

KÄHLERIAN NORMAL COMPLEX SURFACES

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Introduction. By Artin [2] a normal (compact) Moishezon surface X is projective if it has only rational singularities. The purpose of this note is to prove a kählerian analogue of this result. We call a normal isolated surface singularity (X, x) *nondegenerate* if for some (and hence for any) resolution $f: (\tilde{X}, A) \rightarrow (X, x)$ ($A = f^{-1}(x)$) the natural homomorphism $R^1 f_* \mathbf{R} \rightarrow R^1 f_* \mathcal{O}_{\tilde{X}}$ is surjective. In particular a rational singularity is nondegenerate. Then our result is as follows.

THEOREM. *Let X be a normal compact complex surface in \mathcal{E} with only nondegenerate isolated singularities. Then X is Kähler.*

Recall that in general a compact complex space X is said to be in \mathcal{E} if it is a meromorphic image of a compact Kähler manifold [9]. In our case of a surface, it turns out, however, that this is equivalent to the condition that any nonsingular model of X is Kähler (Proposition 2). On the other hand, in general a complex space X is said to be *Kähler* if there exists an open covering $\{U_\mu\}$ of X and a system of C^∞ strictly plurisubharmonic functions $\{u_\mu\}$ with each u_μ defined on U_μ such that $u_\mu - u_\nu$ is pluriharmonic on $U_\mu \cap U_\nu$ (cf. [19], [9]). In this case $dd^c u_\mu = dd^c u_\nu$ on $U_\mu \cap U_\nu$ ($d^c = \sqrt{-1}(\bar{\partial} - \partial)$) where $d = \partial + \bar{\partial}$) and hence we get a globally defined real closed $(1, 1)$ -form α on X which is called a *Kähler form* on X .

In Section 1 we shall prove a lemma on extension of a strictly plurisubharmonic function across an isolated singular point (Lemma 1) as well as draw some of its consequences (including Proposition 2 mentioned above). Using this lemma, we shall derive in Section 2 a necessary and sufficient condition for a normal compact complex surface to be Kähler in terms of its resolution (Proposition 4); indeed Theorem above is an immediate consequence of this proposition. Next in Section 3 we shall give some characterization of nondegenerate singularities. In particular, it turns out at once that nondegenerate singularities are nothing but pararational singularities of Brenton [6]. Finally in Section 4 we shall note that every small deformation of a normal compact Kähler surface with only nondegenerate singularities is again Kähler by merely checking

the condition of Moishezon [19], in which the study of singular Kähler surfaces was initiated.

1. Extension of a strictly plurisubharmonic function. (1.1) Let Y be a complex space. Let $u: Y \rightarrow \mathbf{R} \cup \{-\infty\}$ be a map. Then u is said to be plurisubharmonic if for any point $y \in Y$ there exist a neighborhood $U \ni y$, an embedding $j: U \rightarrow G$ of U into a domain G in \mathbf{C}^N and a plurisubharmonic function u_G on G such that $j^*u_G = u|_U$. On the other hand, u is said to be *weakly plurisubharmonic* if for any holomorphic map $h: D \rightarrow Y$ of the unit disc $D: = \{t \in \mathbf{C}; |t| < 1\}$ into Y , its pull-back h^*u is either subharmonic or $= -\infty$. A plurisubharmonic function clearly is weakly plurisubharmonic. Conversely, it was recently shown by Fornaess-Narasimhan [7] that every weakly plurisubharmonic function is plurisubharmonic. Using this result we shall show the following lemma, which is indeed a main ingredient of our proof of Theorem.

LEMMA 1. *Let (X, x) be a normal isolated singularity with $\dim(X, x) \geq 2$. Let u be a strictly plurisubharmonic function on $X - x$. Then for any sufficiently small neighborhood V of x there exists a strictly plurisubharmonic function v on X such that $u = v$ on $X - V$.*

PROOF. By Grauert-Remmert [11] u extends to a unique weakly plurisubharmonic function on X which we shall still denote by the same letter u . Then by the result of Fornaess-Narasimhan cited above there exists a neighborhood U of x , an embedding $j: U \rightarrow B$ of U into an open ball B in $\mathbf{C}^N = \mathbf{C}^N(z)$ with center $j(x)$ and plurisubharmonic function \hat{u} on B such that $j^*\hat{u} = u$. We may assume that $j(x)$ is the origin of \mathbf{C}^N . Let $B_1 \subset B$ be another concentric ball. Then it suffices to show the lemma for $V = B_1 \cap X$ where we identify X with $j(X)$. Take another concentric ball $B_2 \subset B_1$. Let λ be a nonnegative C^∞ function on \mathbf{C}^N with support contained in B_1 such that $\lambda(z)$ is a positive constant on B_2 , λ depends only on $|z|$ and $\int_{\mathbf{C}^N} \lambda(z) dV(z) = 1$, where $dV(z)$ is the standard volume form on \mathbf{C}^N . Then as in Miyaoka [17] we set

$$\tilde{u}_\varepsilon(z) = \int_{\mathbf{C}^N} \hat{u}(z - \varepsilon\lambda(z)\zeta)\lambda(\zeta)dV(\zeta)$$

for any real number $\varepsilon > 0$. Then for any sufficiently small $\varepsilon > 0$, we see that $\tilde{u}_\varepsilon \equiv \hat{u}$ on $B - B_1$, $\tilde{u}_\varepsilon|_{U - B_2 \cap U}$ is strictly plurisubharmonic and \tilde{u}_ε is plurisubharmonic on B_2 (cf. [17]). Then for any such ε we set

$$v_\delta(z) = \tilde{u}_\varepsilon(z) + \delta|z|^2\lambda(z), \quad \delta > 0.$$

Then $v_\delta(z) \equiv \hat{u}(z)$ on $B - B_1$ and $v_\delta(z)$ is strictly plurisubharmonic on B_2 .

Moreover it is easy to see that if δ is sufficiently small, $v_\delta(z)|_{U-B_2 \cap U}$ also is strictly plurisubharmonic. For such a δ we can define a C^∞ strictly plurisubharmonic function v on X by setting $v = u$ on $X - V$ and $v = v_\delta|_V$ on V . q.e.d.

COROLLARY. *Let (X, x) be as above. Let α be a Kähler form on $X - x$. Suppose that α is written on $X - x$ in the form $\alpha = dd^c u$ for some C^∞ function u on $X - x$. Then for any sufficiently small neighborhood V of x there exists a Kähler form α' on X such that $\alpha' = \alpha$ on $X - V$.*

PROOF. Since α is a Kähler form, u is strictly plurisubharmonic. Hence we have only to set $\alpha' = dd^c v$ where v is as in Lemma 1. q.e.d.

