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OPTIMAL SWITCHING PROBLEMS OF TANDEM TYPE

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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1. Introduction. In this article, we consider the stochastic switching problem of tandem type, related to optimal stopping. The model is described as follows.

Let X_i, $i = 1, 2, \cdots, N$, be a sequence of measurable processes. For each sequence $\hat{T} = (T_1, T_2, \cdots, T_N)$ of N stopping times $T_1 \leqq T_2 \leqq \cdots \leqq T_N$, we define the following process $X_{\hat{T}}$:

$$X_{\hat{T}}(t) = \begin{cases} X_i(t) & \text{if} \quad T_{i-1} \leqq t < T_i \\ 0 & \text{if} \quad t \geqq T_N , \end{cases}$$

where $T_0 = 0$. Then the process $X_{\hat{T}}$ starts with $X_{\hat{T}}(t) = X_1(t)$ if $0 \leqq t \leqq T_1$ and it switches in tandem from X_i to X_{i+1} at the time T_i for $i = 1, 2, \cdots, N$. The object is to maximize the profit:

$$J(\hat{T}) = E\left[\int_0^\infty e^{-\alpha t} f(X_{\hat{T}}(t))\,dt\right] \cong \sum_i E\left[\int_{T_{i-1}}^{T_i} e^{-\alpha t} f(X_i(t))\,dt\right],$$

where f is a given bounded measurable function and a strategy is the sequence $\hat{T}$ of stopping times of switches. The problem is reduced to the optimal stopping problem when $N = 1$.

The content is as follows: In § 2 we formulate the general switching problem of tandem type as Problem (I) in precise terms and we recall some results on the optimal stopping problem [2], [9] and [12]. In § 3, extending the Snell envelope in optimal stopping, we shall define the generalized Snell envelope. In § 4 we show the existence of an optimal strategy by a constructive method. In § 5 we give a necessary and sufficient condition for optimality, which is different from that of [4]. Finally, in § 6 we give a penalty method ([1], [9], [12]) to find the optimal strategy from the computational point of view.

2. Preliminaries and formulation. Let (Ω, F, P) be a complete probability space equipped with an increasing and right-continuous family of sub-σ-fields $(F_t)_{t \geqq 0}$ such that $\bigvee_{t \geqq 0} F_t = F$ and F_0 contains all null sets.

Let W be the Banach space of all right-continuous (F_t)-adapted processes X with its norm $\|X\| = \|\sup_t |X_t|\|_{L^\infty} < \infty$.

Now we formulate the switching problem of tandem type with finite steps.

PROBLEM (I). *For a given positive integer N, let $(f_i(t))$, $i = 1, 2, \cdots, N, N+1$, be a sequence of processes from W with $f_{N+1}(t) = 0$. Let C be the class of all sequences $\hat{T} = (T_1, T_2, \cdots, T_N)$ of N stopping times such that $0 = T_0 \leqq T_1 \leqq T_2 \leqq \cdots \leqq T_N$. We define the profit:*

$$J(\hat{T}) = \sum_{i=1}^{N} E\left[\int_{T_{i-1}}^{T_i} e^{-\alpha t} f_i(t) dt\right], \quad \hat{T} \in C, \quad (\alpha > 0).$$

Find an optimal strategy $\hat{T}^* \in C$, that is, $J(\hat{T}^*) = \sup_{\hat{T} \in C} J(\hat{T})$, and characterize its maximum.

We give some results on optimal stopping problems which will be needed below. Let $X \in W$. A stopping time S is said to be optimal if $E[X_S] = \sup_T E[X_T]$, where the supremum is taken over all stopping times T and $X_\infty = \limsup_{t\to\infty} X_t$. There exists a right-continuous supermartingale Y, called the Snell envelope, majorizing X and satisfying:

$$\lim_{t\to\infty} Y_t = \limsup_{t\to\infty} Y_t = \limsup_{t\to\infty} X_t, \tag{2.1}$$

$$Y_t = \operatorname*{ess\,sup}_{T\geqq t} E[X_T | F_t], \tag{2.2}$$

$$E[Y_0] = \sup_T E[X_T]. \tag{2.3}$$

Moreover, for any stopping time S,

$$Y_S = \operatorname*{ess\,sup}_{T\geqq S} E[X_T | F_S], \qquad E[Y_S] = \sup_{T\geqq S} E[X_T]. \tag{2.4}$$

Suppose that X satisfies the additional hypothesis:

(2.5) For any increasing sequence of stopping times (T_n) with limit T, $E[X_{T_n}] \to E[X_T]$.

