Tôhoku Math. Journ.
35 (1983), 77-99.

ZETA FUNCTIONS IN SEVERAL VARIABLES ASSOCIATED WITH PREHOMOGENEOUS VECTOR SPACES II: A CONVERGENCE CRITERION

FUMIHIRO SATO

(Received March 8, 1982)

In the previous paper [14], we introduced zeta functions associated with prehomogeneous vector spaces and proved their functional equations with respect to a $\boldsymbol{Q}$-regular subspace. For application of the results in [14], it is desirable to find a practical criterion for convergence of zeta functions. The purpose of the present paper is to give a certain sufficient condition for absolute convergence of zeta functions, which is a generalization of the method used by Suzuki [22].

In § 1, we recall the definition of zeta functions associated with prehomogeneous vector spaces and formulate the main result (Theorem 1). The proof of Theorem 1 is given in § 2. Our argument is based upon the techniques in adele geometry developed by Ono [10], [12] and [13]. We shall give some applications of Theorem 1 in § 3 and the forthcoming paper [15].

The author would like to thank T. Suzuki for many stimulating discussions.

In what follows, we denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For a prime ν (finite or infinite) of $\boldsymbol{Q}$, $\boldsymbol{Q}_\nu$ is the completion of $\boldsymbol{Q}$ with respect to ν. For a finite prime p, $\boldsymbol{Z}_p$ is the ring of p-adic integers and $\boldsymbol{F}_p$ is the finite field with p elements. We use the standard notation in Galois cohomology and adele geometry. In particular for any affine algebraic set X defined over $\boldsymbol{Q}$, $X_{\boldsymbol{Q}_\nu}$ (resp. $X_{\boldsymbol{Z}_p}$) are the set of $\boldsymbol{Q}_\nu$-rational (resp. $\boldsymbol{Z}_p$-integral) points of X. The adelization of X over $\boldsymbol{Q}$ is denoted by X_A. For a $\boldsymbol{Q}$-rational gauge form ω on X and a prime ν of $\boldsymbol{Q}$, $|\omega|_\nu$ is the measure on $X_{\boldsymbol{Q}_\nu}$ induced by ω. We denote by $\mathscr{S}(V_A)$ the Schwartz-Bruhat space on the adelization V_A of a $\boldsymbol{Q}$-vector space V. The cardinality of a set X is denoted by $\sharp(X)$. For a linear algbraic group G, we denote by $\mathscr{D}(G)$ and $R_u(G)$ its derived group and its unipotent radical, respectively.

1. Statement of the main results.

1.1. First we recall the difinition of zeta functions associated with prehomogeneous vector spaces (for more detailed treatment, see [14, § 1 and § 4]). Let (G, ρ, V) be a prehomogeneous vector space (briefly a p.v.) defined over $\boldsymbol{Q}$ and S be its singular set. The singular set S is, by definition, a proper algebraic subset of V such that $V-S$ is a single G-orbit. The algebraic set S is defined over $\boldsymbol{Q}$. Let $S_1, \cdots, S_n$ be the $\boldsymbol{Q}$-irreducible components of S with codimension 1. Let $P_1, \cdots, P_n$ be $\boldsymbol{Q}$-irreducible polynomials defining $S_1, \cdots, S_n$, respectively. Then $P_1, \cdots, P_n$ are relative invariants of (G, ρ, V) and there exist $\boldsymbol{Q}$-rational characters $\chi_1, \cdots, \chi_n$ of G such that

$$P_i(\rho(g)x) = \chi_i(g)P_i(x) \quad (g \in G,\ x \in V,\ 1 \leqq i \leqq n) .$$

Let $G_{\boldsymbol{R}}^+$ be a subgroup of $G_{\boldsymbol{R}}$ containing the identity component and let $V_{\boldsymbol{R}} - S_{\boldsymbol{R}} = V_1 \cup \cdots \cup V_\nu$ be the $G_{\boldsymbol{R}}^+$-orbit decomposition. We fix a basis of V and a matrix expression of G compatible with the given $\boldsymbol{Q}$-structure and such that $\rho(G_{\boldsymbol{Z}})V_{\boldsymbol{Z}} \subset V_{\boldsymbol{Z}}$. Put

$$\Gamma = \{g \in G_{\boldsymbol{Z}} \cap G_{\boldsymbol{R}}^+;\ \chi_i(g) = 1\ (1 \leqq i \leqq n)\} .$$

For any $x \in V$, denote by G_x the isotropy subgroup of G at x:

$$G_x = \{g \in G;\ \rho(g)x = x\} .$$

Let G_x° be the identity component of G_x. Set $G_x^+ = G_x \cap G_{\boldsymbol{R}}^+$ and $\Gamma_x = G_x \cap \Gamma$. Let $V'_{\boldsymbol{Q}}$ be the subset of $V_{\boldsymbol{Q}} - S_{\boldsymbol{Q}}$ consisting of all elements x such that G_x° has no non-trivial $\boldsymbol{Q}$-rational character. We assume that $V'_{\boldsymbol{Q}}$ is non-empty.

Let Ω be a right invariant $\boldsymbol{Q}$-rational gauge form on G. Then there exists a $\boldsymbol{Q}$-rational character Δ of G such that $L_h^*\Omega = \Delta(h)\Omega$ $(h \in G)$, where $L_h^*\Omega$ is the pull back of Ω by the left translation $L_h(g) = hg$. For some integer d, the character $(\det \rho/\Delta)^d$ corresponds to a relative invariant of (G, ρ, V) and we can find a $\delta = (\delta_1, \cdots, \delta_n)$ in $\boldsymbol{Q}^n$ such that

$$\{\det \rho(g)/\Delta(g)\}^d = \chi_1(g)^{d\delta_1} \cdots \chi_n(g)^{d\delta_n} .$$

Let dg be a right invariant measure on $G_{\boldsymbol{R}}^+$ and dx be a Euclidean measure on $V_{\boldsymbol{R}}$. Put

$$\omega(x) = |P_1(x)|^{-\delta_1} \cdots |P_n(x)|^{-\delta_n} dx .$$

For any x in $V'_{\boldsymbol{Q}}$, the group G_x^+ is a unimodular Lie group. Normalize a Haar measure $d\mu_x$ on G_x^+ by the following formula:

$$\text{(1-1)} \qquad \int_{G_{\boldsymbol{R}}^+} F(g)\,dg = \int_{G_{\boldsymbol{R}}^+/G_x^+} \omega(\rho(g)x) \int_{G_x^+} F(gh)\,d\mu_x(h) \quad (F \in L^1(G_{\boldsymbol{R}}^+, dg)) .$$

The volume

$$\mu(x) = \int_{G_x^+/\Gamma_x} d\mu_x$$

is finite for any x in V'_Q.

Let L be a Γ-invariant lattice in V_Q and set $L' = L \cap V'_Q$ and $L_i = L' \cap V_i$ $(1 \leqq i \leqq \nu)$. The subset L_i is also Γ-stable and we denote by $\Gamma \backslash L_i$ the set of all Γ-orbits in L_i. We put

$$\xi_i(L; s) = \sum_{x \in \Gamma \backslash L_i} \mu(x) |P_1(x)|^{-s_1} \cdots |P_n(x)|^{-s_n} \quad (s \in \boldsymbol{C}^n,\ 1 \leqq i \leqq \nu) .$$

The Dirichlet series $\xi_1, \cdots, \xi_\nu$ are called the *zeta functions associated with* (G, ρ, V).

1.2. A p.v. (G, ρ, V) is said to be *split over a field* K if it is defined over K and every rational character of G corresponding to a relative invariant is also defined over K. Now the following lemma is an easy consequence of [14, Lemma 1.2 (ii) and Lemma 1.3].

LEMMA 1.1. *The following assertions are equivalent*:

(1) (G, ρ, V) *is split over* K.

(2) *Every absolutely irreducible component of* S *with codimension* 1 *is defined over* K.

(3) *Any relative invariant coincides with a rational function with coefficients in* K *up to a constant multiple.*

In the rest of this paper, we are exclusively concerned with p.v.'s split over $\boldsymbol{Q}$.

Set $G_1 = \{g \in G; \chi_i(g) = 1\ (1 \leqq i \leqq n)\}$. Since we are assuming that (G, ρ, V) is split over $\boldsymbol{Q}$, the group G_1 coincides with the group generated by $\mathscr{D}(G)$, $R_u(G)$ and a generic isotropy subgroup G_x for an $x \in V - S$ (cf. [16, § 4 Proposition 19]). Denote by H the connected component of the identity element of G_1. Then H is the group generated by $\mathscr{D}(G)$, $R_u(G)$ and G_x° for an $x \in V - S$. Put $H_x = H \cap G_x$. Obviously H_x contains G_x°. We always assume that

(S) H_x *is a connected semi-simple algebraic group for any* $x \in V - S$.

It follows from (S) that $V - S \cong G/G_x$ is an affine variety (see, e.g., [1, p. 579]). Hence the singular set S is a hypersurface defined by the polynomial $P_1 \cdots P_n$.