(1.2) As an immediate application of the above corollary we shall show the following:

PROPOSITION 1. *Let X be an irreducible normal complex space with $\dim X \geq 2$. Suppose that X has only isolated quotient singularities. Let U be a smooth Zariski open subset of X such that $\dim(X - U) = 0$. Let α be a Kähler form on U . Then for any sufficiently small neighborhood V of $X - U$ in X we can find a Kähler form $\tilde{\alpha}$ on X such that $\tilde{\alpha}|_{U - U \cap V} = \alpha|_{U - U \cap V}$. In particular, X is a Kähler space, if so is U .*

EXAMPLE. Let X be a Kähler manifold and G a finite group of biholomorphic automorphisms of X . Suppose that the set of those points which are fixed by some elements of G is isolated. Then X/G is a Kähler space.

Now Proposition 1 is clearly a consequence of the following local version of it.

LEMMA 2. *Let (X, x) be an isolated quotient singularity with $\dim(X, x) \geq 2$. Let α be a Kähler form on $X - x$. Then for any sufficiently small neighborhood V of x there exists a Kähler form α' on X such that $\alpha' = \alpha$ on $X - V$.*

PROOF. Let $n = \dim(X, x)$. We can find a neighborhood D of the origin o of C^n and a finite subgroup $G \subseteq GL(n, C)$ leaving D invariant such that (1) $X \cong D/G$ for a suitable representative X of (X, x) and (2) the natural map $\pi: D \rightarrow X$ is unramified over $X' := X - x$ (cf. Prill [20]). Let $\pi': D' \rightarrow X'$ be the induced map, where $D' = D - \{o\}$. Let $\tilde{\alpha} = \pi'^* \alpha$. Then $\tilde{\alpha}$ is a G -invariant Kähler form on D' . Since $n \geq 2$, there exists a C^∞ strictly plurisubharmonic function \tilde{u} on D' such that $\tilde{\alpha} = dd^c \tilde{u}$ by Shiffman [22]. Replacing \tilde{u} by $(1/|G|) \sum_{g \in G} g^* \tilde{u}$ we may assume that \tilde{u}

also is G -invariant. Then \tilde{u} is a pull-back of a C^∞ function u on X' and $\alpha = dd^c u$ on X' . The lemma then follows from Corollary to Lemma 1. q.e.d.

(1.3) As an application of Proposition 1 we shall show that any compact smooth analytic surface X in \mathcal{E} is Kähler. The following special case is essential:

LEMMA 3. *Let X be a compact smooth analytic surface which is bimeromorphic to a K3 surface of algebraic dimension zero. Then X is Kähler if $X \in \mathcal{E}$.*

PROOF. We may assume that X is minimal; otherwise X is obtained by a blowing up of the minimal one, say X_0 , and hence is Kähler if X_0 is (cf. [9]). Then X contains only finite number of irreducible curves C_i , $i = 1, \dots, m$, and C_i are nonsingular rational curves with self-intersection number $C_i \cdot C_i = -2$ (cf. [13]). Moreover the intersection matrix $(C_i \cdot C_j)$ is negative definite. Hence there is a contraction $f: X \rightarrow X'$ of $C := \bigcup_{i=1}^m C_i$ to a finite number of rational double points p_1, \dots, p_s of X' . In particular, X' has only isolated quotient singularities. We show that X' is a Kähler space. Since f is projective, this would show that X is Kähler (cf. [9]). Now by our assumption, there exists a compact Kähler manifold Y and a surjective holomorphic map $h: Y \rightarrow X$. Let $h' = fh$. Since X' contains no curve, h' must be smooth over a Zariski open subset $U \subseteq X'$ whose complement consists of a finite number of points. Let α be a Kähler form on Y . By integration along the fibers, $h'_* \alpha^{r+1}$ defines a Kähler form α' on U , where $r = \dim h$. The result then follows from Proposition 1. (Since a rational singularity is nondegenerate, we can also use Proposition 3 below instead of Proposition 1.) q.e.d.

REMARK 1. The proof actually shows that for any tubular neighborhood of C in X there exists a Kähler form $\hat{\alpha}$ on X such that $\hat{\alpha} = h'_* \alpha^{r+1}$ on its complement in the notation above.

PROPOSITION 2. *Let X be a compact complex manifold of dimension 2. Suppose that $X \in \mathcal{E}$. Then X is Kähler.*

PROOF. Since $b_1(X)$ is even, we have only to consider the case where X is a K3 surface of algebraic dimension zero (cf. Miyaoka [18]). In this case X is Kähler by Lemma 3. q.e.d.

2. Extension of a Kähler form. (2.0) First we shall fix some notations, terminologies and conventions.

(a) Let (X, x) be a normal isolated singularity. Let $f: (\tilde{X}, A) \rightarrow (X, x)$ be a resolution of (X, x) where $A = f^{-1}(x)$. We usually identify \tilde{X} with any tubular neighborhood of A in \tilde{X} ; in this note a *tubular neighborhood* shall mean a relatively compact open neighborhood U with C^∞ strongly pseudoconvex boundary such that A is a strong deformation retract of U . (Note that X is then necessarily Stein.) In particular, the restriction maps $H^i(\tilde{X}, \mathbf{R}) \rightarrow H^i(A, \mathbf{R})$ are isomorphic for all $i \geq 0$, and $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong (R^i f_* \mathcal{O}_{\tilde{X}})_x$ for all $i > 0$. Thus (when $\dim(X, x) = 2$), (X, x) is nondegenerate if and only if the natural map $e: H^2(\tilde{X}, \mathbf{R}) \rightarrow H^2(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is surjective.

Take a local embedding $(X, x) \rightarrow (\mathbf{C}^n, o)$. Then we can always take U as above in such a way that $H^i(U - A, \mathbf{R}) \cong H^i(U - f^{-1}(\bar{B}_\epsilon), \mathbf{R}) \cong H^i(K_\epsilon, \mathbf{R})$ for all sufficiently small $\epsilon > 0$, where \bar{B}_ϵ is the closed ball of radius ϵ with center o and K_ϵ is the boundary of \bar{B}_ϵ . In fact, we may set $U = f^{-1}(B_{\epsilon_0})$ for some $\epsilon_0 > 0$, where B_{ϵ_0} is the interior of \bar{B}_{ϵ_0} .