Then $D = \inf\{t | X_t = Y_t\}$ is an optimal stopping time. Also, for any stopping time S, the stopping time $D(S)$ defined by

$$D(S) = \inf\{t | t \geqq S, X_t = Y_t\} \tag{2.6}$$

is optimal, that is, $E[X_{D(S)}] = \sup_{T\geqq S} E[X_T]$. Furthermore, Y satisfies (2.5).

Next we give the penalty method, approximating the Snell envelope Y of X when X is of the form:

$$X_t = e^{-\alpha t}f_t + \int_0^t e^{-\alpha s}g_s ds\ , \quad f, g \in W\ , \quad \alpha > 0\ . \tag{2.7}$$

Note that Y can be rewritten as follows:

$$Y_t = e^{-\alpha t}z(t) + \int_0^t e^{-\alpha s}g_s ds\ , \tag{2.8}$$

where $z(t) = \text{ess sup}_{T\geqq t} E\left[e^{-\alpha(T-t)}f_T + \int_t^T e^{-\alpha(s-t)}g_s ds \middle| F_t\right]$. We define the linear operators $(G_\alpha)_{\alpha>0}$ from W into itself by

$$G_\alpha x(t) = E\left[\int_t^\infty e^{-\alpha(s-t)}x_s ds \middle| F_t\right]\ , \quad x \in W\ . \tag{2.9}$$

G_α is one-to-one and satisfies the resolvent equation. Let A be the generator from $D[A]$ into W, defined by

$$A = \alpha - G_\alpha^{-1}\ , \qquad D[A] = G_\alpha(W)\ , \tag{2.10}$$

where G_α^{-1} is the inverse of G_α. Let us consider the solution z^ε of the penalty equation

$$(\alpha - A)z^\varepsilon - (f - z^\varepsilon)^+/\varepsilon = g\ , \quad \varepsilon > 0\ . \tag{2.11}$$

Let V_ε be the class of all progressively measurable processes $v = (v_t)$ such that $0 \leqq v_t \leqq 1/\varepsilon$. For $v \in V_\varepsilon$, we set

$$J_t(v) = E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(g_s + v_s f_s)ds \middle| F_t\right]\ . \tag{2.12}$$

Then the solution z^ε is given by

$$z^\varepsilon(t) = \underset{v\in V_\varepsilon}{\text{ess sup}}\, J_t(v)\ . \tag{2.13}$$

Letting $\varepsilon \downarrow 0$, we obtain that $z^\varepsilon(t)$ converges to $z(t)$ almost surely for each $t \geqq 0$.

3. The generalized Snell envelope. For each stopping time S, let $C_i(S)$, $i = 1, 2, \cdots, N$, denote the classes of all sequences $\bar{T} = (T_{i-1}, T_i, \cdots, T_N)$ of stopping times such that $S = T_{i-1} \leqq T_i \leqq \cdots \leqq T_N$.

THEOREM 1. *There exists a sequence $z = (z_1, z_2, \cdots, z_N, z_{N+1})$ of right continuous adapted processes z_i such that for each stopping time S and $i = 1, 2, \cdots, N$,*

$$z_i(S) = \underset{\bar{T}\in C_i(S)}{\text{ess sup}}\, E\left[\sum_{j=i}^N \int_{T_{j-1}}^{T_j} e^{-\alpha(t-S)}f_j(t)dt \middle| F_S\right]\ , \tag{3.1}$$

$$z_1(t) \geqq z_2(t) \geqq \cdots \geqq z_N(t) \geqq z_{N+1}(t) = 0\ , \quad t \geqq 0\ , \tag{3.2}$$

$$e^{-\alpha\cdot}z_i \in W\ , \tag{3.3}$$

(3.4) $$\lim_{t\to\infty} e^{-\alpha t} z_i(t) = 0 ,$$

(3.5) $$\left(e^{-\alpha t} z_i(t) + \int_0^t e^{-\alpha s} f_i(s)ds\right) \text{ is a supermartingale ,}$$