For any semi-simple algebraic group A defined over $\boldsymbol{Q}$, we denote by $\tilde{A} = (\tilde{A}, \pi)$ the universal covering group of A defined over $\boldsymbol{Q}$: $\pi: \tilde{A} \to A$. It is known that $H^1(\boldsymbol{Q}_p, \tilde{A})$ is trivial for any finite prime p (cf. [21, Theorem 3.3]). Consider the following property for such a group A:

(H) *For every inner* $\boldsymbol{Q}$*-form* A' *of* A,

$$H^1(\boldsymbol{Q}, \tilde{A}') \to \prod_{\nu} H^1(\boldsymbol{Q}_\nu, \tilde{A}') = H^1(\boldsymbol{R}, \tilde{A}')$$

is a bijection.

We shall say that (G, ρ, V) *has the property* (H) if the group H_x has the property (H) for any x in $V_{\boldsymbol{Q}} - S_{\boldsymbol{Q}}$.

We further consider the following condition:

(W) *For any $x \in V_{\boldsymbol{Q}} - S_{\boldsymbol{Q}}$, the Tamagawa number $\tau(\tilde{H}_x)$ of $\tilde{H}_x$ does not exceed some positive constant independent of x.*

The main theorem of this paper is as follows:

THEOREM 1. *If a p.v. (G, ρ, V) split over $\boldsymbol{Q}$ has the properties* (S), (H) *and* (W), *then the Dirichlet series $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ are absolutely convergent for* $\operatorname{Re} s_1 > \delta_1, \cdots, \operatorname{Re} s_n > \delta_n$.

If the group H_x is trivial for some $x \in V - S$, we may consider that (G, ρ, V) satisfies (S), (H) and (W).

COROLLARY. *Let (G, ρ, V) be a p.v. split over $\boldsymbol{Q}$. If the group H_x is trivial for some $x \in V - S$, then the Dirichlet series $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ are absolutely convergent for* $\operatorname{Re} s_1 > \delta_1, \cdots, \operatorname{Re} s_n > \delta_n$.

REMARK 1. If H_x has no simple component of type E_8, the condition (S) implies the condition (H) (cf. [3]). By the classification of irreducible p.v.'s ([16]), no simple component of type E_8 appears in H_x ($x \in V - S$) for any irreducible regular p.v. The so-called Weil conjecture asserts that the Tamagawa number of any simply connected algebraic group defined over $\boldsymbol{Q}$ is equal to 1. This conjecture is established for a fairly wide class of semi-simple algebraic groups (cf. [7], [8], [9] and [24]). For such groups, we can take 1 as a positive constant in (W). These remarks show that the most essential condition is (S). Notice that this condition is concerned only with the structure of (G, ρ, V) over $\boldsymbol{C}$.

REMARK 2. Theorem 1 and Corollary are partial affirmative answers to the conjecture proposed in [14, § 4].

1.3. Let (G, ρ, V) be a p.v. split over $\boldsymbol{Q}$ with the properties (S), (H) and (W). Assume that (G, ρ, V) is decomposed over $\boldsymbol{Q}$ into a direct sum as $(G, \rho, V) = (G, \rho_1 \oplus \rho_2, E \oplus F)$ and F is a $\boldsymbol{Q}$-regular subspace. Note that, by the assumption that (G, ρ, V) is split over $\boldsymbol{Q}$, any regular subspace is necessarily a $\boldsymbol{Q}$-regular subspace. Let F^* be the vector space dual to F and ρ_2^* the representation of G on F^* contragredient to ρ_2. Set $\rho^* = \rho_1 \oplus \rho_2^*$ and $V^* = E \oplus F^*$.

PROPOSITION 1.2. *The p.v.* (G, ρ^*, V^*) *is also a p.v. split over* $\boldsymbol{Q}$ *with the properties* (S), (H) *and* (W).

PROOF. By [14, Lemma 2.4, (iii)], the group of all characters corresponding to relative invariants of (G, ρ, V) coincides with that of (G, ρ^*, V^*). Hence (G, ρ, V) is split over $\boldsymbol{Q}$ if and only if so is (G, ρ^*, V^*). Let P be a relative invariant of (G, ρ, V) with coefficients in $\boldsymbol{Q}$ such that the Hessian

$$H_{P,y} = \det\left(\frac{\partial^2 P}{\partial y_i \partial y_j}(x, y)\right) \quad (x \in E,\ y \in F)$$

with respect to F does not vanish identically. Then the mapping $\phi_P\colon V - S \to V^* - S^*$ introduced in [14, (2-3)] is a G-equivariant biregular rational mapping defined over $\boldsymbol{Q}$ (cf. [14, Lemma 2.4, (iv)]). Moreover ϕ_P induces a one-to-one correspondence between $V_Q - S_Q$ and $V_Q^* - S_Q^*$. For any $\xi \in V_Q - S_Q$, we have $G_\xi = G_{\phi_P(\xi)}$ and hence $H_\xi = H_{\phi_P(\xi)}$ (cf. [14, Lemma 2.4, (ii)]). Thus the conditions (S), (H) and (W) are satisfied also by (G, ρ^*, V^*).

Let $(G, \rho, V) = (G, \rho_1 \oplus \rho_2, E \oplus F)$ be a p.v. split over $\boldsymbol{Q}$ with a $\boldsymbol{Q}$-regular subspace F satisfying the conditions (S), (H) and (W). Then the condition (S) yields the condition (6-1) of [14]. As is remarked in the preceding paragraph, (G, ρ, V) satisfies (5-2) of [14]. The condition (6-2) follows immediately from Proposition 1.2 and Theorem 1. Hence the results in [14, § 6] can be applied to such a p.v. and we are able to obtain functional equations of associated zeta functions.

THEOREM 2. *Let* (G, ρ, V) *be a p.v. split over* $\boldsymbol{Q}$ *with a reductive algebraic group* G *satisfying the conditions* (S), (H) *and* (W). *Then the Dirichlet series* $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ *have analytic continuations to meromorphic functions of* s *in the whole of* $\boldsymbol{C}^n$.

PROOF. Since G is reductive, the condition (S) implies that V is regular over $\boldsymbol{Q}$ ([16, § 4 Remark 26]). Hence the theorem follows from Theorem 1 and [14, Corollary 1 to Theorem 2].

1.4. As examples, consider the following two p.v.'s which were studied in [14, § 7].

(1) $G = SL(2) \times GL(1)^3$, $V = \boldsymbol{C}^2 \oplus \boldsymbol{C}^2 \oplus \boldsymbol{C}^2$, $\rho(g, t_1, t_2, t_3)(x, y, z) = (gxt_1^{-1}, gyt_2^{-1}, gzt_3^{-1})$,

(2) $G = GL(2) \times GL(1)$, $V = \{x \in M(2; \boldsymbol{C});\ {}^t x = x\} \oplus \boldsymbol{C}^2$, $\rho(g_2, g_1)(x, y) = (g_2 x\, {}^t g_2, {}^t g_2^{-1} y g_1)$.

In these two cases, we have

(1) $H = SL(2) \times \{1\}^3$, $\delta = (1, 1, 1)$, $H_x =$ trivial for all x in $V - S$,

(2) $H = SL(2) \times \{1\}$, $\delta = (1, 1)$, $H_x =$ trivial for all x in $V - S$.

Hence, by Corollary to Theorem 1, we see that the associated zeta functions are absolutely convergent for $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2, \mathrm{Re}\, s_3 > 1$ in the former case and for $\mathrm{Re}\, s_1, \mathrm{Re}\, s_2 > 1$ in the latter case. The explicit formulas (7-4) and (7-5) of [14] of the zeta functions for the standard lattices $V_{\boldsymbol{Z}}$ show that our result is the best possible.

We shall present another application of Theorem 1 in § 3 (see also [15]).

2. Proof of Theorem 1. We devide the proof into several steps.

2.1. Let $\phi: V \to \boldsymbol{C}^n$ be the polynomial mapping defined by $\phi(x) = (P_1(x), \cdots, P_n(x))$. For any $t \in (\boldsymbol{C}^\times)^n$, we put $V(t) = \phi^{-1}(t)$. Since $V - S = \phi^{-1}((\boldsymbol{C}^\times)^n)$ is a G-orbit, G_1 acts on $V(t)$ $(t \in (\boldsymbol{C}^\times)^n)$ transitively. Take a point x in $V(t)$. Then $G_1 = \mathscr{D}(G)R_u(G)G_x$ and hence $V(t)$ is a $\mathscr{D}(G)R_u(G)$-orbit. In particular $V(t)$ is a homogeneous space of H and is irreducible. It is clear that ϕ is submersive at any point in $V - S$. Hence we have a $\boldsymbol{Q}$-rational gauge form $\theta_t(x) = dx/dP_1 \wedge \cdots \wedge dP_n$ on $V(t)$ for any $t \in (\boldsymbol{Q}^\times)^n$ (cf. [25, I.5.]). It is clear that the gauge form θ_t is H-invariant. For a $\boldsymbol{Q}$-rational point ξ in $V(t)$, we define a morphism $\pi_\xi: H \to V(t)$ by $\pi_\xi(h) = \rho(h)\xi$. Let dh be a $\boldsymbol{Q}$-rational invariant gauge form on H and $d\nu_\xi$ be the $\boldsymbol{Q}$-rational invariant gauge form on H_ξ given by $d\nu_\xi = dh/(\pi_\xi)^*(\theta_t)$. It is easy to check that we can normalize a Haar measure dg on $G^+_{\boldsymbol{R}}$ such that

$$d\mu_\xi = \prod_{i=1}^{n} |t_i|^{\delta_i - 1} |d\nu_\xi|_\infty \quad (\xi \in V(t)_{\boldsymbol{Q}}) \tag{2-1}$$

on $H_{\xi,\boldsymbol{R}} \cap G^+_\xi$, where $d\mu_\xi$ is the Haar measure on G^+_ξ normalized by the formula (1-1).