(b) Let Y be a reduced complex space. We denote by \mathcal{P}_Y the sheaf of germs of pluriharmonic functions on Y . Then we have the short exact sequence of abelian groups on Y ;

$$(1) \quad 0 \rightarrow \mathbf{R} \xrightarrow{\iota} \mathcal{O}_Y \xrightarrow{\mu_Y} \mathcal{P}_Y \rightarrow 0,$$

where ι is the natural inclusion and $\mu(g) = -(\text{imaginary part of } g)$. (When Y is smooth, this is well-known and the general case follows from the smooth case readily.) From this we get the long exact sequence of cohomology

$$(2) \quad \rightarrow H^1(Y, \mathbf{R}) \rightarrow H^1(Y, \mathcal{O}_Y) \rightarrow H^1(Y, \mathcal{P}_Y) \xrightarrow{\delta} H^2(Y, \mathbf{R}).$$

On the other hand, let $\mu^*: \mathcal{O}_Y^* \rightarrow \mathcal{P}_Y$ be defined by $\mu^*(g) = -(1/2\pi) \log |g|$. Let $\hat{c}: H^1(Y, \mathcal{O}_Y^*) \rightarrow H^1(Y, \mathcal{P}_Y)$ be the resulting homomorphism. Then for any line bundle L on Y we call $\hat{c}(L)$ the *refined Chern class* of L . One checks readily that $\delta \hat{c}(L) = c(L)$, where $c(L)$ is the real Chern class of L .

(c) Let Y be as in (b). Let $\alpha = \{dd^c u_\mu\}$ be a Kähler form on Y with respect to an open covering $\mathfrak{U} = \{U_\mu\}$ as in the introduction. Then α defines a class $\bar{\alpha} \in H^1(Y, \mathcal{P}_Y)$ which is the image of the class $\{u_\mu - u_\nu\} \in H^1(\mathfrak{U}, \mathcal{P}_Y)$ under the natural homomorphism $H^1(\mathfrak{U}, \mathcal{P}_Y) \rightarrow H^1(Y, \mathcal{P}_Y)$. We call $\bar{\alpha}$ the *Kähler class* defined by α .

When Y is nonsingular, we have the natural isomorphism $H^1(Y, \mathcal{P}_Y) \cong Z^{1,1}/dd^c E$, where $Z^{1,1}$ is the real vector space of real d -closed $C^\infty(1, 1)$ -forms on Y and E is that of real C^∞ functions on Y (cf. [22]). Under this isomorphism the Kähler class $\bar{\alpha}$ above corresponds precisely to the Kähler form α considered modulo $dd^c E$.

(2.1) Let (X, x) be a normal isolated surface singularity. Let $f: (\tilde{X}, A) \rightarrow (X, x)$ be a resolution. For a Kähler form α on \tilde{X} we shall consider the following condition (E):

(E) Regard α as a Kähler form on $X - x$ by restriction to $\tilde{X} - A$ followed by the isomorphism $\tilde{X} - A \cong X - x$. Then for any sufficiently small neighborhood V of x in X there exists a Kähler form α' on X such that $\alpha' = \alpha$ on $X - V$ (i.e., the conclusion of Corollary to Lemma 1 holds true for α).

On the other hand, let A_1, \dots, A_m be the irreducible components of A . Then we shall denote by $H(A)$ the (real) linear subspace of $H^1(\tilde{X}, \mathcal{P}_{\tilde{X}})$ generated by the refined Chern classes $\hat{c}([A_i])$ of $[A_i]$, where $[A_i]$ is the line bundle defined by A_i . Further, recall the exact sequence (2) for $Y = \tilde{X}$;

$$(3) \quad \rightarrow H^1(\tilde{X}, \mathbf{R}) \xrightarrow{e} H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \xrightarrow{d} H^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) \rightarrow H^2(\tilde{X}, \mathbf{R}).$$

PROPOSITION 3. *Let (X, x) be a normal isolated surface singularity. Let $f: (\tilde{X}, A) \rightarrow (X, x)$ be a resolution. Then:*

(1) *we have the natural direct sum decomposition $H^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) = H(A) \oplus \text{Im } d$, where Im denotes the image, and*

(2) *for any Kähler form α on \tilde{X} , the following conditions are equivalent;*

- (a) *α satisfies the condition (E) above, and*
- (b) *the Kähler class $\bar{\alpha} \in H^1(\tilde{X}, \mathcal{P}_{\tilde{X}})$ defined by α belongs to $H(A)$.*

PROOF. (1) We consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 H_A^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H_A^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) & \xrightarrow{h} & H_A^2(\tilde{X}, \mathbf{R}) & \xrightarrow{b} & H_A^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \\
 \downarrow & & \downarrow c & & \downarrow a & & \downarrow \\
 H^1(\tilde{X}, \mathbf{R}) & \xrightarrow{e} & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \xrightarrow{d} & H^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) & \xrightarrow{\delta} & H^2(\tilde{X}, \mathbf{R}) \longrightarrow H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \\
 & & & & \downarrow r & & \\
 & & & & H^1(\tilde{X} - A, \mathcal{P}_{\tilde{X}}) & &
 \end{array}$$

where the horizontal sequences come from (1) for \tilde{X} and the vertical lines are part of the local cohomology exact sequences. We first show that a is isomorphic. First, we have the natural isomorphisms $H_A^2(\tilde{X}, \mathbf{R}) \cong \bigoplus_{i=1}^m H_{A_i}^2(\tilde{X}, \mathbf{R}) \cong \bigoplus_{i=1}^m H^0(A_i, \mathbf{R}) \cong \mathbf{R}^{\oplus m}$ and $H^2(\tilde{X}, \mathbf{R}) \cong H^2(A, \mathbf{R}) \cong \bigoplus_{i=1}^m H^2(A_i, \mathbf{R}) \cong \mathbf{R}^{\oplus m}$, where A_1, \dots, A_m are the irreducible components of A . Then a is given by; $a(y_1, \dots, y_m) = (\dots, \sum_{i=1}^m y_i(A_i \cdot A_j), \dots)$ where $(A_i \cdot A_j)$ is the intersection number of A_i and A_j . Indeed, the image of