(3.6) $$e^{-\alpha t} z_i(t) = \operatorname*{ess\,sup}_{T \geqq t} E\left[\int_t^T e^{-\alpha s} f_i(s)ds + e^{-\alpha T} z_{i+1}(T) \,\middle|\, F_t\right],$$

(3.7) $$E[z_1(0)] = \sup_{\hat{T} \in C} J(\hat{T}) .$$

PROOF. We define inductively $z = (z_1, z_2, \cdots, z_N, z_{N+1})$ as follows: we set $z_{N+1}(t) = 0$, and if $z_{i+1}(t)$ is given, we define $\hat{z}_i$ by

$$e^{-\alpha t} \hat{z}_i(t) = \operatorname*{ess\,sup}_{T \geqq t} E\left[\int_t^T e^{-\alpha s} f_i(s)ds + e^{-\alpha T} z_{i+1}(T) \,\middle|\, F_t\right].$$

Since the process $\left(e^{-\alpha t}\hat{z}_i(t) + \int_0^t e^{-\alpha s} f_i(s)ds\right)$ is the Snell envelope of the process $\left(e^{-\alpha t} z_{i+1}(t) + \int_0^t e^{-\alpha s} f_i(s)ds\right)$, it has a right continuous modification by (2.2), denoted by $\left(e^{-\alpha t} z_i(t) + \int_0^t e^{-\alpha s} f_i(s)ds\right)$. It is clear that (3.6) and (3.3) are verified. (3.2) follows immediately from (3.6), and we obtain (3.5), combining (3.6) with (2.2). By (2.1) and (3.6),

$$\lim_{t\to\infty}\left(e^{-\alpha t} z_i(t) + \int_0^t e^{-\alpha s} f_i(s)ds\right) = \limsup_{t\to\infty}\left(e^{-\alpha t} z_{i+1}(t) + \int_0^t e^{-\alpha s} f_i(s)ds\right),$$

from which

$$\lim_{t\to\infty} e^{-\alpha t} z_i(t) \leqq \limsup_{t\to\infty} e^{-\alpha t} z_{N+1}(t) = 0 .$$

This implies (3.4) by (3.2). We show that (3.1) holds. We denote by $y_i(S)$, $i = 1, 2, \cdots, N$, the right hand side of (3.1), and for each stopping time S, we set

$$Y_i(S) = e^{-\alpha S} y_i(S) + \int_0^S e^{-\alpha t} f_i(t)dt = \operatorname*{ess\,sup}_{\bar{T} \in C_i(S)} G(S; T_i, T_{i+1}, \cdots, T_N) ,$$

where $G(S; T_i, T_{i+1}, \cdots, T_N) = E\left[\int_0^{T_i} e^{-\alpha t} f_i(t)dt + \sum_{j=i+1}^{N} \int_{T_{j-1}}^{T_j} e^{-\alpha t} f_j(t)dt \,\middle|\, F_S\right]$ for $\bar{T} = (T_{i-1}, T_i, \cdots, T_N) \in C_i(S)$. Then for any stopping times $S \leqq T$, we have

(3.8) $$E[Y_i(T) \mid F_S] \leqq Y_i(S) .$$

Indeed, let Γ be the class of all F_T-measurable functions $G(T; T_i, T_{i+1}, \cdots, T_N)$ for each $\bar{T} = (T_{i-1}, T_i, \cdots, T_N) \in C_i(T)$. For $\bar{T}(k) \in C_i(T)$ with $\bar{T}(k) = (T_{i-1}(k), T_i(k), \cdots, T_N(k))$ $(k = 1, 2)$, we define $\bar{T} = (T_{i-1}, T_i, \cdots, T_N) \in C_i(T)$

by $T_j = T_j(1)I_B + T_j(2)I_{B^c}$, where $B = \{G(T; T_i(1), T_{i+1}(1), \cdots, T_N(1)) \geqq G(T; T_i(2), T_{i+1}(2), \cdots, T_N(2))\}$. Then it is easy to see that

$$G(T; T_i(1), \cdots, T_N(1)) \vee G(T; T_i(2), \cdots, T_N(2)) = G(T; T_i, \cdots, T_N)\ ,$$

that is, Γ is closed under the operation "sup". By Proposition VI-1-1 of [10], there exists a sequence $\bar{T}(n) = (T_{i-1}(n), T_i(n), \cdots, T_N(n)) \in C_i(T)$ such that

$$Y_i(T) = \lim_{n\to\infty} \uparrow G(T; T_i(n), T_{i+1}(n), \cdots, T_N(n))\ .$$

