}$ and hence $\boldsymbol{U}' \subset \boldsymbol{P} \cap \boldsymbol{P}'$. In view of Borel and Tits [2, Prop. 4.4 (b)], $\boldsymbol{U}' \cdot (\boldsymbol{P} \cap \boldsymbol{P}') = \boldsymbol{P} \cap \boldsymbol{P}'$ is an F-parabolic subgroup of $\boldsymbol{G}$. Hence $\boldsymbol{P}$ and $\boldsymbol{P}'$ contain a common minimal parabolic subgroup $\boldsymbol{P}_0$ of $\boldsymbol{G}$. Let $\boldsymbol{S}_0$ be a maximal F-split torus of $\boldsymbol{G}$ contained in $\boldsymbol{P}_0$. Let Σ_0' be the root system of $\boldsymbol{G}$ relative to $\boldsymbol{S}_0$. Denote by $\{b_1, \cdots, b_r\}$ the set of the simple roots of Σ_0' relative to $\boldsymbol{P}_0$. Then there exists a root $b_0 = n_1 b_1 + \cdots + n_r b_r$ with $n_i > 0$ $(1 \leqq i \leqq r)$. It is known that the root subgroup $\boldsymbol{U}_{b_0}$ corresponding to b_0 is contained in the unipotent radical of every proper parabolic subgroup, especially in $\boldsymbol{U}'$. Hence the assumption $\boldsymbol{U}' \subset \boldsymbol{L}$ implies that

$$\boldsymbol{U}_{b_0} \subset \boldsymbol{L} \,. \tag{4.6}$$

On the other hand, since $\boldsymbol{P}$ is a proper parabolic subgroup of $\boldsymbol{G}$, there exists a non-empty subset μ_P of $\{1, \cdots, r\}$ which satisfies the following condition:

For a root $b = m_1 b_1 + \cdots + m_r b_r$ ($m_i \in \boldsymbol{Z}$) in Σ_0', assume that $\boldsymbol{U}_b \subset \boldsymbol{L}$. Then $m_i = 0$ for any $i \in \mu_P$. This fact contradicts the inclusion relation (4.6) and the assumption $n_i > 0$ for all i. q.e.d.

As a consequence, we have proved:

LEMMA 11. *If we put* $\Lambda_{w,P} = \sum_{i=1}^r m_i(w)\cdot a_i$, *then* $m_i(w) > 0$ *for any* i *and* $w \in W_0$.

Since $\bar{C}_0 = \{x \in V \mid a_i(x) \geqq 0\,(1 \leqq i \leqq r)\}$ and since $\Phi_{ws_P}(x)$ is a polynomial function on V, the function given by $x \mapsto \Phi_{ws_P}(x)\cdot \ell^{-\Lambda_{w,P}(x)}$ vanishes at infinity on $\bar{C}_0$. Thus we have proved:

PROPOSITION 3. *If* $\boldsymbol{P}$ *is a proper parabolic subgroup of* $\boldsymbol{G}$, *the zonal spherical function* ω_P *associated with* $P = \boldsymbol{P}_F$ *vanishes at infinity on* $G = \boldsymbol{G}_F$.

5. In this section, we suppose that $\boldsymbol{G}$ is a connected, simply connected and absolutely almost simple linear algebraic group defined over $\boldsymbol{Q}$.

5–1. For each finite place p of $\boldsymbol{Q}$, we can apply results of §3 and §4 to $\boldsymbol{G}_{\boldsymbol{Q}_p}$. We write M_p for K, the maximal compact subgroup of $\boldsymbol{G}_{\boldsymbol{Q}_p}$ defined in 3–1. Then, except for a finite number of p, we have $M_p = \boldsymbol{G}_{\boldsymbol{Z}_p}$ if we fix a suitable embedding $\boldsymbol{G}_{\boldsymbol{Q}} \hookrightarrow \mathrm{GL}(N, \boldsymbol{Q})$. Hence this choice of M_p is consistent with the choice of the maximal compact subgroups of $\boldsymbol{G}_{\boldsymbol{Q}_p}$ in 2–1. From now on we fix the Haar measure dx_f on G_f normalized by

$$\int_{M_f} dx_f = 1 \quad \Big(M_f = \prod_{p<\infty} M_p\Big)\,.$$

Let $\{S(j)\}_{j=1}^{\infty}$, K and ξ_j be the same as in §1.