$(0, \dots, 1, \dots, 0)$ (1 in the i -th place) in $H^2(\tilde{X}, \mathbf{R})$ by a is nothing but the real Chern class $c([A_i])$ of $[A_i]$. Since the intersection matrix is negative definite [10], a is isomorphic as was desired. Next we show that b is the zero map. First note that the (topological) dual map b' of b is given by $b': H^0(A, \Omega_{\tilde{X}|A}^2) \rightarrow H^2(A, \mathbf{R})$ (cf. [3]). But since the restriction of any holomorphic 2-form α on \tilde{X} to A vanishes (so that the integral of α over each A_i is zero), b' is the zero map. Hence so is b . Thus h is isomorphic, since $H_A^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ by Grauert-Riemenschneider [12]. Then δc also is isomorphic and we have the direct sum decomposition $H^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) = \text{Im } c \oplus \text{Im } d$. It suffices to show that $\text{Im } c = H(A)$. Consider the following commutative diagram

$$\begin{CD} H_A^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*) @>\hat{c}_A>> H_A^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) @>h>> H_A^2(\tilde{X}, \mathbf{R}) \\ @V k VV @VV c V @VV a V \\ H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*) @>\hat{c}>> H^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) @>\delta>> H^2(\tilde{X}, \mathbf{R}), \end{CD}$$

where \hat{c}_A is defined in the same way as \hat{c} . (The sequences are in general not exact.) Obviously the line bundles $[A_i]$ are in the image of k and hence $H(A) \subseteq \text{Im } c$. On the other hand, the above description of the map a shows that $\delta(H(A)) = H^2(\tilde{X}, \mathbf{R})$. Thus $H(A) = \text{Im } c$ as was desired.

(2) (b) \Rightarrow (a). Suppose that $\bar{\alpha} \in H(A) = \text{Im } c$. Then $r(\bar{\alpha}) = 0$ and hence $\alpha = dd^c u$ for some C^∞ function u on $\tilde{X} - A$. Then α satisfies the condition (E) by Corollary to Lemma 1. (a) \Rightarrow (b). Suppose that α satisfies (E). For any V and α' as in (E), write $\alpha' = dd^c v$ on X for some C^∞ function v . Then with respect to some local embedding $(X, x) \rightarrow (\mathbf{C}^n, 0)$, there exists a small open ball B with center 0 in \mathbf{C}^n such that $\alpha = dd^c v$ on $X - X \cap B$. Write $B' = X \cap B$. We show that the restriction map $j: H^1(X - x, \mathcal{P}_x) \rightarrow H^1(X - B', \mathcal{P}_x)$ is injective. In view of the commutative diagram of exact sequences

$$\begin{CD} H^1(X - x, \mathbf{R}) @>>> H^1(X - x, \mathcal{O}_x) @>>> H^1(X - x, \mathcal{P}_x) @>>> H^2(X - x, \mathbf{R}) \\ @. @. @VV j V @. @VV s V \\ H^1(X - B', \mathbf{R}) @>>> H^1(X - B', \mathcal{O}_x) @>>> H^1(X - B', \mathcal{P}_x) @>>> H^2(X - B', \mathbf{R}) \end{CD}$$

coming from the sequence (1) on $X - x$, we see that it suffices to show that i is injective (cf. (2.0) (a)). But this is shown in Andreotti and Grauert [1, Théorème 15]. Now by the injectivity of j we can write $\alpha = dd^c v'$ for some C^∞ function v' on $X - x$. This implies that $r(\bar{\alpha}) = 0$ in $H^1(\tilde{X} - A, \mathcal{P}_{\tilde{X}}) = H^1(X - x, \mathcal{P}_x)$, or $\bar{\alpha} \in \text{Im } c = H(A)$. q.e.d.

REMARK 2. Let (X, x) be a normal isolated singularity of dimension \geq

3. Let $f: (\tilde{X}, A) \rightarrow (X, x)$ be a resolution such that $\dim A = 1$. Then the same proof as above shows that no Kähler form on X satisfies the condition (E), unless (X, x) is smooth.

(2.2) Next we globalize Proposition 3.

PROPOSITION 4. *Let X be a normal complex surface with isolated singular points x_1, x_2, \dots, x_s . Let $f: \tilde{X} \rightarrow X$ be a resolution. Let $A_\nu = f^{-1}(x_\nu)$ and U_ν a tubular neighborhood of A_ν in \tilde{X} . Then the following conditions are equivalent:*

- (1) X is a Kähler space.
- (2) \tilde{X} is Kähler and there exists a Kähler form α on \tilde{X} such that for each ν the Kähler class $\bar{\alpha}_\nu \in H^1(U_\nu, \mathcal{P}_{\tilde{X}})$ defined by $\alpha|_{U_\nu}$ belongs to $H(A_\nu)$.

Note that Theorem in the introduction follows immediately from Propositions 2, 3 and 4 together with (3), in view of the definition of nondegeneracy.

For the implication (1) \Rightarrow (2) we need lemmas.

LEMMA 4. *Let L be a line bundle on a complex manifold X . Let b_1, \dots, b_m be C^∞ sections of L on X which have no common zeroes. Then the refined Chern class $\hat{c}(L)$ of L is represented by the real closed (1, 1)-form (cf. 2.0) (c)*

$$\gamma = (-1/4\pi) dd^c \log \left(\sum_{i=1}^m |b_i|^2 \right).$$

Here if $b_j(x) \neq 0$ at $x \in X$, then by definition $dd^c \log (\sum_{i=1}^m |b_i|^2) = dd^c \log ((\sum_{i=1}^m |b_i|^2) / |b_j|^2)$ at x , which is independent of the choice of such a j . The proof of the lemma is straightforward and is omitted.

LEMMA 5. *Let \tilde{X} be a smooth complex surface and $A \subseteq \tilde{X}$ an exceptional divisor. Let $U \supseteq U' \supseteq A$ be tubular neighborhoods of A . Let A_1, \dots, A_m be the irreducible components of A . Then there exists a line bundle F on \tilde{X} of the form $F = \bigotimes_{i=1}^m [A_i]^{n_i}$ whose refined Chern class $\hat{c}(F)$ is represented by a real closed $C^\infty(1, 1)$ -form β which is positive definite (i.e., Kähler) on U' and vanishes identically on $\tilde{X} - U$.*

PROOF. (cf. [21, §4, 3]). By [14, Lemma 4.10] there exist integers n_i such that $(\sum_{i=1}^m n_i A_i \cdot A_j) < 0$ for any j . Set $L = \bigotimes_{i=1}^m [A_i]^{-n_i}$. Then L is positive, and hence is ample, on U' (cf. [8, Lemmas 4 and 3]). So for a sufficiently large $b > 0$ we can find holomorphic sections ψ_1, \dots, ψ_s of $L^{\otimes b}$ on U' which embeds U' into the projective space P^{s-1} . Let σ be the canonical meromorphic section of $L = \bigotimes [A_i]^{-n_i}$, which has thus no zeroes and poles outside A . (Actually $n_i > 0$ for all i , and σ is holomor-

phic.) Let ρ be a C^∞ function on \tilde{X} with support contained in U such that $\rho \equiv 1$ on U' . Then $\rho\psi_1, \dots, \rho\psi_s, (1 - \rho)\sigma^b$ are C^∞ sections of $L^{\otimes b}$ which have no common zeroes on \tilde{X} . Hence by Lemma 4

$$\gamma := -(1/4\pi b)dd^c \log \left(\sum_{i=1}^s |\rho\psi_i|^2 + |(1 - \rho)\sigma^b|^2 \right)$$

represents $\hat{c}(L)$. Since $\gamma = -(1/4\pi b)dd^c \log (\sum_{i=1}^s |\psi_i|^2)$ on U' it is negative definite there and clearly $\gamma \equiv 0$ on $\tilde{X} - U$. Then it suffices to set $F = L^*$ (the dual of L) and $\beta = -\gamma$. q.e.d.