Hence, for any $\bar{B} \in F_S$,

$$E[Y_i(T)I_{\bar{B}}] = \sup_{\bar{T}\in C_i(T)} E[G(T; T_i, T_{i+1}, \cdots, T_N)I_{\bar{B}}]\ . \tag{3.9}$$

Since $C_i(T) \subset C_i(S)$, the right hand side of (3.9) is less than $\sup_{\bar{T}\in C_i(S)} E[G(S; T_i, T_{i+1}, \cdots, T_N)I_{\bar{B}}] = E[Y_i(S)I_{\bar{B}}]$ by (3.9). Therefore we obtain (3.8).

Now, let us note that $z_N(S) = y_N(S)$ for any stopping time S by the definition of y_N, applying (2.4) to (3.6). Suppose that $y_j(S') \geqq z_j(S')$ for any stopping time S' and $j = i + 1$. Since $y_i(S') \geqq y_{i+1}(S')$ by definition, (3.8) yields that

$$\begin{aligned} e^{-\alpha S}y_i(S) + \int_0^S e^{-\alpha t}f_i(t)dt &\geqq E\Big[\int_0^T e^{-\alpha t}f_i(t)dt + e^{-\alpha T}y_i(T)\Big|F_S\Big] \\ &\geqq E\Big[\int_0^T e^{-\alpha t}f_i(t)dt + e^{-\alpha T}z_{i+1}(T)\Big|F_S\Big]\ . \end{aligned}$$

Thus, by (2.4) and (3.6),

$$y_i(S) \geqq \operatorname*{ess\,sup}_{T\geqq S} E\Big[\int_S^T e^{-\alpha(t-S)}f_i(t)dt + e^{-\alpha(T-S)}z_{i+1}(T)\Big|F_S\Big] = z_i(S)\ .$$

Conversely, let us suppose that $y_j(S') \leqq z_j(S')$ for any stopping time S' and $j = i+1$. By the definition of y_{i+1}, for $\bar{T}' = (T'_i, T'_{i+1}, \cdots, T'_N) \in C_{i+1}(S')$, we have

$$e^{-\alpha S'}y_{i+1}(S') \geqq E\Big[\sum_{j=i+1}^N \int_{T'_{j-1}}^{T'_j} e^{-\alpha t}f_j(t)dt\Big|F_{S'}\Big]\ .$$

Hence, by (2.4) and (3.6), for any $\bar{T} = (T_{i-1}, T_i, \cdots, T_N) \in C_i(S)$,

$$\begin{aligned} &E\Big[\sum_{j=i}^N \int_{T_{j-1}}^{T_j} e^{-\alpha(t-S)}f_i(t)dt\Big|F_S\Big] \\ &\quad = E\Big[\int_S^{T_i} e^{-\alpha(t-S)}f_i(t)dt + e^{\alpha S}E\Big[\sum_{j=i+1}^N \int_{T_{j-1}}^{T_j} e^{-\alpha t}f_j(t)dt\Big|F_{T_i}\Big]\Big|F_S\Big] \\ &\quad \leqq E\Big[\int_S^{T_i} e^{-\alpha(t-S)}f_i(t)dt + e^{-\alpha(T_i-S)}y_{i+1}(T_i)\Big|F_S\Big] \\ &\quad \leqq E\Big[\int_S^{T_i} e^{-\alpha(t-S)}f_i(t)dt + e^{-\alpha(T_i-S)}z_{i+1}(T_i)\Big|F_S\Big] \leqq z_i(S)\ . \end{aligned}$$

Consequently, we have (3.1). (3.7) follows from (3.1) and (3.9). The theorem is established.