Let

$$\nu(\xi) = \int_{H_{\xi,\boldsymbol{R}}/H_{\xi,\boldsymbol{Z}}} |d\nu_\xi|_\infty \quad (\xi \in V_{\boldsymbol{Q}} - S_{\boldsymbol{Q}})\,. \tag{2-2}$$

Obviously the indices $[H_{\xi,\boldsymbol{R}}: H_{\xi,\boldsymbol{R}} \cap G^+_\xi]$ and $[G^+_\xi: G^+_\xi \cap H_{\xi,\boldsymbol{R}}]$ are finite and depend only upon the $G^+_{\boldsymbol{R}}$-orbit of ξ. Hence we can find two positive constants A and B such that

$$A \prod_{i=1}^{n} |t_i|^{\delta_i - 1} \nu(\xi) < \mu(\xi) < B \prod_{i=1}^{n} |t_i|^{\delta_i - 1} \nu(\xi) \quad (\xi \in V(t)_{\boldsymbol{Q}})\,. \tag{2-3}$$

It is sufficient to prove Theorem 1 for $L = V_{\boldsymbol{Z}}$. Moreover we may assume that $P_1, \cdots, P_n$ have coefficients in $\boldsymbol{Z}$. Then we have

$$\sum_{i=1}^{\nu} \xi_i(L; s) = \sum_t \{\sum_\xi \mu(\xi)\} \prod_{i=1}^{n} |t_i|^{-s_i}\,,$$

where $t=(t_1, \cdots, t_n)$ runs through all n-tuples of non-zero integers and the summation with respect to ξ is taken over a complete set of representatives of $\Gamma\backslash V(t)_Z$. Now we consider the sum $A(t)=\sum_{\xi\in H_Z\backslash V(t)_Z}\nu(\xi)$. The group Γ and H_Z are commensurable. Hence, by (2-3), the domain of absolute convergence of $\sum_{i=1}^{\nu}\xi_i(L; s)$ coincides with that of the Dirichlet series

$$\sum_t A(t)\prod_{i=1}^{n}|t_i|^{-s_i+\delta_i-1}\,. \tag{2-4}$$

So we concentrate our attention to the estimation of $A(t)$.

2.2. Let A be an algebraic group defined over $\boldsymbol{Q}$ or a Galois module over $\boldsymbol{Q}$. We use the following two symbols:

$$i^1(A)=\#(\mathrm{Ker}\,\{H^1(\boldsymbol{Q}, A)\to\prod_\nu H^1(\boldsymbol{Q}_\nu, A)\})\,,\qquad h^1(A)=\#(H^1(\boldsymbol{Q}, A))\,.$$

LEMMA 2.1. *Let A be a connected semi-simple algebraic group defined over $\boldsymbol{Q}$ with the property* (H). *Let $(\tilde{A}, \pi)$ be the universal covering group of A defined over $\boldsymbol{Q}$. Denote by M the kernel of π and put*

$$\hat{M}=\mathrm{Hom}\,(M, GL(1))\,.$$

The group $\hat{M}$ is a Galois module over $\boldsymbol{Q}$ in a natural manner. Then we have

$$i^1(A)\leqq i^1(\hat{M})h^1(\tilde{A})\,.$$

REMARK. By the condition (H) and [2, Theorems 6.1 and 7.1], the right hand side of the inequality is finite.

PROOF OF LEMMA 2.1. Consider the following commutative diagram:

$$\begin{array}{ccccc}
H^1(\boldsymbol{Q}, \tilde{A}) & \xrightarrow{\pi_1} & H^1(\boldsymbol{Q}, A) & \xrightarrow{\Delta} & H^2(\boldsymbol{Q}, M)\\
\downarrow{\scriptstyle \tilde{p}_1} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2}\\
\prod_\nu H^1(\boldsymbol{Q}_\nu, \tilde{A}) & \xrightarrow{\pi_1'} & \prod_\nu H^1(\boldsymbol{Q}_\nu, A) & \xrightarrow{\Delta'} & \prod_\nu H^2(\boldsymbol{Q}_\nu, M)\,.
\end{array}$$

Both of the horizontal sequences are exact. Let $\gamma\in H^1(\boldsymbol{Q}, A)$ be a cohomology class in $\mathrm{Ker}\,p_1$. Then we have $\#(\Delta^{-1}(\Delta(\gamma)))\leqq h^1({}_\gamma\tilde{A})$ where ${}_\gamma A$ is the inner $\boldsymbol{Q}$-form of A corresponding to γ (cf. [18, Chap. 1, § 5, Prop. 44, Cor.]). Since γ is in $\mathrm{Ker}\,p_1$, ${}_\gamma\tilde{A}$ is isomorphic to $\tilde{A}$ over $\boldsymbol{R}$. Hence, by (H), $\#(\Delta^{-1}(\Delta(\gamma)))\leqq h^1(\tilde{A})$. Therefore, by the duality theorem of Tate ([23, Th. 3.1 (a)]), we obtain

$$i^1(A)\leqq h^1(\tilde{A})\#(\mathrm{Ker}\,p_2)=i^1(\hat{M})h^1(\tilde{A})\,.$$

2.3. We return to the situation in § 1 and § 2.1. Let H_A (resp. V_A, $V(t)_A$) be the adelization of H (resp. V, $V(t)$) over $\boldsymbol{Q}$. The representation

ρ induces an action of H_A on V_A and hence on $V(t)_A$. We denote them also by ρ. Two elements x and y in V_Q are said to be *globally* (resp. *locally*) *equivalent* if they are in the same H_Q- (resp. H_A-) orbit. Denote by Θ_x the set of all elements in V_Q locally equivalent to $x \in V_Q$: $\Theta_x = V_Q \cap \rho(H_A)x$. We write $\sim\backslash\Theta_x$ for the set of all global equivalence classes in Θ_x. Put $\tau(\Theta_x) = \sum_{\xi \in \sim\backslash\Theta_x} \tau(H_\xi)$ where $\tau(H_\xi)$ is the Tamagawa number of the semi-simple algebraic group H_ξ.

LEMMA 2.2. *The numbers $\tau(\Theta_x)$ $(x \in V_Q - S_Q)$ are bounded.*

PROOF. Let $(\tilde{H}_\xi, \pi)$ be the universal covering group of H_ξ defined over $\boldsymbol{Q}$ and put $M_\xi = \operatorname{Ker} \pi$ and $\hat{M}_\xi = \operatorname{Hom}(M_\xi, GL(1))$. By [10, Theorem 2.3.1],

$$\tau(H_\xi) = \sharp(\hat{M}_\xi^{\mathfrak{G}})\tau(\tilde{H}_\xi)/i^1(\hat{M}_\xi)$$

where $\hat{M}_\xi^{\mathfrak{G}}$ is the set of all fixed elements in $\hat{M}_\xi$ under the cannonical action of $\mathfrak{G} = \operatorname{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. Set

$$\tau = \operatorname{Sup}\{\tau(\tilde{H}_\xi); \xi \in V_Q - S_Q\} .$$

The condition (W) asserts that τ is finite. Hence $\tau(H_\xi) \leqq \tau m/i^1(\hat{M}_\xi)$ where $m = \sharp(\hat{M}_\xi) = \sharp(M_\xi)$. By the prehomogeneity, the constant m does not depend on ξ. Let y be an element in Θ_x such that $i^1(\hat{M}_y) \leqq i^1(\hat{M}_\xi)$ for any $\xi \in \Theta_x$. Then, by [12, Lemma 6.2] and Lemma 2.1,

$$\tau(\Theta_x) \leqq \tau m\, i^1(H_y)/i^1(\hat{M}_y) \leqq \tau m\, h^1(\tilde{H}_y) .$$

The condition (H) implies that $h^1(\tilde{H}_y)$ depends only on the isomorphism class of H_y over $\boldsymbol{R}$. Since the number of $G_{\boldsymbol{R}}^{+}$-orbits in $V_{\boldsymbol{R}} - S_{\boldsymbol{R}}$ is finite,

$$h^1 = \operatorname{Sup}\{h^1(\tilde{H}_y); y \in V_Q - S_Q\} < +\infty .$$

Thus we have the inequality $\tau(\Theta_x) \leqq \tau m\, h^1$ $(x \in V_Q - S_Q)$. The right hand side of this inequality is independent of x.