PROOF OF PROPOSITION 4. We may assume that $U_\mu \cap U_\nu = \emptyset$ for $\mu \neq \nu$.

(2) \Rightarrow (1). Let α_ν be the restriction of α to U_ν . By further restriction, regard α_ν as a Kähler form on $f(U_\nu) - x_\nu \cong U_\nu - A_\nu$. Then by Proposition 3 after modification of α_ν within a relatively compact open subset of $f(U_\nu)$, α_ν extends to a Kähler form α'_ν on $f(U_\nu)$. Define a C^∞ form α' on X by: $\alpha' = \alpha'_\nu$ on $f(U_\nu)$ and $\alpha' = \alpha$ on $X - \bigcup_\nu f(U_\nu) = \tilde{X} - \bigcup_\nu U_\nu$. Then α' is a Kähler form on X .

(1) \Rightarrow (2). Let α' be a Kähler form on X . Take a tubular neighborhood U'_ν of A_ν with $U'_\nu \subset U_\nu$. Let $A_{\nu,i}, i = 1, \dots, m_\nu$, be the irreducible components of A_ν . Then by Lemma 5 for each ν there exists a line bundle F_ν on \tilde{X} of the form $\bigotimes_{i=1}^{m_\nu} [A_{\nu,i}]^{k'_i}$ whose refined Chern class is represented by a real closed (1, 1)-form β_ν , which is positive definite on U'_ν and vanishes identically on $\tilde{X} - U_\nu$. Since $f^*\alpha'$ is semipositive on \tilde{X} and positive on $\tilde{X} - \bigcup_\nu A_\nu$, $\alpha := Mf^*\alpha' + \sum_{\nu=1}^s \beta_\nu$ is a Kähler form on \tilde{X} if we take a real number M sufficiently large. Moreover $\bar{\alpha}_\nu = \bar{\beta}_\nu$ on each U_ν , where $\bar{\beta}_\nu$ is the class of β_ν in $H^1(U_\nu, \mathcal{S}_{\tilde{X}})$. Hence $\bar{\alpha}_\nu \in H(A_\nu)$. q.e.d.

REMARK 3. Let \tilde{X} be a Kähler manifold (of arbitrary dimension) and A an exceptional (compact) connected submanifold of codimension 1 in \tilde{X} in the sense of Grauert [10]. Let $f: \tilde{X} \rightarrow X$ be the contraction of A to a normal point x of a complex space X . Then a modification of the proof of Theorem shows that if $h^{1,1}(A) = 1$ and $H^1(A, \mathcal{N}^{*\nu}) = 0$ for all $\nu > 0$, X also is Kähler where \mathcal{N}^* is the conormal sheaf of A in \tilde{X} . This is a Kählerian analogue of Grauert's criterion for projective contraction ([10, Satz 8]).

(2.3) The following proposition shows that the condition of Theorem, i.e., the nondegeneracy of the singularity, is a consequence of a certain global condition on the variety X .

PROPOSITION 5. Let X be a normal compact complex surface in \mathcal{E} . Then X has only nondegenerate singularities if the natural homomorphism

$f^*: H^2(X, \mathcal{O}_X) \rightarrow H^2(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is injective for some (and hence for any) resolution $f: \tilde{X} \rightarrow X$ of X . In particular, X is Kähler. Moreover, under the same condition, \tilde{X} is projective if X is projective.

PROOF. By Theorem, the second assertion follows from the first. The first assertion follows from the commutative diagram

$$\begin{CD} H^1(\tilde{X}, \mathbf{R}) @>>> H^0(X, R^1f_*\mathbf{R}) @>>> H^2(X, \mathbf{R}) @>>> H^2(\tilde{X}, \mathbf{R}) \\ @VVaV @VVV @VVV @VVV \\ H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) @>>> H^0(X, R^1f_*\mathcal{O}_{\tilde{X}}) @>>> H^2(X, \mathcal{O}_X) @>>> H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \end{CD}$$

of exact sequences coming from the Leray spectral sequences for f , if we note the surjectivity of a ($X \in \mathcal{C}$). To show the last assertion, recall from the proof of Theorem that for any Kähler form β on \tilde{X} we can find a Kähler form α on X such that $\beta = f^*\alpha$ on a complement of a tubular neighborhood of $A := f^{-1}(\text{Sing } X)$. Let $\bar{\beta} \in H^2(\tilde{X}, \mathbf{R})$ (resp. $\bar{\alpha} \in H^2(X, \mathbf{R})$) be the class defined by β (resp. α). Then by the above remark, $\bar{\beta} - f^*\bar{\alpha}$ is contained in the image of the natural map $H^2_A(\tilde{X}, \mathbf{R}) \rightarrow H^2(\tilde{X}, \mathbf{R})$, which is generated by the real Chern classes $c(A_i) := c([A_i])$ of $[A_i]$, where A_i are the irreducible components of A . Namely $\bar{\beta} - f^*\bar{\alpha} = \sum_i r_i c(A_i)$ for some real numbers r_i . Suppose now that \tilde{X} is projective and $\bar{\beta}$ is a real Chern class of an ample line bundle L on \tilde{X} . Then r_i are all rational numbers; indeed $0 = f^*\bar{\alpha} \cdot c(A_j) = \bar{\beta} \cdot c(A_j) - \sum_i r_i (A_i \cdot A_j)$ for any j , $\bar{\beta} \cdot c(A_j)$ is an integer and $\{(A_i \cdot A_j)\}$ is a negative definite integral matrix. Thus $m f^*\bar{\alpha} = f^*(m\bar{\alpha})$ is a real Chern class of a line bundle for some $m > 0$. Then from our assumption and from the commutative diagram of exact sequences

$$\begin{CD} H^1(X, \mathcal{O}_X^*) @>>> H^2(X, \mathbf{Z}) @>>> H^2(X, \mathcal{O}_X) \\ @Vf^*VV @Vf^*VV @VVV \\ H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*) @>>> H^2(\tilde{X}, \mathbf{Z}) @>>> H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \end{CD}$$

it follows that $m\bar{\alpha}$ itself is a real Chern class of a line bundle, say L' . Then by Grauert [10, Satz 3] L' is ample and hence X is projective.