4. Existence of optimal strategies.

THEOREM 2. *The strategy* $\hat{T}^* = (T_1^*, T_2^*, \cdots, T_N^*) \in C$ *given by*

$$T_i^* = \inf\{t \geqq T_{i-1}^* \mid z_i(t) = z_{i+1}(t)\}, \quad T_0^* = 0, \tag{4.1}$$

is optimal in Problem (I).

PROOF. Let us note that the process z_i satisfies (2.5) for each $i = 1, 2, \cdots, N, N+1$. Indeed, clearly z_{N+1} satisfies (2.5). Suppose that z_{i+1} satisfies (2.5). Then, by (3.6), the process $\left(e^{-\alpha t}z_i(t) + \int_0^t e^{-\alpha s}f_i(s)ds\right)$ is the Snell envelope of the process $\left(e^{-\alpha t}z_{i+1}(t) + \int_0^t e^{-\alpha s}f_i(s)ds\right)$, which satisfies (2.5). Thus, as is described in § 2, the process $\left(e^{-\alpha t}z_i(t) + \int_0^t e^{-\alpha s}f_i(s)ds\right)$ satisfies (2.5), and so does z_i.

Next, we show that for T_i^* of (4.1), $i = 1, 2, \cdots, N$, we have

$$\sup_{\hat{T} \in C} J(\hat{T}) = E\left[\int_0^{T_1^*} e^{-\alpha s}f_1(s)ds + e^{-\alpha T_1^*}z_2(T_1^*)\right], \tag{4.2}$$

$$E[e^{-\alpha T_{i-1}^*}z_i(T_{i-1}^*)] = E\left[\int_{T_{i-1}^*}^{T_i^*} e^{-\alpha s}f_i(s)ds + e^{-\alpha T_i^*}z_{i+1}(T_i^*)\right]. \tag{4.3}$$

By (3.7) and (3.6), we have

$$\begin{aligned}\sup_{\hat{T} \in C} J(\hat{T}) = E[z_1(0)] &= E\left[\operatorname*{ess\,sup}_{T \geqq 0} E\left[\int_0^T e^{-\alpha t}f_1(t)\,dt + e^{-\alpha T}z_2(T) \,\middle|\, F_0\right]\right] \\ &= \sup_T E\left[\int_0^T e^{-\alpha t}f_1(t)\,dt + e^{-\alpha T}z_2(T)\right],\end{aligned}$$

which follows from (2.3). Since the process $\left(e^{-\alpha t}z_1(t) + \int_0^t e^{-\alpha s}f_1(s)ds\right)$ is the Snell envelope of the process $\left(e^{-\alpha t}z_2(t) + \int_0^t e^{-\alpha s}f_1(s)ds\right)$, we have (4.2) by (2.6). On the other hand, by (3.6) and (2.4), we have

$$e^{-\alpha T_{i-1}^*}z_i(T_{i-1}^*) = \operatorname*{ess\,sup}_{T \geqq T_{i-1}^*} E\left[\int_{T_{i-1}^*}^T e^{-\alpha s}f_i(s)ds + e^{-\alpha T}z_{i+1}(T) \,\middle|\, F_{T_{i-1}^*}\right].$$

Hence, by (2.4),

$$E[e^{-\alpha T_{i-1}^*}z_i(T_{i-1}^*)] = \sup_{T \geqq T_{i-1}^*} E\left[\int_{T_{i-1}^*}^T e^{-\alpha s}f_i(s)ds + e^{-\alpha T}z_{i+1}(T)\right]. \tag{4.4}$$

Also, by (2.6) and the above arguments,

$$(4.5)\qquad E\left[\int_0^{T_i^*} e^{-\alpha s} f_i(s)ds + e^{-\alpha T_i^*} z_{i+1}(T_i^*)\right] = \sup_{T \geqq T_{i-1}^*} E\left[\int_0^T e^{-\alpha s} f_i(s)ds + e^{-\alpha T} z_{i+1}(T)\right].$$

Thus (4.3) follows from (4.4) and (4.5). Combining (4.2) with (4.3), we establish the theorem by induction.