5–2. We fix throughout 5–2, 5–3 and 5–4 a proper $\boldsymbol{Q}$-parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}$. In this subsection, we let the notation be the same as in 2–2. Denote by $\mathscr{H}_P^K$ the closed subspace of $\mathscr{H}_P$ consisting of right K-invariant functions. Let $V(P, K)$ be the linear space spanned by all elements of $\mathscr{H}_P^K$ that are continuous, bounded and M_∞-finite as functions on $\boldsymbol{G}_A$. We easily see that $V(P, K)$ is dense subspace of $\mathscr{H}_P^K$.

Define a function $\omega_{P,f}$ on G_f by

$$\omega_{P,f}(x_f) = \int_{M_f} \exp(\langle \rho_P, H_P(m_f x_f)\rangle)dm_f\,, \tag{5.1}$$

where dm_f is the Haar measure on M_f normalized by $\int_{M_f} dm_f = 1$. Then

$\omega_{P,f}$ is a zonal spherical function on G_f relative to M_f. Denote by $\mathfrak{H}(G_f, K)$ the $\boldsymbol{C}$-module of $\varphi \in C_0^\infty(G_f)$ satisfying $\varphi(kgk') = \varphi(g)$ for any $g \in G_f$ and $k, k' \in K$. Observe that ξ_j belongs to $\mathfrak{H}(G_f, K)$.

LEMMA 12. *For any $\Phi \in V(P, K)$, there exists a positive constant C_Φ such that*

$$\| I_P(\Lambda, \varphi)\Phi \| \leqq C_\Phi \int_{G_f} \omega_{P,f}(x_f) |\varphi(x_f)| dx_f \tag{5.2}$$

for any $\Lambda \in i\mathfrak{a}_P$, $\varphi \in \mathfrak{H}(G_f, K)$.

PROOF. Denote by $M^\wedge$ the set of all equivalence classes of irreducible unitary representations of M. For $\tau \in M^\wedge$, let V_τ be the representation space of τ. For $T \in \mathrm{End}(V_\tau)$, set $\|T\|_\tau^2 = \dim\tau \cdot \mathrm{tr}(T \cdot T^*)$, where T^* denotes the adjoint of T with respect to the scalar product of V_τ. Then $T \to \|T\|_\tau$ defines a norm of $\mathrm{End}(V_\tau)$. For $\tau \in M^\wedge$ and $\Psi \in \mathscr{H}_P$, set

$$\Psi(x, \tau) = \int_M \Psi(xm) \cdot \tau(m^{-1}) dm \quad (x \in \boldsymbol{G}_A) , \tag{5.3}$$

where dm is the Haar measure on M normalized by $\int_M dm = 1$. By virtue of the Peter-Weyl theorem, we have

$$\|\Psi\|^2 = \int_M \int_{\mathscr{L}} |\Psi(lm)|^2 dl dm = \sum_{\tau \in M^\wedge} \int_{\mathscr{L}} \|\Psi(l, \tau)\|_\tau^2 dl ,$$

where $\mathscr{L}$ denotes the quotient space $(\boldsymbol{L}_P)_Q(\boldsymbol{A}_P)_R^0 \backslash (\boldsymbol{L}_P)_A$. Applying this formula to $\Psi = I_P(\Lambda, \varphi)\Phi$ ($\varphi \in \mathfrak{H}(G_f, K)$, $\Phi \in V(P, K)$), we obtain

$$\| I_P(\Lambda, \varphi)\Phi \|^2 = \sum_{\tau \in M^\wedge} \int_{\mathscr{L}} \| I_P(\Lambda, \varphi)\Phi(l, \tau) \|_\tau^2 dl . \tag{5.4}$$

In view of definitions (5.3), (2.3) and (2.4), we have

$$\begin{aligned} & I_P(\Lambda, \varphi)\Phi(l, \tau) \\ &= \int_M \int_{G_f} \varphi(x_f)\Phi(lmx_f) \exp(\langle \Lambda + \rho_P, H_P(lmx_f) - H_P(lm) \rangle) \tau(m^{-1}) dx_f dm \\ &= \int_M \int_{G_f} \varphi(x_f)\Phi(lmx_f) \exp(\langle \Lambda + \rho_P, H_P(mx_f) \rangle) \tau(m^{-1}) dx_f dm \end{aligned}$$