REMARK 4. The last assertion is originally due Brenton (cf. [5, Prop. 10]).

3. Criteria for nondegeneracy. We use the convention and notation of (2.0) (a).

(3.1) We first note the following:

LEMMA 6. Let (X, x) be a normal isolated singularity. Let $f: (\tilde{X}, A) \rightarrow$

(X, x) be a resolution. Then the natural map $H^1(\tilde{X}, \mathbf{R}) \rightarrow H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is injective.

PROOF. We consider the commutative diagram of exact sequences (2) for \tilde{X} and A ;

$$\begin{array}{ccccccc} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^0(\tilde{X}, \mathcal{P}_{\tilde{X}}) & \xrightarrow{b} & H^1(\tilde{X}, \mathbf{R}) & \longrightarrow & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \\ \downarrow & & \downarrow & & \downarrow r & & \downarrow \\ H^0(A, \mathcal{O}_A) & \xrightarrow{d} & H^0(A, \mathcal{P}_A) & \xrightarrow{\bar{b}} & H^1(A, \mathbf{R}) & \longrightarrow & H^1(A, \mathcal{O}_A), \end{array}$$

where the vertical arrows are the restriction maps. It suffices to show that b is the zero map. Since r is isomorphic, this follows, if \bar{b} is the zero map, or d is surjective. Since A is compact and connected, by the maximum principle d is isomorphic to the map $d': C \rightarrow \mathbf{R}$ defined by $d'(s) = -(\text{imaginary part of } s)$. Thus d is surjective as was desired. q.e.d.

Let (X, x) be a normal isolated surface singularity and $f: (\tilde{X}, A) \rightarrow (X, x)$ a resolution. Then we set $b_1(X, x) = b_1(\tilde{X})$ and call it the *first Betti number* of (X, x) . This is independent of the chosen resolution f . Let $p_g(X, x) := \dim(R^1 f_* \mathcal{O}_{\tilde{X}})_x = \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ be the geometric genus of (X, x) . Then the above lemma shows the inequality $b_1(X, x) \leq 2p_g(X, x)$. From the definitions we obtain:

PROPOSITION 6. A normal isolated surface singularity (X, x) is nondegenerate if and only if $b_1(X, x) = 2p_g(X, x)$.

(3.2) According to Brenton [6], a normal isolated surface singularity (X, x) is called *pararational*, if, for some (and hence for any) resolution $f: (\tilde{X}, A) \rightarrow (X, x)$ of (X, x) such that A is of normal crossings in \tilde{X} , we have (1) (the dual graph of) A contains no cycles and (2) $R^1 f_* \mathcal{I} = 0$ where \mathcal{I} is the ideal sheaf of A with the reduced structure.

PROPOSITION 7. Let (X, x) be a normal isolated surface singularity. Then (X, x) is nondegenerate if and only if (X, x) is pararational.

PROOF. Let A_1, \dots, A_m be the irreducible components of A and p_1, \dots, p_n the singular points of A . Since A is a curve with normal crossings, we have $b_1(A) = n - m + 1 + \sum_{i=1}^m 2g(A_i)$ and $g(A) := \dim H^1(A, \mathcal{O}_A) = n - m + 1 + \sum_{i=1}^m g(A_i)$, where b_1 is the first Betti number and $g(A_i)$ is the genus of A_i . Hence we see that the following conditions are equivalent: (a) $\dim_{\mathbf{R}} H^1(A, \mathbf{R}) \geq 2 \dim H^1(A, \mathcal{O}_A)$, (b) $\dim_{\mathbf{R}} H^1(A, \mathbf{R}) = 2 \dim H^1(A, \mathcal{O}_A)$ and (c) $n - m + 1 = 0$. Note further that (c) is equivalent to the condition (1) of (3.2) above. Now we consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & H^1(\tilde{X}, \mathbf{R}) & \xrightarrow{\sim} & H^1(A, \mathbf{R}) & & \\
 & & \downarrow e & & \downarrow & & \\
 0 & \longrightarrow & H^1(\tilde{X}, \mathcal{S}) & \longrightarrow & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^1(A, \mathcal{O}_A) \longrightarrow 0,
 \end{array}$$

where the bottom sequence is exact, since $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow H^0(A, \mathcal{O}_A) \cong \mathbf{C}$ is surjective and $H^2(\tilde{X}, \mathcal{S}) = 0$. Then in view of the above remark together with the injectivity of e (Lemma 6), it follows that (1) and (2) imply the surjectivity of e . Conversely if e is surjective, then the above equivalent conditions are satisfied, and $\dim H^1(\tilde{X}, \mathcal{S}) + \dim H^1(A, \mathcal{O}_A) = \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \leq (1/2) \dim_{\mathbf{R}} H^1(\tilde{X}, \mathbf{R}) = \dim H^1(A, \mathcal{O}_A)$ by (b). Hence $H^1(\tilde{X}, \mathcal{S}) = 0$ as was desired. q.e.d.

EXAMPLE. Let A be a compact Riemann surface and L a negative line bundle on A . Let (X, x) be the normal isolated singularity obtained by contracting the zero section of L . Then (X, x) is nondegenerate if and only if $H^1(A, L^{*\otimes m}) = 0$ for any $m > 0$, where L^* denotes the dual of L . In particular simple-elliptic singularity of K. Saito are all nondegenerate.

(3.3) Modifying Laufer's argument [15], we shall give a criterion which makes no use of a resolution.

PROPOSITION 8. *Let (X, x) be a normal isolated surface singularity. Then (X, x) is nondegenerate if and only if the following condition is satisfied; any holomorphic 2-form ω on $X - x$ is L^2 -integrable whenever the class $[\omega + \bar{\omega}] \in H^2(X - x, \mathbf{R})$ defined by $\omega + \bar{\omega}$ vanishes. Here $\bar{\omega}$ denotes the complex conjugate of ω .*

First we recall some results needed for the proof from the duality theory.