5. Conditions for optimality. Let us consider the following problem.

PROBLEM (II). *Let U be the class of all processes $u \in W$ defined by*

$$u(t) = u_{\hat{T}}(t) = \sum_{n=0}^{N} I_{(T_n \leqq t)}, \qquad \hat{T} = (T_1, T_2, \cdots, T_N) \in C, \qquad T_0 = 0.$$

We define the profit:

$$I(u) = E\left[\int_0^\infty e^{-\alpha s} f(s, u(s))ds\right], \quad u \in U, \quad \alpha > 0,$$

where $f(s, i) = f_i(s)$ is as in Problem (I). *Find an optimal strategy $u^* \in U$, i.e., $I(u^*) = \sup_{u \in U} I(u)$, and characterize its maximum.*

Problems (I) and (III) are identical in the following sense:

$$(5.1)\qquad \sup_{\hat{T} \in C} J(\hat{T}) = \sup_{u \in U} I(u).$$

Indeed, taking into account $f(s, N+1) = 0$, we have

$$I(u_{\hat{T}}) = E\left[\int_0^\infty e^{-\alpha s} f(s, u_{\hat{T}}(s))ds\right] = E\left[\sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} e^{-\alpha s} f(s, u_{\hat{T}}(s))ds\right]$$
$$= E\left[\sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} e^{-\alpha s} f(s, i)ds\right] = J(\hat{T}),$$

which implies (5.1). Thus, Problem (I) is a kind of stochastic control problem with the profit $I(u)$. We would like to obtain dynamic programming conditions for optimality. But U is not closed under concatenation, i.e., if $u, v \in U$, then (u, v, t) does not necessarily belong to U where $(u, v, t)(s) = u(s)$ for $s \leqq t$, $= v(s)$ for $s > t$. Therefore we cannot apply the technique of [3] and [4] to $I(u)$. Here we show that Theorem 1 enables us to give an optimality condition.

THEOREM 3. *For each $\hat{T} = (T_1, T_2, \cdots, T_N) \in C$ and $i = 1, 2, \cdots, N$,*

$$(5.2)\qquad E[e^{-\alpha T_{i-1}} z_i(T_{i-1})] \geqq E\left[\int_{T_{i-1}}^{T_i} e^{-\alpha s} f_i(s)ds + e^{-\alpha T_i} z_{i+1}(T_i)\right].$$

Furthermore, $\hat{T}^* = (T_1^*, T_2^*, \cdots, T_N^*) \in C$ is optimal if and only if for each $i = 1, 2, \cdots, N$,

$$(5.3)\qquad E[e^{-\alpha T_{i-1}^*} z_i(T_{i-1}^*)] = E\left[\int_{T_{i-1}^*}^{T_i^*} e^{-\alpha s} f_i(s)ds + e^{-\alpha T_i^*} z_{i+1}(T_i^*)\right].$$

PROOF. (5.2) is immediate from (3.6) and (2.4). Suppose that (5.3) holds. Then, by induction, we have

$$E[z_1(0)] = E\left[\sum_{j=1}^{N} \int_{T_{j-1}^*}^{T_j^*} e^{-\alpha s} f_j(s) ds\right].$$

Thus $\hat{T}^*$ is optimal by (3.7). Conversely, let $\hat{T}^* \in C$ be optimal. To prove (5.3), it suffices to show that for $i = 1, 2, \cdots, N$,

$$E[e^{-\alpha T_{i-1}^*} z_i(T_{i-1}^*)] = E\left[\sum_{j=i}^{N} \int_{T_{j-1}^*}^{T_j^*} e^{-\alpha s} f_j(s) ds\right]. \tag{5.4}$$

By the optimality of $\hat{T}^*$, (5.4) holds for $i = 1$. Suppose that (5.4) holds for $i = k$. By (3.1),