2.4. By the condition (S), the group H has no non-trivial rational character. Hence, for any $t \in (\boldsymbol{Q}^\times)^n$, $V(t)$ is a special homogeneous space defined over $\boldsymbol{Q}$ in the sence of Ono [12]. The formal product $\prod_\nu |\theta_t|_\nu$ well-defines a measure on $V(t)_A$ (cf. [12, § 4]). The Tamagawa measure on H_A (resp. $H_{\xi,A}$, $\xi \in V_Q - S_Q$) is given by

$$|dh|_A = \prod_\nu |dh|_\nu \quad (\text{resp. } |d\nu_\xi|_A = \prod_\nu |d\nu_\xi|_\nu) .$$

LEMMA 2.3. *Let f be an everywhere non-negative function in $L^1(V(t)_A; |\theta_t|_A)$. Then*

$$I(f, t) = \int_{H_A/H_Q} \sum_{\xi \in V(t)_Q} f(\rho(h)\xi)|dh|_A < c_1 \int_{V(t)_A} f(x)|\theta_t(x)|_A$$

for some positive constant c_1 independent of t and f.

PROOF. It is easy to see that

$$I(f, t) = \sum_{\xi}{}' \tau(H_\xi) \int_{\rho(H_A)\xi} f(x)|\theta_t(x)|_A$$

where the summation is taken over all the global equivalence classes ξ in $V(t)_Q$. Since the integral on the right hand side depends only on the local equivalence class of ξ, we have

$$I(f, t) = \sum_{\xi}{}'' \tau(\Theta_\xi) \int_{\rho(H_A)\xi} f(x)|\theta_t(x)|_A$$

where the summation is taken over all the local equivalence classes ξ in $V(t)_Q$. By Lemma 2.2,

$$I(f, t) < c_1 \int_{\rho(H_A)V(t)_Q} f(x)|\theta_t(x)|_A \leqq c_1 \int_{V(t)_A} f(x)|\theta_t(x)|_A$$

for some positive constant c_1 independent of t and f.

LEMMA 2.4. *We have the inequality*

$$A(t) < c_2 \prod_p \int_{V(t)_{Z_p}} |\theta_t(x)|_p \quad (t \in (Q^\times)^n)$$

for some positive constant c_2 independent of t, where the product with respect to p is taken over all finite primes of Q.

PROOF. Set $\Phi = \bigotimes_v \Phi_v$ where Φ_p is the characteristic function of V_{Z_p} for any finite prime p and Φ_∞ is an everywhere non-negative smooth function on V_R with the compact support contained in $V_R - S_R$. Then the restriction of Φ to $V(t)_A$ is an L^1-function with respect to the measure $|\theta_t|_A$ and

$$I(\Phi, t) = I(\Phi|_{V(t)_A}, t) \geqq \prod_p \int_{H_{Z_p}} |dh|_p \times \int_{H_R/H_Z} \sum_{\xi \in V(t)_Z} \Phi_\infty(\rho(h)\xi)|dh|_\infty .$$

Since H is special in the sense of Ono [12], the product

$$\prod_p \int_{H_{Z_p}} |dh|_p$$

is finite. Let $V(t)_R = V(t)_{1,R} \cup \cdots \cup V(t)_{m,R}$ be the H_R-orbit decomposition. For any G_R-orbit $\mathcal{O}$ in $V_R - S_R$ and for $t \in (R^\times)^n$ such that $V(t)_R \cap \mathcal{O} \neq \emptyset$, the number of H_R-orbits in $V(t)_R \cap \mathcal{O}$ depends only on $\mathcal{O}$, since H_R is a normal subgroup of G_R. This shows that the number m of H_R-orbits in $V(t)_R$ does not exceed some positive constant M. Put $V(t)_{i,Z} = V(t)_{i,R} \cap V(t)_Z$. Assuming that $(\operatorname{Supp}\Phi_\infty) \cap V(t)_R \subset V(t)_{i,R}$, we obtain

$$I(\Phi, t) \geqq A(t)_i \cdot \left\{\prod_p \int_{H_{Z_p}} |dh|_p\right\} \cdot \int_{V(t)_R} \Phi_\infty(x) |\theta_t(x)|_\infty .$$

Here we put $A(t)_i = \sum_\xi \nu(\xi)$ where ξ runs through a complete set of representatives of $H_Z \backslash V(t)_{i,Z}$. Hence, by Lemma 2.3,

$$A(t)_i < c_1 \left\{\prod_p \int_{H_{Z_p}} |dh|_p\right\}^{-1} \prod_p \int_{V(t)_{Z_p}} |\theta_t(x)|_p$$

for any i. Therefore the inequality in the lemma is valid for

$$c_2 = M \cdot c_1 \cdot \left\{\prod_p \int_{H_{Z_p}} |dh|_p\right\}^{-1} .$$

2.5. For any algebraic object X defined over $\boldsymbol{Q}$ or $\boldsymbol{Q}_p$, we denote by $X^{(p)}$ the reduction of X modulo a finite prime p. The following lemma is easily proved by the theory of reduction of constant fields (cf. [19, Chap. III]).

LEMMA 2.5. *There exists a finite set $\boldsymbol{P}_1$ of primes of $\boldsymbol{Q}$ such that, for any finite prime $p \notin \boldsymbol{P}_1$,*

(1) *$G^{(p)}$ is a connected linear algebraic group defined over $\boldsymbol{F}_p$,*

(2) *the reduction $\rho^{(p)}$ of ρ is a representation of $G^{(p)}$ on $V^{(p)}$ defined over $\boldsymbol{F}_p$ and $\rho^{(p)}(G^{(p)})$ acts on $V^{(p)} - S^{(p)}$ transitively,*

(3) *all the coefficients of $P_1, \cdots, P_n$ are in $\boldsymbol{Z}_p$ and $S^{(p)}$ is given by*

$$S^{(p)} = \bigcup_{i=1}^{n} \{x \in V^{(p)};\, P_i^{(p)}(x) = 0\} .$$

Take a $\boldsymbol{Q}$-subgroup H_s of H such that H_s is semi-simple and H is a semi-direct product of H_s and $R_u(H)$. Since H has no non-trivial character, such an H_s exists (cf. [12, Theorem 2.1]).

LEMMA 2.6. *There exists a finite set $\boldsymbol{P}_2$ of primes of $\boldsymbol{Q}$ such that*

(1) *$\boldsymbol{P}_2 \supset \boldsymbol{P}_1$,*

(2) *if $p \notin \boldsymbol{P}_2$, then $H^{(p)}$ is a connected linear algebraic group defined over $\boldsymbol{F}_p$ and is a semi-direct product of $R_u(H)^{(p)}$ and $H_s^{(p)}$,*

(3) *for any $t \in \boldsymbol{Z}^n$, if $(p, t_1 \cdots t_n) = 1$ and $p \notin \boldsymbol{P}_2$, then $H_{F_p}^{(p)}$ acts transitively on $V(t)_{F_p}^{(p)}$.*

PROOF. Fix a $\xi \in (V - S) \cap V_Z$ and put $\tau = (\tau_1, \cdots, \tau_n) = (P_1(\xi), \cdots, P_n(\xi))$. Let $\boldsymbol{P}_2$ be a finite set of primes which, in addition to (1) and (2), satisfies the conditions

(4) if $p \notin \boldsymbol{P}_2$, then $(p, \tau_1 \cdots \tau_n) = 1$ and $H^{(p)}$ acts transitively on $V(\tau)^{(p)}$, and

(5) if $p \notin \boldsymbol{P}_2$, then $(H_\xi)^{(p)}$ is a connected semi-simple algebraic group and coincides with the group

$$H_{\bar{\xi}}^{(p)} = \{g \in H^{(p)};\ \rho^{(p)}(g)\bar{\xi} = \bar{\xi}\}$$

where $\bar{\xi} = \xi \pmod p$. Let us prove that these four conditions imply the condition (3). Let p be a prime which is not contained in $\boldsymbol{P}_2$ and let $t_1, \cdots, t_n$ be rational integers such that $(p, t_1 \cdots t_n) = 1$. Since $p \notin \boldsymbol{P}_1$, the group $G^{(p)}$ acts transitively on $V^{(p)} - S^{(p)}$. Hence, for an $\eta \in V(t)_{F_p}^{(p)}$, there exists a $g \in G^{(p)}$ such that $\rho^{(p)}(g)\bar{\xi} = \eta$. By (4), $gH^{(p)}g^{-1} = H^{(p)}$ acts transitively on $V(t)^{(p)}$. By (5), the group $H_\eta^{(p)} = gH_{\bar{\xi}}^{(p)}g^{-1}$ is also connected. Therefore, by [5, Theorem 2], the principal homogeneous spaces

$$\{g \in H^{(p)};\ \rho^{(p)}(g)\eta = x\} \quad (x \in V(t)_{F_p}^{(p)})$$

over $H_\eta^{(p)}$ defined over $\boldsymbol{F}_p$ have non-empty sets of $\boldsymbol{F}_p$-rational points. This shows that $\boldsymbol{P}_2$ satisfies the condition (3).

LEMMA 2.7. *If $p \notin \boldsymbol{P}_2$ and $t_1, \cdots, t_n \in \boldsymbol{Z}_p^\times$,*

$$\int_{V(t)_{Z_p}} |\theta_t|_p = p^{-(\dim V - n)} \#(H_{F_p}^{(p)}) / \#(H_{\eta, F_p}^{(p)})$$

for an $\eta \in V(t)_{F_p}^{(p)}$.