(note that $H_P(lgm) = H_P(l) + H_P(g)$ for any $l \in (\boldsymbol{L}_P)_A$, $g \in \boldsymbol{G}_A$ and $m \in M$). Put $C'_\Phi = \sup_{g \in \boldsymbol{G}_A} |\Phi(g)|$. If $\Lambda \in i\mathfrak{a}_P$, we have

$$\begin{aligned} & \sup_{l \in (\boldsymbol{L}_P)_A} \| I_P(\Lambda, \varphi)\Phi(l, \tau) \| \\ & \leqq \sup_{l \in (\boldsymbol{L}_P)_A} \int_M \int_{G_f} |\varphi(x_f)| \exp(\mathrm{Re}\langle \rho_P + \Lambda, H_P(mx_f) \rangle) \\ & \qquad \times |\Phi(lmx_f)| \cdot \| \tau(m^{-1}) \|_\tau dx_f dm \end{aligned} \tag{5.5}$$

$$\leqq C'_\Phi \dim \tau \cdot \int_{G_f} |\varphi(x_f)|\, dx_f \int_{M_f} \exp(\langle \rho_P, H_P(m_f x_f)\rangle) dm_f$$

$$= C'_\Phi \dim \tau \int_{G_f} |\varphi(x_f)|\, \omega_{P,f}(x_f) dx_f$$

(note that $H_P(m_\infty x_f) = H_P(x_f)$ for any $m_\infty \in M_\infty$ and $x_f \in G_f$). Since $I_P(\Lambda, \varphi)\Phi$ is right K-invariant for any $\varphi \in \mathfrak{H}(G_f, K)$ and since Φ is M_∞-finite, there exists a finite subset N_Φ of $M^\wedge$ such that, if $\tau \in M^\wedge - N_\Phi$, we have $I_P(\Lambda, \varphi)\Phi(g, \tau) = 0$ for any $\Lambda \in i\mathfrak{a}_P$, $\varphi \in \mathfrak{H}(G_f, K)$ and $g \in \boldsymbol{G}_A$. In view of (5.4) and (5.5), we have

$$\| I_P(\Lambda, \varphi)\Phi \|^2 = \sum_{\tau \in N_\Phi} \int_{\mathscr{L}} \| I_P(\Lambda, \varphi)\cdot\Phi(l, \tau) \|_\tau^2 dl$$

$$= \sum_{\tau \in N_\Phi} \int_{\mathscr{L}} dl \cdot \sup_{l \in L_A} \| I_P(\Lambda, \varphi)\cdot\Phi(l, \tau) \|_\tau^2$$

$$= \sum_{\tau \in N_\Phi} \int_{\mathscr{L}} dl \cdot \Big(C'_\Phi \dim \tau \cdot \int_{G_f} |\varphi(x_f)|\, \omega_{P,f}(x_f) dx_f\Big)^2$$

for $\Lambda \in i\mathfrak{a}_P$, $\varphi \in \mathfrak{H}(G_f, K)$ and $\Phi \in V(P, K)$. $\Big($Note that $\int_{\mathscr{L}} dl$ is finite.$\Big)$ Thus the inequality (5.2) holds if we set

$$C_\Phi = C'_\Phi \Big(\sum_{\tau \in N_\Phi} (\dim \tau)^2 \cdot \int_{\mathscr{L}} dl \Big)^{1/2} . \qquad \text{q.e.d.}$$

5–3. Recall the definition of Δ_{P_A} in 2–1; $\Delta_{P_A}(l) = d(lu_P l^{-1})/du_P$ ($l \in (\boldsymbol{L}_P)_A$) where du_P is a Haar measure on $(\boldsymbol{U}_P)_A$. We extend Δ_{P_A} to a function on $\boldsymbol{G}_A$ by putting $\Delta_{P_A}(g) = \Delta_{P_A}(l_g)$ for $g \in \boldsymbol{G}_A$, where we choose $l_g \in (\boldsymbol{L}_P)_A$ so that $g \in (\boldsymbol{U}_P)_A \cdot l_g \cdot M$. (For simplicity we use the same notation Δ_{P_A}.)