(a) Let $U = X - x$. Then $H^1(U, \mathcal{O}_U)$ has the natural QFS structure (QFS = quotient of Frechet-Schwartz) which is separated in our case where X is Stein (cf. [3, p. 82, Theorem 6.1]). On the other hand, by a theorem of Malgrange [16] $H^2(U, \Omega_U^2) = 0$ because $\dim U = 2$. Hence by Serre duality, (1) the natural QDFS structure (QDFS = quotient of dual of FS) on $H_c^1(U, \Omega_U^2)$ also is separated and (2) $H^1(U, \mathcal{O}_U)$ and $H_c^1(U, \Omega_U^2)$ are in topological duality, where c denotes the compact supports (cf. [3]). Let $\omega_x := j_* \Omega_U^2$ be the Grothendieck dualizing sheaf on X where $j: U \rightarrow X$ is the inclusion (cf. [12]). Then $\text{depth } \omega_x = 2$ so that $H_c^0(X, \omega_x) = H_c^1(X, \omega_x) = 0$ (cf. [3, p. 41, Corollary 3.10]). Therefore from the exact sequence

$$H_c^0(X, \omega_x) \rightarrow \omega_{x,x} \rightarrow H_c^1(U, \Omega_U^2) \rightarrow H_c^1(X, \omega_x)$$

we have the natural (topological) isomorphism $\delta: \omega_{x,z} \cong H_c^1(U, \Omega_U^2)$.

We now proceed to describe the induced duality between $H^1(U, \mathcal{O}_U)$ and $\omega_{x,z}$, or the pairing \langle, \rangle giving this duality. Let $\alpha \in H^1(U, \mathcal{O}_U)$ and $\beta \in \omega_{x,z}$ be arbitrary. Let $\tilde{\alpha}$ (resp. $\tilde{\beta}$) be a $\bar{\partial}$ -closed C^∞ (0, 1)-form on U (resp. a holomorphic 2-form defined on $V - x$ for some open neighborhood V of x) representing α (resp. β). Let ρ be a C^∞ function with compact support on X such that $\rho \equiv 1$ on a neighborhood V' of x with $V' \subset V$. Then $\bar{\partial}(\rho\tilde{\beta})$ has compact support on U and hence $\int_U \tilde{\alpha} \wedge \bar{\partial}(\rho\tilde{\beta})$ is finite. Further, the integral is independent of the choice of $\tilde{\alpha}$, $\tilde{\beta}$ and ρ as above depending only on α and β . Then from the definition of δ it follows that $\langle \alpha, \beta \rangle = \int_U \tilde{\alpha} \wedge \bar{\partial}(\rho\tilde{\beta})$.

Let B_ε be the ball of radius ε with center x with respect to a local embedding of X into \mathbb{C}^N . Let S_ε be the boundary sphere B_ε . Let $U_\varepsilon = X - B_\varepsilon \cap X$ and $K_\varepsilon = X \cap S_\varepsilon$. Take ε so small that $B_\varepsilon \cap X \subseteq V'$. Then using Stokes' theorem we obtain

$$\int_U \tilde{\alpha} \wedge \bar{\partial}(\rho\tilde{\beta}) = \int_{U_\varepsilon} \tilde{\alpha} \wedge \bar{\partial}(\rho\tilde{\beta}) = \int_{U_\varepsilon} d(\tilde{\alpha} \wedge \rho\tilde{\beta}) - \int_{K_\varepsilon} d\tilde{\alpha} \wedge \rho\tilde{\beta} = \int_{K_\varepsilon} \tilde{\alpha} \wedge \tilde{\beta}.$$

Since these are independent of ε as above, we write $K = K_\varepsilon$ symbolically.

We have thus $\langle \alpha, \beta \rangle = \int_K \tilde{\alpha} \wedge \tilde{\beta}$.

(b) Let $f: (\tilde{X}, A) \rightarrow (X, x)$ be resolution. Consider the coboundary map $\zeta: H^1(\tilde{X} - A, \mathcal{O}_{\tilde{X}}) \rightarrow H_A^2(\tilde{X}, \mathcal{O}_{\tilde{X}})$ in the local cohomology exact sequence. Note that ζ is surjective since $H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. Both terms have the natural QFS structures for which ζ is continuous (cf. [3, pp. 82, 287]). Since $H^1(A, \Omega_A^2)$ is finite dimensional, $H_A^2(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is separated and $H_A^2(\tilde{X}, \mathcal{O}_{\tilde{X}})$ and $H^0(A, \Omega_A^2)$ are in natural topological duality. On the other hand, since $H^1(\tilde{X} - A, \mathcal{O}_{\tilde{X}}) \cong H^1(X - x, \mathcal{O}_X)$, the transpose ζ' of ζ gives rise to a homomorphism $H^0(A, \Omega_A^2) \rightarrow \omega_{x,z}$ by (a). Further, with a little more efforts this homomorphism is identified with the natural homomorphism $(f_*\Omega_{\tilde{X}}^2)_x \rightarrow \omega_{x,z}$ obtained by the restriction to $\tilde{X} - A \cong X - x$, which we shall still denote by ζ' .

(c) Let $K = K_\varepsilon$ be as in (a). Then for $K = \mathbb{C}$ or \mathbb{R} we have $H^i(\tilde{X} - A, K) \cong H^i(X - x, K) \cong H^i(K, K)$ for all i (cf. (2.0) (a)) and they are finite dimensional over K . So they have the natural FS structure with its (topological) dual given by $H^i(\tilde{X} - A, K)' \cong H^i(K, K)' \cong H^{3-i}(K, K) \cong H^{3-i}(\tilde{X} - A, K)$ by the Poincaré duality.

The natural map $\gamma_c: H^1(\tilde{X} - A, \mathbb{C}) \rightarrow H^1(\tilde{X} - A, \mathcal{O}_{\tilde{X}})$ is given by $\gamma_c(\alpha) =$ the class of $\tilde{\alpha}_1$, where $\tilde{\alpha}_1$ is the (0, 1)-component of a representing $\bar{\partial}$ -closed 1-form $\tilde{\alpha}$ of α . Then from the description of the duality

in (a) it follows that the dual map $\gamma'_c: \omega_{X,x} \rightarrow H^2(\tilde{X} - A, \mathbf{C})$ is given by $\gamma'_c(\beta) = \tilde{\beta}$ modulo d -closed forms via the natural isomorphism $H^2(\tilde{X} - A, \mathbf{C}) \cong H^2(V - x, \mathbf{C})$, where $\tilde{\beta}$ and V are as in (a) with V suitably restricted.