PROOF. If $p \notin \boldsymbol{P}_2$ and $t_1, \cdots, t_n \in \boldsymbol{Z}_p^\times$, we have, by Lemma 2.6 (3),

$$\#(V(t)_{F_p}^{(p)}) = \#(H_{F_p}^{(p)}) / \#(H_{\eta, F_p}^{(p)}) \tag{2-5}$$

for an $\eta \in V(t)_{F_p}^{(p)}$. Since $H^{(p)}$ acts on $V(t)^{(p)}$ transitively, every point in $V(t)_{F_p}^{(p)}$ is a simple point. Hence, by the same argument as in the proof of [24, Theorem 2.2.5], we obtain

$$\int_{V(t)_{Z_p}} |\theta_t(x)|_p = p^{-(\dim V - n)} \#(V(t)_{F_p}^{(p)}) .$$

Combining this equality with (2-5), we get the lemma.

LEMMA 2.8. *Let t be an n-tuple of non-zero integers. Then, for some positive constant c_3 independent of t,*

$$\prod_p{}' \int_{V(t)_{Z_p}} |\theta_t|_p \leqq c_3 \prod_p{}' \int_{\Gamma_p(1)} (1 - p^{-1})^{-n} |dx|_p$$

where $\Gamma_p(1) = \{x \in V_{Z_p};\ P_i(x) \in \boldsymbol{Z}_p^\times\ (1 \leqq i \leqq n)\}$ and the product is taken over all finite primes such that $(p, t_1) = \cdots = (p, t_n) = 1$ and $p \notin \boldsymbol{P}_2$.

PROOF. Since $H_s^{(p)}$ and $H_\eta^{(p)}$ $(\eta \in V(t)_{F_p}^{(p)})$ are semi-simple for $p \notin \boldsymbol{P}_2$, it is known that

$$\prod_{i=1}^{r} (1 - p^{-a(i)}) \leqq p^{-\dim H^{(p)}} \#(H_{F_p}^{(p)}) = p^{-\dim H_s^{(p)}} \#(H_{s, F_p}^{(p)}) \leqq \prod_{i=1}^{r} (1 + p^{-a(i)})$$

and

$$\prod_{i=1}^{r'} (1 - p^{-b(i)}) \leqq p^{-\dim H_\eta^{(p)}} \#(H_{\eta, F_p}^{(p)}) \leqq \prod_{i=1}^{r'} (1 + p^{-b(i)})$$

where $r = \operatorname{rank} H_s^{(p)}$, $r' = \operatorname{rank} H_\eta^{(p)}$ and $a(i)$, $b(i) \geqq 2$ (cf. [11] and [10, Appendix II]). The constants $b(1), \cdots, b(r')$ and r' are independent of η and p. By Lemma 2.7, we have

$$(2\text{-}6)\quad \left\{\prod_{i=1}^{r}(1-p^{-a(i)})\right\}\Big/\left\{\prod_{i=1}^{r'}(1+p^{-b(i)})\right\} \leqq \int_{V(\tau)_{Z_p}} |\theta_\tau|_p \leqq \left\{\prod_{i=1}^{r}(1+p^{-a(i)})\right\}\Big/\left\{\prod_{i=1}^{r'}(1-p^{-b(i)})\right\}$$

for any $p \notin \boldsymbol{P}_2$ and any $\tau \in (\boldsymbol{Z}_p^\times)^n$. Hence

$$\int_{V(t)_{Z_p}} |\theta_t|_p \leqq \left\{\prod_{i=1}^{r}\frac{(1+p^{-a(i)})}{(1-p^{-a(i)})}\right\}\left\{\prod_{i=1}^{r'}\frac{(1+p^{-b(i)})}{(1-p^{-b(i)})}\right\}\int_{V(\tau)_{Z_p}} |\theta_\tau|_p$$

for any $p \notin \boldsymbol{P}_2$ such that $(p, t_1) = \cdots = (p, t_n) = 1$ and for any $\tau \in (\boldsymbol{Z}_p^\times)^n$. Put

$$c_3 = \prod_p \left\{\prod_{i=1}^{r}\frac{(1+p^{-a(i)})}{(1-p^{-a(i)})}\right\}\left\{\prod_{i=1}^{r'}\frac{(1+p^{-b(i)})}{(1-p^{-b(i)})}\right\}$$

where the product is over all the finite primes. Then

$$\begin{aligned}\prod_p{}' \int_{V(t)_{Z_p}} |\theta_t|_p &\leqq c_3 \prod_p{}' \int_{(Z_p^\times)^n} (1-p^{-1})^{-n} |d\tau_1|_p \cdots |d\tau_n|_p \int_{V(\tau)_{Z_p}} |\theta_\tau|_p \\ &= c_3 \prod_p{}' \int_{\Gamma_p(1)} (1-p^{-1})^{-n} |dx|_p .\end{aligned}$$

2.6. Let T be the torus part of the radical of G. Since (G, ρ, V) is split over $\boldsymbol{Q}$ and has the property (S), T is a $\boldsymbol{Q}$-split torus of dimension n. Let $\psi_1, \cdots, \psi_n$ be a system of generators of the group of rational characters of T. Then there exists an n by n integral matrix $D = (d_{ij})$ of rank n such that $\chi_i = \prod_{j=1}^n \psi_j^{d_{ij}}$ $(1 \leqq i \leqq n)$ on T. We identify T with $GL(1)^n$ via the isomorphism $\psi\colon T \to GL(1)^n$ defined by $\psi(g) = (\psi_1(g), \cdots, \psi_n(g))$. For any prime number p, we put $T_{Z_p} = \psi^{-1}((\boldsymbol{Z}_p^\times)^n)$. Let i_p be the index of $\rho(T_{Z_p}) \cap GL(V)_{Z_p}$ in $\rho(T_{Z_p})$. The index i_p is finite for all p and is equal to 1 for almost all p. Set

$$V_{t,Z_p} = \{\gamma x;\, x \in V(t)_{Z_p},\, \gamma \in \rho(T_{Z_p}) \cap GL(V)_{Z_p}\} .$$

Denote by $d_1, \cdots, d_n$ the elementary divisors of D and set

$$v_p = \prod_{i=1}^{n} \int_{U_p(d_i)} |d\tau|_p ,$$

where $U_p(d_i) = \{\tau = u^{d_i};\, u \in \boldsymbol{Z}_p^\times\}$. For a $u \in (\boldsymbol{Z}_p^\times)^n$ and a $t \in (\boldsymbol{Q}^\times)^n$, we write

$$u^D = (\chi_1(\psi^{-1}(u)), \cdots, \chi_n(\psi^{-1}(u))) = \left(\prod_{j=1}^{n} u_j^{d_{1j}}, \cdots, \prod_{j=1}^{n} u_j^{d_{nj}}\right)$$

and

$$u^D t = (\chi_1(\psi^{-1}(u))t_1, \cdots, \chi_n(\psi^{-1}(u))t_n) .$$

LEMMA 2.9. *For any finite prime p and any $t \in (\boldsymbol{Z} - \{0\})^n$,*

$$\int_{V(t)_{\boldsymbol{Z}_p}} |\theta_t|_p \leqq (i_p/v_p)|t_1 \cdots t_n|_p^{-1} \int_{V_{t,\boldsymbol{Z}_p}} |dx|_p .$$

PROOF. For a $u \in (\boldsymbol{Z}_p^\times)^n$ such that $\rho \circ \psi^{-1}(u) \in GL(V)_{\boldsymbol{Z}_p}$, $\rho \circ \psi^{-1}(u)$ induces a homeomorphism of $V(t)_{\boldsymbol{Z}_p}$ onto $V(\tau)_{\boldsymbol{Z}_p}$ and we have

$$\int_{V(t)_{\boldsymbol{Z}_p}} |\theta_t|_p = \int_{V(\tau)_{\boldsymbol{Z}_p}} |\theta_\tau|_p$$

where $\tau = u^D t$. Further we obtain

$$\int_\tau |d\tau_1|_p \cdots |d\tau_n|_p \geqq |t_1 \cdots t_n|_p v_p/i_p$$

where the integral is taken over the set

$$\{\tau = u^D t;\, u \in (\boldsymbol{Z}_p^\times)^n,\, \rho \circ \psi^{-1}(u) \in GL(V)_{\boldsymbol{Z}_p}\} .$$

Hence

$$\begin{aligned}\int_{V(t)_{\boldsymbol{Z}_p}} |\theta_t|_p &\leqq (i_p/v_p)|t_1 \cdots t_n|_p^{-1} \int_\tau |d\tau_1|_p \cdots |d\tau_n|_p \int_{V(\tau)_{\boldsymbol{Z}_p}} |\theta_\tau|_p \\ &= (i_p/v_p)|t_1 \cdots t_n|_p^{-1} \int_{V_{t,\boldsymbol{Z}_p}} |dx|_p .\end{aligned}$$

COROLLARY. *If $(p, d_1) = \cdots = (p, d_n) = 1$,*

$$\int_{V(t)_{\boldsymbol{Z}_p}} |\theta_t|_p \leqq i_p \prod_{i=1}^n (d_i, p-1)|t_1 \cdots t_n|_p^{-1} \int_{V_{t,\boldsymbol{Z}_p}} (1 - p^{-1})^{-n} |dx|_p .$$

PROOF. If $(p, d_i) = 1$, then

$$\int_{U_p(d_i)} |d\tau|_p = (1 - p^{-1})/(d_i, p-1) .$$

This proves the assertion.