$$\begin{aligned} E[e^{-\alpha T_{k-1}^*} z_k(T_{k-1}^*)] &= E\left[\sum_{j=k}^{N} \int_{T_{j-1}^*}^{T_j^*} e^{-\alpha s} f_j(s) ds\right] \\ &= E\left[\int_{T_{k-1}^*}^{T_k^*} e^{-\alpha s} f_k(s) ds + \sum_{j=k+1}^{N} \int_{T_{j-1}^*}^{T_j^*} e^{-\alpha s} f_j(s) ds\right] \\ &\leqq E\left[\int_{T_{k-1}^*}^{T_k^*} e^{-\alpha s} f_k(s) ds + \operatorname*{ess\,sup}_{\hat{T}^* \in C_{k+1}(T_k^*)} E\left[\sum_{j=k+1}^{N} \int_{T_{j-1}^*}^{T_j^*} e^{-\alpha s} f_j(s) ds \,\middle|\, F_{T_k^*}\right]\right] \\ &= E\left[\int_{T_{k-1}^*}^{T_k^*} e^{-\alpha s} f_k(s) ds + e^{-\alpha T_k^*} z_{k+1}(T_k^*)\right]. \end{aligned} \tag{5.5}$$

On the other hand, by (3.6) and (2.4), we obtain the converse inequality of (5.5). Thus, (5.4) holds for $i = k + 1$.

6. Penalization. Let Φ be the class of all sequences $x = (x_1, x_2, \cdots, x_N)$ of right continuous adapted processes x_i, $i = 1, 2, \cdots, N$, such that

$$e^{-\alpha \cdot} x_i \in W \quad \text{and} \quad \lim_{t\to\infty} e^{-\alpha t} x_i(t) = 0 , \tag{6.1}$$

$$x_1(t) \geqq x_2(t) \geqq \cdots \geqq x_N(t) \geqq 0 , \quad t \geqq 0 , \tag{6.2}$$

$$\left(e^{-\alpha t} x_i(t) + \int_0^t e^{-\alpha s} f_i(s) ds\right) \text{ is a supermartingale .} \tag{6.3}$$

Let us consider the penalty equation:

$$(\alpha - A) z_i^\varepsilon - (z_{i+1}^\varepsilon - z_i^\varepsilon)^+/\varepsilon = f_i , \qquad z_{N+1}^\varepsilon = 0 , \quad i = 1, 2, \cdots, N . \tag{6.4}$$

By (2.11) and induction, the equation (6.4) has a solution $z^\varepsilon = (z_1^\varepsilon, z_2^\varepsilon, \cdots, z_N^\varepsilon)$, and then we can obtain the following theorem.

THEOREM 4. *Let $z = (z_1, z_2, \cdots, z_N)$ be a sequence of the processes z_i, $i = 1, 2, \cdots, N$, given in Theorem 1. Then $z_i^\varepsilon(t)$ converges to $z_i(t)$ almost surely for each $t \geqq 0$ and $i = 1, 2, \cdots, N$, as $\varepsilon \downarrow 0$.*

LEMMA 5. *For each* $i = 1, 2, \cdots, N$, *we have*

$$z_i^\varepsilon(t) \leqq z_i(t) , \tag{6.5}$$

$$z_i^\varepsilon(t) \leqq z_i^{\varepsilon'}(t) \quad for \quad \varepsilon \geqq \varepsilon' . \tag{6.6}$$

PROOF. By definition, (6.5) holds for $i = N + 1$. Suppose that (6.5) holds for $i = k + 1$. Then we have $z_{k+1}^\varepsilon \leqq z_{k+1} \leqq z_k$ by (3.2). Thus, by (2.13),

$$\begin{aligned} z_k^\varepsilon(t) &= \operatorname*{ess\,sup}_{v \in V_\varepsilon} E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(f_k(s) + v_s z_{K+1}^\varepsilon(s))\, ds \,\middle|\, F_t\right] \\ &\leqq \operatorname*{ess\,sup}_{v \in V_\varepsilon} E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(f_k(s) + v_s z_k(s))\, ds \,\middle|\, F_t\right] , \end{aligned}$$

which is less than $z_k(t)$ by virtue of Lemma 5 in [9]. This implies (6.5). Next, taking into account $z_{N+1}^\varepsilon(t) = z_{N+1}^{\varepsilon'}(t) = 0$, we assume that $z_{i+1}^\varepsilon \leqq z_{i+1}^{\varepsilon'}$ for $\varepsilon \geqq \varepsilon'$. Then, by (2.13),

$$\begin{aligned} z_i^\varepsilon(t) &= \operatorname*{ess\,sup}_{v \in V_\varepsilon} E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(f_i(s) + v_s z_{i+1}^\varepsilon(s)) ds \,\middle|\, F_t\right] \\ &\leqq \operatorname*{ess\,sup}_{v \in V_{\varepsilon'}} E\left[\int_t^\infty \exp\left(-\int_t^s \alpha + v_r dr\right)(f_i(s) + v_s z_{i+1}^{\varepsilon'}(s)) ds \,\middle|\, F_t\right] = z_i^{\varepsilon'}(t) , \end{aligned}$$

which implies (6.6).