In view of (2.5), we have $\exp(\langle \rho_P, H_P(g)\rangle) = \Delta_{P_A}(g)^{1/2}$ ($g \in \boldsymbol{G}_A$), and hence

$$\omega_{P,f}(x_f) = \int_{M_f} \Delta_{P_A}(m_f x_f)^{1/2} dm_f \quad (x_f \in G_f) .$$

For $x_f = (x_p)_{p<\infty} \in G_f$ ($x_p \in \boldsymbol{G}_{Q_p}$), we easily see $\Delta_{P_A}(x_f) = \prod_{p<\infty} \Delta_{P_p}(x_p)$ where we write P_p for $\boldsymbol{P}_{Q_p}$ and hence

(5.6) $$\omega_{P,f}(x_f) = \prod_{p<\infty} \int_{M_p} \Delta_{P_p}(m_p x_p)^{1/2} dm_p = \prod_{p<\infty} \omega_{P_p}(x_p) .$$

(For the definition of Δ_{P_p} and ω_{P_p}, see (4.1) and (4.2), respectively.) Since ω_{P_p} is positive definite (see Lemma 5), we have

(5.7) $$|\omega_{P_p}(x_p)| \leqq 1 \quad (x_p \in \boldsymbol{G}_{Q_p})$$

in view of Macdonald [8, Lemma (1.4.1)]. Combining (5.6), (5.7) and

Proposition 3, we obtain:

LEMMA 13. *The function* $\omega_{P,f}$ *vanishes at infinity on* $G_f(\mathscr{S})$ *for any finite set* $\mathscr{S}$ *of finite places of* $\boldsymbol{Q}$.

5-4. The following lemma was proved in Murase [9, Lemma 5].

LEMMA 14. *Let* X *be a locally compact group and let* dx *be a Haar measure on* X. *Let* $\{S(j)\}_{j=1}^{\infty}$ *be a sequence of open compact subsets of* X. *Assume that*

$$\lim_{j\to\infty}\int_{S_j} dx = \infty .$$

If a continuous function f *on* X *vanishes at infinity, we have*

$$\lim_{j\to\infty}\int_{S_j} f(x)dx \Big/ \int_{S_j} dx = 0 .$$

We can now prove the following fundamental result.

PROPOSITION 4. *Let the notation be the same as in* 5-1. *Assume*

$$\lim_{j\to\infty}\deg\xi_j = \infty . \tag{5.8}$$

Then, for any $\Phi\in\mathscr{H}_P^K$ *and* $\Lambda\in i\mathfrak{a}_P$,

$$\lim_{j\to\infty}\|I_P(\Lambda, \check{\xi}_j)\Phi\|/\deg\xi_j = 0 . \tag{5.9}$$

PROOF. The equality (5.9) for $\Phi\in V(P, K)$ is an immediate consequence of Lemmas 12, 13 and 14. Since $V(P, K)$ is dense in $\mathscr{H}_P^K$ and since

$$\|I_P(\Lambda, \check{\xi}_j)\Phi\|/\deg\xi_j \leqq \|\Phi\|$$

for $\Phi\in\mathscr{H}_P^K$ and $\Lambda\in i\mathfrak{a}_P$, the equality (5.9) is also valid for $\Phi\in\mathscr{H}_P^K$.

q.e.d.

5-5. In this section we employ the notation in 2-3 and 2-4. Let $\mathscr{P}$ be an associated class of (standard) parabolic subgroups of $\boldsymbol{G}$. Assume $\mathscr{P}\neq\{\boldsymbol{G}\}$. Let $\Xi_{\mathscr{P}}^K$ be the subspace of $\Xi_{\mathscr{P}}$ consisting of $F=\{F_P\}_{P\in\mathscr{P}}\in\Xi_{\mathscr{P}}$ which satisfies $I_P(\Lambda, k)F_P(\Lambda)=F_P(\Lambda)$ for any $P\in\mathscr{P}$, $\Lambda\in i\mathfrak{a}_p$ and $k\in K$. Then $\Theta_{\mathscr{P}}$ defines a norm-preserving linear operator from $\Xi_{\mathscr{P}}^K$ onto the subspace $L^2_{\mathscr{P}}(\boldsymbol{G}_Q\backslash\boldsymbol{G}_A/K)$ of $L^2_{\mathscr{P}}(\boldsymbol{G}_Q\backslash\boldsymbol{G}_A)$ consisting of right K-invariant functions. In view of (2.7)-(2.11) we obtain

$$\|\Theta_{\mathscr{P}}(F)*\xi_j/\deg\xi_j\|^2 = \|\Theta_{\mathscr{P}}(F*\xi_j)/\deg\xi_j\|^2 = \|F*\xi_j/\deg\xi_j\|^2$$

$$= \sum_{P\in\mathscr{P}}(2\pi i)^{-\dim\mathfrak{a}_P}\cdot c_P^{-1}\int_{i\mathfrak{a}_P}\|I_P(\Lambda, \check{\xi}_j)F_P(\Lambda)/\deg\xi_j\|^2 d\Lambda$$