2.6. The following lemma is a generalization of a part of [13, Theorem 1].

LEMMA 2.10. (1) *Put*

$$\lambda_\nu = \begin{cases} (1 - p^{-1})^n & \text{for } \nu = \text{a finite prime } p , \\ 1 & \text{for } \nu = \infty . \end{cases}$$

Then $\{\lambda_\nu\}$ is a convergence factor for $V - S$, namely,

$$\prod_p \lambda_p^{-1} \int_{(V-S)_{\boldsymbol{Z}_p}} |dx|_p < \infty .$$

(2) *For any* $f \in \mathscr{S}(V_A)$, *the integral*

$$\int_{(V-S)_A} \prod_{i=1}^{n} |P_i(x)|_A^{s_i} f(x) |\lambda^{-1} dx|_A$$

is absolutely convergent for $\operatorname{Re} s_1, \cdots, \operatorname{Re} s_n > 0$, *where*

$$|\lambda^{-1} dx|_A = \prod_{\nu} \lambda_\nu^{-1} |dx|_\nu .$$

PROOF. Since we are assuming that (G, ρ, V) is split over $\boldsymbol{Q}$, the polynomials $P_1, \cdots, P_n$ are absolutely irreducible and algebraically independent. We take a finite set $\boldsymbol{P}$ of primes of $\boldsymbol{Q}$ satisfying the following three conditions:

(1) $\boldsymbol{P} \ni \infty$.

(2) If $p \notin \boldsymbol{P}$, then $P_1, \cdots, P_n$ have coefficients in $\boldsymbol{Z}_p$. Moreover their reductions $P_1^{(p)}, \cdots, P_n^{(p)}$ modulo p remain to be absolutely irreducible and algebraically independent.

(3) If $p \notin \boldsymbol{P}$, then

$$\int_{(V-S)_{\boldsymbol{Z}_p}} |dx|_p = p^{-\dim V} \#[(V-S)_{\boldsymbol{F}_p}^{(p)}] .$$

Let p be a prime such that $p \notin \boldsymbol{P}$. In the following, we denote by $c_1, c_2, \cdots$ positive constants independent of p. For any subset I of $\{1, 2, \cdots, n\}$, we put

$$N_I^{(p)} = \#\{x \in \boldsymbol{F}_p^{\dim V};\, P_i^{(p)}(x) = 0 \text{ for all } i \in I\} .$$

In particular, for $I = \varnothing$, $N_\varnothing^{(p)} = p^{\dim V}$. Then $\#[(V-S)_{\boldsymbol{F}_p}^{(p)}] = \sum_I (-1)^{\#(I)} N_I^{(p)}$. Since $P_1^{(p)}, \cdots, P_n^{(p)}$ are algebraically independent, by [6, Lemma 1],

$$N_I^{(p)} \leqq c_1 p^{\dim V - \#(I)} . \tag{2-7}$$

If $\#(I) = 1$, by [6, Theorem 1] and the fact that $P_i^{(p)}$'s are absolutely irreducible, we have

$$|N_I^{(p)} - p^{\dim V - 1}| \leqq c_2 p^{\dim V - 3/2} \quad (\#(I) = 1) . \tag{2-8}$$

By (3), we get

$$\lambda_p^{-1} \int_{(V-S)_{\boldsymbol{Z}_p}} |dx|_p = (1 - p^{-1})^{-n} \sum_I (-1)^{\#(I)} p^{-\dim V} N_I^{(p)} .$$

Hence, by (2-7) and (2-8),

$$\left| 1 - \lambda_p^{-1} \int_{(V-S)_{\boldsymbol{Z}_p}} |dx|_p \right| < c_3 p^{-3/2} . \tag{2-9}$$

This implies the first assertion. It is enough to prove the second assertion under the additional assumption that f is of the form $f = \bigotimes_\nu f_\nu$ where $f_\nu \in \mathscr{S}(V_{\boldsymbol{Q}_\nu})$ and f_p is the characteristic function of $V_{\boldsymbol{Z}_p}$ for almost all p. So we may assume that, if $p \notin \boldsymbol{P}$, f_p is the characteristic function of V_{Z_p}. For a $p \notin \boldsymbol{P}$, put

$$I^{(p)} = \int_{V_{Z_p}} \prod_{i=1}^{n} |P_i(x)|_p^{s_i} \lambda_p^{-1} |dx|_p .$$

Also put

$$E_0 = \{x \in V_{Z_p};\ P_i(x) \not\equiv 0 \pmod p \text{ for all } i\}$$

and $E_1 = V_{Z_p} - E_0$. Since $|P_i(x)|_p = 1$ $(1 \leqq i \leqq n)$ on E_0, we have by the assumption (3)

$$\text{(2-10)} \qquad \int_{E_0} \prod_{i=1}^{n} |P_i(x)|_p^{s_i} \lambda_p^{-1} |dx|_p = \lambda_p^{-1} \int_{(V-S)_{Z_p}} |dx|_p .$$

Assume that $\operatorname{Re} s_1, \cdots, \operatorname{Re} s_n \geqq \varepsilon$. Then $|\prod_{i=1}^{n} |P_i(x)|_p^{s_i}| \leqq p^{-\varepsilon}$ for $x \in E_1$. Hence

$$\left| \int_{E_1} \prod_{i=1}^{n} |P_i(x)|_p^{s_i} \lambda_p^{-1} |dx|_p \right| \leqq \lambda_p^{-1} p^{-\dim V - \varepsilon} \#[E_1 \colon \operatorname{mod} p] .$$

It is obvious that $\#[E_1 \colon \operatorname{mod} p] = \sum_{I \neq \varnothing} (-1)^{\#(I)-1} N_I^{(p)}$. By (2-7), we get

$$\text{(2-11)} \qquad \left| \int_{E_1} \prod_{i=1}^{n} |P_i(x)|_p^{s_i} \lambda_p^{-1} |dx|_p \right| < c_4 p^{-1-\varepsilon} .$$

Since the integral over V_{Z_p} is the sum of those over E_1 and E_0, it follows from (2-9), (2-10) and (2-11) that

$$\left| 1 - \int_{V_{Z_p}} \prod_{i=1}^{n} |P_i(x)|_p^{s_i} \lambda_p^{-1} |dx|_p \right| < c_5 \operatorname{Max}(p^{-3/2}, p^{-1-\varepsilon})$$

$(p \notin \boldsymbol{P},\ \operatorname{Re} s_1, \cdots, \operatorname{Re} s_n \geqq \varepsilon)$. This shows that the integral

$$\int_{(V-S)_A} \prod_{i=1}^{n} |P_i(x)|_A^{s_i} f(x) |\lambda^{-1} dx|_A$$

converges absolutely for $\operatorname{Re} s_1, \cdots, \operatorname{Re} s_n > 0$ and is equal to the product

$$\prod_{\nu} \int_{(V-S)_{Q_\nu}} \prod_{i=1}^{n} |P_i(x)|_\nu^{s_i} f_\nu(x) \lambda_\nu^{-1} |dx|_\nu .$$

2.7. Now we are ready to prove Theorem 1. Set

$$\boldsymbol{P}_3 = \boldsymbol{P}_2 \cup \{p;\ p \mid d_i \text{ for some } i\} \cup \{p;\ i_p \geqq 2\} ,$$

where $\boldsymbol{P}_2$ is a finite set of primes given by Lemma 2.7. By Lemma 2.8, Lemma 2.9 and its corollary, we obtain

$$\text{(2-12)} \qquad \prod_p \int_{V(t)_{Z_p}} |\theta_t|_p < c_3 \{ \prod_{p \in \boldsymbol{P}_3} i_p (1 - p^{-1})^n / v_p \} \Big\{ \prod_{p \mid t_1 \cdots t_n} \prod_{i=1}^{n} (d_i, p-1) \Big\}$$
$$\times \prod_p |t_1 \cdots t_n|_p^{-1} \int_{\Gamma_p(t)} \lambda_p^{-1} |dx|_p$$

where $\Gamma_p(t) = \{x \in V_{Z_p}; |P_i(x)|_p = |t_i|_p\ (1 \leqq i \leqq n)\}$ and c_3 is the constant given by Lemma 2.8.

LEMMA 2.11. *Let d be a non-zero integer. Then, for any $\varepsilon > 0$, there exists a constant c_ε such that*

$$\prod_{p|t} (d, p-1) < c_\varepsilon |t|^\varepsilon$$

for all $t \in \boldsymbol{Z} - \{0\}$.

PROOF. Take a prime number p_0 such that $\log d < \varepsilon \log p_0$. Let m_0 be the number of primes smaller than p_0. Let m be the number of primes which divide t. If $m \leqq m_0$, then $\prod_{p|t} (d, p-1) \leqq d^m \leqq d^{m_0}$. Assume that $m > m_0$. Let

$$|t| = p_1^{r_1} \cdots p_m^{r_m} \quad (p_1 < p_2 < \cdots < p_m,\ r_i \geqq 1)$$

be the decomposition of $|t|$ into the product of primes. Then we have

$$\log |t| = \sum_{i=1}^{m} r_i \log p_i > m_0 \log 2 + (m - m_0) \log p_0 \,.$$

Hence

$$\prod_{p|t} (d, p-1) \leqq d^m < \exp\{(\log d/\log p_0) \log |t| + m_0 \log d\} < d^{m_0}|t|^\varepsilon \,.$$

Thus we get $\prod_{p|t} (d, p-1) < d^{m_0}|t|^\varepsilon$ for any $t \in \boldsymbol{Z} - \{0\}$.