PROOF OF PROPOSITION 8. Let $f: (\tilde{X}, A) \rightarrow (X, x)$ be a resolution. Consider the commutative diagram of local cohomology exact sequences

$$\begin{array}{ccccccc} H^1(\tilde{X}, \mathbf{R}) & \xrightarrow{b} & H^1(\tilde{X} - A, \mathbf{R}) & \longrightarrow & H^2_A(\tilde{X}, \mathbf{R}) & \xrightarrow{a} & H^2(\tilde{X}, \mathbf{R}) \\ & & \downarrow e & & \downarrow \gamma & & \downarrow \\ H^1_A(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \xrightarrow{\varepsilon} & H^1(\tilde{X} - A, \mathcal{O}_{\tilde{X}}) & \xrightarrow{\zeta} & H^2_A(\tilde{X}, \mathcal{O}_{\tilde{X}}), \end{array}$$

where the vertical arrows are the natural maps. First by Grauert-Riemenschneider [12] $H^1_A(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ so that ε is injective. Since a is injective (cf. the proof of Proposition 3), b is surjective. This implies that (X, x) is nondegenerate, i.e., e is surjective if and only if $\text{Im } \gamma \cong \text{Im } \varepsilon = \text{Ker } \zeta$, where Im and Ker denote the image and the kernel respectively; in other words, the complex

$$H^1(\tilde{X} - A, \mathbf{R}) \xrightarrow{\gamma} H^1(\tilde{X} - A, \mathcal{O}_{\tilde{X}}) \xrightarrow{\zeta} H^2_A(\tilde{X}, \mathcal{O}_{\tilde{X}})$$

is exact. In view of the remarks preceding the proof, the topological dual of this sequence is given by

$$H^2(\tilde{X} - A, \mathbf{R}) \xleftarrow{\gamma'} \omega_{X,x} \xleftarrow{\zeta'} (f_* \Omega^2_{\tilde{X}})_x$$

and this sequence is exact if and only if the original sequence is exact (cf. [3, p. 248 (C)]). Further, $\text{Im } \zeta'$ is precisely the space of L^2 -integrable holomorphic 2-forms on $X - x$ (cf. [12], [15]). On the other hand, with K as in (c) above, by the description of γ'_c above we see that γ' is given by the map $\gamma'(\beta) = \text{the class of } (1/2)(\tilde{\beta} + \bar{\tilde{\beta}})$ where $\tilde{\beta}$ is as in (a) and $\bar{\tilde{\beta}}$ denotes its complex conjugate. The proposition thus follows.

4. Small deformations and nondegeneracy. We show that every small deformation of a normal compact Kähler surface with only nondegenerate isolated singularities is again Kähler.

PROPOSITION 9. *Let X be a normal compact complex surface in \mathcal{E} . Suppose that X has only nondegenerate singularities so that X is Kähler (cf. Theorem). Then every (flat) small deformation X_t of X is Kähler.*

This is a consequence of the following theorem of Moishezon [19] in view of Lemma 7 below.

THEOREM (Moishezon). *Let X be a normal compact Kähler space.*

If the natural map $H^2(X, \mathbf{R}) \rightarrow H^2(X, \mathcal{O}_X)$ is surjective, then every small deformation X_t of X is again Kähler.

LEMMA 7. Let X be as in Proposition 9. Then the natural map $H^2(X, \mathbf{R}) \rightarrow H^2(X, \mathcal{O}_X)$ is surjective.

PROOF. Let $f: \tilde{X} \rightarrow X$ be a resolution. We consider the following commutative diagram of exact sequences derived from the Leray spectral sequence for f

$$\begin{array}{ccccccc}
 & & & & H^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) & & \\
 & & & & \downarrow e & & \\
 H^0(X, R^1f_*\mathbf{R}) & \longrightarrow & H^2(X, \mathbf{R}) & \longrightarrow & H^2(\tilde{X}, \mathbf{R}) & \xrightarrow{d} & H^0(X, R^2f_*\mathbf{R}) \\
 \downarrow a & & \downarrow b & & \downarrow c & & \downarrow \\
 H^0(X, R^1f_*\mathcal{O}_{\tilde{X}}) & \longrightarrow & H^2(X, \mathcal{O}_X) & \longrightarrow & H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & 0.
 \end{array}$$

Our assumption is that a is surjective and we have to show that b is surjective. Since $\tilde{X} \in \mathcal{E}$, c is surjective. Hence by diagram chasing it suffices to show that $de: H^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) \rightarrow H^0(X, R^2f_*\mathbf{R})$ is surjective. Let x_1, \dots, x_m be the singular points of X . Let $A_\nu = f^{-1}(x_\nu)$ and $A = \bigcup_{\nu=1}^m A_\nu$. Then we show even that the composite map $H^1_A(\tilde{X}, \mathcal{P}_{\tilde{X}}) \rightarrow H^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) \rightarrow H^0(X, R^2f_*\mathbf{R})$ is surjective. First, we have the natural isomorphisms $H^1_A(\tilde{X}, \mathcal{P}_{\tilde{X}}) \cong \bigoplus_{\nu=1}^m H^1_{A_\nu}(\tilde{X}, \mathcal{P}_{\tilde{X}})$ and $H^0(X, R^2f_*\mathbf{R}) \cong \bigoplus_{\nu=1}^m H^2(A_\nu, \mathbf{R})$ and thus it suffices to show that the composite map $H^1_{A_\nu}(\tilde{X}, \mathcal{P}_{\tilde{X}}) \rightarrow H^1(\tilde{X}, \mathcal{P}_{\tilde{X}}) \rightarrow H^2(A_\nu, \mathbf{R})$ is surjective for any ν . This is indeed proved in the course of the proof of Proposition 3. q.e.d.

EXAMPLE. Let C be any nonsingular cubic curve in the complex projective plane P^2 . Take ten points q_1, \dots, q_{10} on C arbitrarily and blow up P^2 with center q_1, \dots, q_{10} . Let \tilde{X} be the resulting surface and E the proper transform of C in \tilde{X} . Then the self-intersection number $E \cdot E = -1$ so that E can be blown down to a simple-elliptic singularity x of a normal Moishezon surface X . Then by our Theorem together with example in (3.2) X is Kähler, though X is not projective for 'general' choice of q_i .

This example of kählerian non-projective Moishezon surface is originally due to Moishezon [19], where he has proved that the above $X = X(q_1, \dots, q_{10})$ is Kähler if it is sufficiently near to a projective one by using the above theorem. (Note that his proof amounts to showing that (X, x) is nondegenerate in our sense.) Many other examples of kählerian non-projective normal Moishezon surfaces can now be obtained by using our criterion. For instance, let $h: Y \rightarrow C$ be a projective nonsingular

elliptic surface with $q(Y) = g(C)$, where $q(Y)$ and $g(C)$ are the irregularity and the genus of Y and C , respectively. Let $c \in C$ be a general point and $p \in Y_c$ the general point of Y_c . Let X be the normal surface obtained by blowing up p and then contracting the proper transform of Y_c to a unique singular point $x \in X$. Then X satisfies the desired condition.

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