for $F\in\Xi_{\mathscr{P}}^K$. Observe that $F_P(\Lambda)\in\mathscr{H}_P^K$ for any $P\in\mathscr{P}$ and $\Lambda\in i\mathfrak{a}_P$ if $F\in\Xi_{\mathscr{P}}^K$. By virtue of Proposition 4, we have

$$\lim_{j\to\infty} \| I_P(\Lambda, \check{\xi_j}) F_P(\Lambda)/\deg \xi_j \| = 0$$

for any $\Lambda \in i\mathfrak{a}_P$ under the assumption (5.8). Observe that we have $\| I_P(\Lambda, \check{\xi_j}) F_P(\Lambda)/\deg \xi_j \| \leqq \| F_P(\Lambda) \|$ for any $\Lambda \in i\mathfrak{a}_P$ and $j \geqq 1$, since $I_P(\Lambda)$ is a unitary representation. Since $\Lambda \mapsto \| F_P(\Lambda) \|^2$ is integrable on $i\mathfrak{a}_P$, the Lebesgue convergence theorem applies and we have

$$\lim_{j\to\infty} \int_{i\mathfrak{a}_P} \| I_P(\Lambda, \check{\xi_j}) F_P(\Lambda)/\deg \xi_j \|^2 d\Lambda = \int_{i\mathfrak{a}_P} \lim_{j\to\infty} \| I_P(\Lambda, \check{\xi_j}) F_P(\Lambda)/\deg \xi_j \|^2 d\Lambda$$
$$= 0 ,$$

and hence

$$\lim_{j\to\infty} \| \Theta_P(F) * \xi_j/\deg \xi_j \| = 0$$

for any $F \in \Xi^K_{\mathscr{P}}$ under the assumption (5.8). Observing that $\Theta_{\mathscr{P}}$ maps $\Xi^K_{\mathscr{P}}$ onto $L^2_{\mathscr{P}}(G_Q \backslash G_A/K)$ and that $C \cdot 1$ is orthgonal to $L^2_{\mathscr{P}}(G_Q \backslash G_A/K)$, we obtain the following:

PROPOSITION 5. *Let $\mathscr{P}$ be an associated class of parabolic subgroups of G. If $\mathscr{P} \neq \{G\}$ and if the assumption* (5.8) *is satisfied, we have*

$$\lim_{j\to\infty} \| f * \xi_j/\deg \xi_j - (f, 1)1 \| = 0$$

for any $f \in L^2_{\mathscr{P}}(G_Q \backslash G_A/K)$.

5–6. We shall consider the case $\mathscr{P} = \{G\}$. Recall that $L^2_{\{G\}}(G_Q \backslash G_A)$ decomposes into a discrete direct sum of countably infinite irreducible unitary representations with finite multiplicities;

$$L^2_{\{G\}}(G_Q \backslash G_A) = \sum_{n=1}^{\infty} \mathscr{H}_n . \tag{5.10}$$

The following proposition was essentially proved in Murase [9].

PROPOSITION 6. *Let $\mathscr{H}$ be a G_A-invariant subspace of $L^2(G_Q \backslash G_A)$. Assume that the unitary representation of G_A defined on $\mathscr{H}$ by right translation is irreducible. Denote by $\mathscr{H}^K$ the closed subspace of $\mathscr{H}$ consisting of all right K-invariant functions. Then, if* (5.8) *is satisfied, we have*

$$\lim_{j\to\infty} \| f * \xi_j/\deg \xi_j - (f, 1) \cdot 1 \| = 0$$

for any $f \in \mathscr{H}^K$.

This proposition and the decomposition (5.10) imply that the statement of Proposition 5 also holds for $\mathscr{P} = \{G\}$. Observing that $L^2(G_Q \backslash G_A/K) = \sum_{\mathscr{P}} L^2_{\mathscr{P}}(G_Q \backslash G_A/K)$ where $\mathscr{P}$ runs over all associated classes of parabolic subgroups of G, we obtain the following result, which completes the proof of our theorem in view of Kuga's criterion (Proposition 1).

PROPOSITION 7. *Let the assumption be the same as in Theorem. Then we have, for any* $f \in L^2(G_Q \backslash G_A/K)$,

$$\lim_{j\to\infty} \| f*\xi_j/\deg \xi_j - (f, 1)\cdot 1 \| = 0 .$$

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