For an arbitrary $\varepsilon > 0$, by (2-12) and Lemma 2.11, there exists a constant c'_ε independent of t, such that

$$\prod_p \int_{V(t)_{Z_p}} |\theta_t|_p < c'_\varepsilon \prod_p \left\{ |t_1 \cdots t_n|_p^{-1-\varepsilon} \int_{\Gamma_p(t)} \lambda_p^{-1} |dx|_p \right\} .$$

Therefore, by Lemma 2.4, the Dirichlet series (2-4) is majorized by

$$\begin{aligned} c_2 c'_\varepsilon \sum_t \prod_p & \left\{ \prod_{i=1}^{n} |t_i|_p^{s_i - \delta_i - \varepsilon} \int_{\Gamma_p(t)} \lambda_p^{-1} |dx|_p \right\} \\ & \leqq 2^n c_2 c'_\varepsilon \prod_p \int_{V_{Z_p}} \prod_{i=1}^{n} |P_i(x)|_p^{s_i - \delta_i - \varepsilon} \lambda_p^{-1} |dx|_p \,. \end{aligned}$$

Lemma 2.10 implies that the Dirichlet series (2-4) converges absolutely for $\mathrm{Re}\, s_1 > \delta_1, \cdots, \mathrm{Re}\, s_n > \delta_n$. Thus Theorem 1 is proved.

REMARK. If we remove the assumption that (G, ρ, V) is split over $\boldsymbol{Q}$ in Theorem 1, then we are able to obtain a less precise result that $\xi_1(L; s), \cdots, \xi_\nu(L; s)$ are absolutely convergent for $\mathrm{Re}\, s_1 > \delta_1 + r + 1, \cdots$, $\mathrm{Re}\, s_n > \delta_n + r + 1$ where r is the dimension of the torus part of the radical of H. Moreover, Theorem 2 is valid without the assumption of of splitness of (G, ρ, V).

3. Application. In this section, we give an application of Theorem 1 to the castling transform. The notion of castling transform was introduced by M. Sato and plays an essential role in the classification of irreducible p.v.'s (see [16]).

3.1. Let G_0 be a connected linear algebraic group, V_0 a finite dimensional $\boldsymbol{C}$-vector space and ρ_0 a rational representation of G_0 on V_0. For any positive integer k, we denote by Λ_1 the standard representation of $GL(k)$ (or $SL(k)$) on the k-dimensional vector space $V(k) = \boldsymbol{C}^k$. Put $m = \dim V_0$. For a k $(1 \leqq k \leqq m-1)$, consider the triples

$$(G, \rho, V) = (G_0 \times GL(k), \rho_0 \otimes \Lambda_1, V_0 \otimes V(k))$$

and

$$(G', \rho', V') = (G_0 \times GL(m-k), \rho_0^* \otimes \Lambda_1, V_0^* \otimes V(m-k))$$

where V_0^* is the vector space dual to V_0 and ρ_0^* is the representation of G_0 contragredient to ρ_0.

Let $\bigwedge^k(V_0)$ (resp. $\bigwedge^{m-k}(V_0^*)$) be the k- (resp. $(m-k)$-) fold exterior power of V_0 (resp. V_0^*). The representation ρ_0 (resp. ρ_0^*) canonically induces a representation ρ_k (resp. ρ_{m-k}^*) of G_0 on $\bigwedge^k(V_0)$ (resp. $\bigwedge^{m-k}(V_0^*)$). We may identify $\bigwedge^k(V_0)$ and $\bigwedge^{m-k}(V_0^*)$ via the canonical pairing $\bigwedge^k(V_0) \times \bigwedge^{m-k}(V_0) \to \bigwedge^m(V_0) \cong \boldsymbol{C}$. Fix an identification $\iota\colon \bigwedge^k(V_0) \to \bigwedge^{m-k}(V_0^*)$. Then

$$\iota(\rho_k(g)y) = \det \rho_0(g) \cdot \rho_{m-k}^*(g)\iota(y) \quad (g \in G_0,\ y \in \textstyle\bigwedge^k(V_0)) . \tag{3-1}$$

We also identify V (resp. V') with the direct sum of k (resp. $m-k$) copies of V_0 (resp. V_0^*). Let $\lambda\colon V \to \bigwedge^k(V_0)$ and $\lambda'\colon V' \to \bigwedge^{m-k}(V_0^*)$ be the mappings defined by $\lambda(x_1, \cdots, x_k) = x_1 \wedge \cdots \wedge x_k$ and $\lambda'(x_1^*, \cdots, x_{m-k}^*) = x_1^* \wedge \cdots \wedge x_{m-k}^*$. We get

$$\begin{cases} \lambda(\rho(g, h)x) = (\det h)^{-1}\rho_k(g)\lambda(x) , \\ \lambda'(\rho'(g, h')x') = (\det h')^{-1}\rho_{m-k}^*(g)\lambda'(x') \end{cases} \tag{3-2}$$

$(g \in G_0,\ h \in GL(k),\ h' \in GL(m-k),\ x \in V,\ x' \in V')$.

Set $W = V - \lambda^{-1}(0)$ and $W' = V' - \lambda'^{-1}(0)$.

LEMMA 3.1. *For an $x \in W$ and an $x' \in W'$ such that $\iota(\lambda(x)) = \lambda'(x')$, the isotropy subgroup G_x of G at x is isomorphic to the isotropy subgroup $G'_{x'}$ of G' at x'.*

PROOF. Let p (resp. p') be the projection of G (resp. G') onto G_0. Since the fibre $\lambda^{-1}(\lambda(x))$ (resp. $\lambda'^{-1}(\lambda'(x'))$) is a principal homogeneous space of $SL(k)$ (resp. $SL(m-k)$), we obtain

$$p(G_x) = \{g \in G_0;\ \rho_k(g)\lambda(x) = t\lambda(x) \text{ for some } t \in \boldsymbol{C}^\times\}$$

and

$$p'(G'_{x'}) = \{g \in G_0;\ \rho^*_{m-k}(g)\lambda'(x') = t\lambda'(x') \text{ for some } t \in \boldsymbol{C}^\times\}\,.$$

Hence, by (3-1), $p(G_x) = p'(G'_{x'})$. It can be easily seen that $G_x \cong p(G_x)$ and $G'_{x'} \cong p'(G'_{x'})$.

The next lemma is an immediate consequence of Lemma 3.1.

LEMMA 3.2. *The triple* (G, ρ, V) *is a p.v. if and only if the triple* (G', ρ', V') *is a p.v. In this case, we have* $\iota\lambda(V - S) = \lambda'(V' - S')$, *where* S *and* S' *is the singular sets of* (G, ρ, V) *and* (G', ρ', V'), *respectively.*

We call the triples (G, ρ, V) and (G', ρ', V') the *castling transforms* of each other.

It is well-known that any invariant of $SL(k)$ (resp. $SL(m-k)$) on V (resp. V') is a composite of a rational function on $\bigwedge^k(V_0)$ (resp. $\bigwedge^{m-k}(V_0^*)$) and λ (resp. λ'). Hence we obtain the following lemma:

LEMMA 3.3. *Any relative invariant of* (G, ρ, V) *(resp.* (G', ρ', V')*) is of the form* $Q(\lambda(x))$ *(resp.* $Q(\lambda'(x'))$*), where* Q *is a homogeneous relative invariant of the triple* $(G_0, \rho_k, \bigwedge^k(V_0))$ *(resp.* $(G_0, \rho^*_{m-k}, \bigwedge^{m-k}(V_0^*))$*).*

Note that there exists a natural one-to-one correspondence between the set of homogeneous relative invariants of $(G_0, \rho_k, \bigwedge^k(V_0))$ and that of $(G_0, \rho^*_{m-k}, \bigwedge^{m-k}(V_0^*))$.

Suppose that (G_0, ρ_0, V_0) is defined over a field K. Then (G, ρ, V) and (G', ρ', V') have natural K-structures. In Lemma 3.1, if x and x' are K-rational points, G_x and $G'_{x'}$ are K-isomorphic. Moreover, we have $\iota\lambda(V_K - S_K) = \lambda'(V'_K - S'_K)$. By Lemmas 1.1 and 3.3, (G, ρ, V) is a p.v. split over K if and only if so is (G', ρ', V').

THEOREM 3. *Suppose that* (G_0, ρ_0, V_0) *is defined over* $\boldsymbol{Q}$. *Then the following two assertions are equivalent:*

(1) (G, ρ, V) *is a p.v. split over* $\boldsymbol{Q}$ *with the properties* (S), (H) *and* (W).

(2) (G', ρ', V') *is a p.v. split over* $\boldsymbol{Q}$ *with the properties* (S), (H) *and* (W).