PROOF OF THEOREM 4. First we note that $z = (z_1, z_2, \cdots, z_N)$ of Theorem 1 belongs to Φ. Moreover, it is a minimal element of Φ. Indeed, let $x = (x_1, x_2, \cdots, x_N) \in \Phi$, $\hat{T} = (T_{i-1}, T_i, \cdots, T_N) \in C_i(t)$ and $i = 1, 2, \cdots, N$. By (6.3) and (6.2),

$$\begin{aligned} e^{-\alpha T_{j-1}} x_j(T_{j-1}) &\geqq E\left[e^{-\alpha T_j} x_j(T_j) + \int_{T_{j-1}}^{T_j} e^{-\alpha s} f_j(s) ds \,\middle|\, F_{T_{j-1}}\right] \\ &\geqq E\left[e^{-\alpha T_j} x_{j+1}(T_j) + \int_{T_{j-1}}^{T_j} e^{-\alpha s} f_j(s) ds \,\middle|\, F_{T_{j-1}}\right] , \quad j \geqq i . \end{aligned}$$

Hence, by induction,

$$e^{-\alpha t} x_i(t) \geqq E\left[\int_t^{T_i} e^{-\alpha s} f_i(s) ds + \sum_{j=i+1}^N \int_{T_{j-1}}^{T_j} e^{-\alpha s} f_j(s) ds \,\middle|\, F_t\right] .$$

By (3.1), we obtain $x_i \geqq z_i$ for $i = 1, 2, \cdots, N$.

Next, by (6.6), we can define the process $z_i^*(t) = \lim_{\varepsilon \downarrow 0} \uparrow z_i^\varepsilon(t)$. Then, by (6.5), we have $z_i^* \leqq z_i$. To prove the theorem, it suffices to show that $z^* = (z_1^*, z_2^*, \cdots, z_N^*)$ belongs to Φ. By (6.4) and Lemma 2 of [9],

$$\begin{aligned} A\left(e^{-\alpha t} z_i^\varepsilon(t) + \int_0^t e^{-\alpha s} f_i(s) ds\right) &= -e^{-\alpha t}(\alpha - A) z_i^\varepsilon(t) + e^{-\alpha t} f_i(t) \\ &= -e^{-\alpha t}(z_{i+1}^\varepsilon - z_i^\varepsilon)^+(t)/\varepsilon \leqq 0 . \end{aligned}$$

By virtue of the corollary to Theorem 5 in [9], $\left(e^{-\alpha t}z_i^\varepsilon(t) + \int_0^t e^{-\alpha s}f_i(s)ds\right)$ is a right continuous supermartingale. By the monotone convergence theorem and Theorem 16 of [8, Chap. VI], $\left(e^{-\alpha t}z_i^*(t) + \int_0^t e^{-\alpha s}f_i(s)ds\right)$ is a right continuous supermartingale. Thus z_i^* is a right continuous adapted process and z^* satisfies (6.3). Since $e^{-\alpha\cdot}z_i^\varepsilon \in W$ and $\lim_{t\to\infty} e^{-\alpha t}z_i^\varepsilon(t) = 0$ by $z_i^\varepsilon \in W$, z_i^* satisfies (6.1). By (6.4) and (6.5),

$$G_\alpha(z_{i+1}^\varepsilon - z_i^\varepsilon)^+(t) = \varepsilon(z_i^\varepsilon - G_\alpha f_i)(t) \leqq \varepsilon(z_i - G_\alpha f_i)(t) ,$$

which goes to zero as $\varepsilon \downarrow 0$. By the bounded convergence theorem, we have $G_\alpha(z_{i+1}^* - z_i^*)^+ = 0$. This implies (6.2). The proof is complete.

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