PROOF. We prove (1) implies (2). By the observation preceding the theorem, (G', ρ', V') is also a p.v. split over $\boldsymbol{Q}$. Let H (resp. H') be the connected component of $G_1 = G_x\mathscr{D}(G)R_u(G)$ (resp. $G'_1 = G'_{x'}\mathscr{D}(G')R_u(G')$), where x (resp. x') is a generic point of (G, ρ, V) (resp. (G', ρ', V')). Since $\iota\lambda(V_Q - S_Q) = \lambda'(V'_Q - S'_Q)$, for any $x' \in V'_Q - S'_Q$, we can find an $x \in V_Q - S_Q$ such that $\iota(\lambda(x)) = \lambda'(x')$. Put $G^\circ_{0,x'} = p(G^\circ_x) = p'(G'^\circ_{x'})$. By the condition (S) for (G, ρ, V), the group $G^\circ_{0,x'}$ is a connected semi-simple algebraic group and has no non-trivial character. Hence, for any $g \in G^\circ_{0,x'}$, we have $\rho_k(g)\lambda(x) = \lambda(x)$ and $\rho^*_{m-k}(g)\lambda'(x') = \lambda'(x')$. This implies that $G^\circ_x \subset G^\circ_{0,x'} \times SL(k)$ and $(G'_{x'})^\circ \subset G^\circ_{0,x'} \times SL(m-k)$. Therefore $H = H_0 \times SL(k)$ and $H' = H_0 \times SL(m-k)$, where we put $H_0 = G^\circ_{0,x'}\mathscr{D}(G_0)R_u(G_0)$. Thus we obtain $H_x \cong \{g \in H_0;\ \rho_k(g)\lambda(x) = \lambda(x)\} = \{g \in H_0;\ \rho^*_{m-k}(g)\lambda'(x') = \lambda'(x')\} \cong H_{x'}$. Since the isomorphisms are all defined over $\boldsymbol{Q}$, the conditions (S), (H) and (W) hold also for (G', ρ', V').

3.2. As is noted in [16, § 2], the castling transform gives us a method to construct a new p.v. from a given p.v. Thanks to Theorems 1 and 3, we are able to make use of the castling transform in order to find new Dirichlet series satisfying certain functional equations. Here is an example:

Let Y be an m by m rational non-degenerate symmetric matrix of signature (p, q) $(p+q=m,\ p, q \geqq 1)$. We assume that $m \geqq 4$. Set $G_0 = SO(Y)$. Denote by ρ_0 the natural representation of G_0 on $V_0 = V(m) = \boldsymbol{C}^m$. Also set $G^{(1)} = SO(Y) \times GL(1)$ and $V^{(1)} = V_0$. Let $\rho^{(1)}$ be the representation of $G^{(1)}$ on $V^{(1)}$ defined by the formula

$$\rho^{(1)}(g, t)x = \rho_0(g)xt^{-1} \quad (g \in SO(Y),\ t \in GL(1),\ x \in V^{(1)})\ .$$

The triple $(G^{(1)}, \rho^{(1)}, V^{(1)})$ is a regular p.v. split over $\boldsymbol{Q}$ and has a unique (up to a constant factor) irreducible relative invariant $P(x) = {}^t x Y x$. The zeta functions associated with this p.v. are the Siegel zeta functions (see [20] and [17, § 2, n° 4]).

It is easy to check that the p.v. $(G^{(1)}, \rho^{(1)}, V^{(1)})$ satisfies (S), (H) and (W). By the repeated use of Theorem 3, the triples

$$(G^{(2)}, \rho^{(2)}, V^{(2)}) = (G^{(1)} \times SL(m-1), \rho^{(1)} \otimes \Lambda_1, V^{(1)} \otimes V(m-1))\ ,$$
$$(G^{(3)}, \rho^{(3)}, V^{(3)}) = (G^{(2)} \times SL(m^2-m-1), \rho^{(2)} \otimes \Lambda_1, V^{(2)} \otimes V(m^2-m-1))\ ,$$
$$\cdots\cdots\cdots\cdots$$

are p.v.'s split over $\boldsymbol{Q}$ with the same properties. Since $G^{(i)}$ is reductive and the generic isotropy subgroup is semi-simple, all these p.v.'s are regular (over $\boldsymbol{Q}$) ([16, § 4, Remark 26]). By Theorems 1 and 2, their associated zeta functions are absolutely convergent in some half plane and are continued meromorphically to the whole complex plane. Applying the result in [17] or [14] to these p.v.'s, we are able to obtain infinitely many new Dirichlet series which have analytic continuations to meromorphic functions in $\boldsymbol{C}$ and satisfy certain functional equations.

Here we give the explicit form of the functional equations of the zeta functions only for $(G^{(2)}, \rho^{(2)}, V^{(2)})$. In the following we omit the superscript (2).

Identify the vector space V with $M(m, m-1)$. The representation ρ is given by

$$\rho(g, t, h)x = gx(th)^{-1} \quad (g \in SO(Y),\ t \in GL(1),\ h \in SL(m-1),\ x \in M(m, m-1)) .$$

We also identify V^* with $V = M(m, m-1)$ via the symmetric bilinear form

$$\langle x, x^* \rangle = \operatorname{tr} {}^t x x^* \quad (x, x^* \in M(m, m-1)) .$$

The representation ρ^* contragradient to ρ is given by $\rho^*(g, t, h)x^* = {}^t g^{-1} x^* (t\, {}^t h)$. The polynomial $P(x) = \det({}^t x Y x)$ (resp. $Q(x^*) = \det({}^t x^* Y^{-1} x^*)$) is an irreducible relative invariant of (G, ρ, V) (resp. (G, ρ^*, V^*)).

Set $G_{\boldsymbol{R}}^+ = SO(Y)_{\boldsymbol{R}} \times \boldsymbol{R}_+ \times SL(m-1)_{\boldsymbol{R}}$ where $\boldsymbol{R}_+$ is the multiplicative group of positive real numbers. We put

$$V_+ = \{x \in V_{\boldsymbol{R}};\ P(x) > 0\} , \qquad V_- = \{x \in V_{\boldsymbol{R}};\ P(x) < 0\} ,$$
$$V_+^* = \{x^* \in V_{\boldsymbol{R}}^*;\ Q(x^*) > 0\} , \qquad V_-^* = \{x^* \in V_{\boldsymbol{R}}^*;\ Q(x^*) < 0\}$$

where $V_{\boldsymbol{R}} = V_{\boldsymbol{R}}^* = M(m, m-1; \boldsymbol{R})$. The orbit decompositions of $V_{\boldsymbol{R}} - S_{\boldsymbol{R}}$ and $V_{\boldsymbol{R}}^* - S_{\boldsymbol{R}}^*$ are as follows:

$$V_{\boldsymbol{R}} - S_{\boldsymbol{R}} = V_+ \cup V_- , \qquad V_{\boldsymbol{R}}^* - S_{\boldsymbol{R}}^* = V_+^* \cup V_-^* .$$

For an $f \in \mathscr{S}(V_{\boldsymbol{R}}) = \mathscr{S}(V_{\boldsymbol{R}}^*)$, set

$$\Phi_\pm(f; s) = \int_{V_\pm} |P(x)|^s f(x) dx \qquad \text{and} \qquad \Phi_\pm^*(f; s) = \int_{V_\pm^*} |Q(x^*)|^s f(x^*) dx^* ,$$

where dx and dx^* are the standard Euclidean measures on $V_{\boldsymbol{R}}$ and $V_{\boldsymbol{R}}^*$, respectively. We define the Fourier transform $\hat{f}$ of f by putting

$$\hat{f}(x) = \int_{V_{\boldsymbol{R}}^*} f(x^*) \exp(2\pi\sqrt{-1}\langle x, x^* \rangle) dx^* .$$

The explicit form of the functional equation in [17, Theorem 1] (or [14, Theorem 1]) is as follows:

LEMMA 3.4. *The functions $\Phi_\pm(f; s)$ and $\Phi_\pm^*(f; s)$ have analytic continuations to meromorphic functions of s in $\boldsymbol{C}$ and satisfy the following functional equations*:

$$\begin{pmatrix} \Phi_+(\hat{f}; s) \\ \Phi_-(\hat{f}; s) \end{pmatrix} = (-1)^m \pi^{-2(m-1)s-(m-1)(m+2)/2} |\det Y|^{(m-1)/2}$$
$$\times \prod_{i=1}^{m-1} \Gamma(s + (i+1)/2)^2 \prod_{i=1}^{m-2} \sin(2s + i)\pi/2$$
$$\times \begin{pmatrix} -\sin(2s + q)\pi/2 & \sin p\pi/2 \\ \sin q\pi/2 & -\sin(2s + p)\pi/2 \end{pmatrix} \begin{pmatrix} \Phi_+^*(f; -s - m/2) \\ \Phi_-^*(f; -s - m/2) \end{pmatrix} .$$

Let L be a $\rho(SO(Y)_{Z} \times SL(m-1)_{Z})$-invariant lattice in $M(m, m-1; \boldsymbol{Q})$ and L^* be the lattice dual to L. Let $\xi_{\pm}(L; s)$ and $\xi^*_{\pm}(L^*; s)$ be the zeta functions introduced in § 1 (or [14, § 4], [17]). Set

$$v(L) = \int_{V_{\boldsymbol{R}}/L} dx .$$

By Lemma 3.4 and [14, Theorem 2] (or [17, Theorem 2 and Additional Remark 2]), we have the following theorem:

THEOREM 4.

$$\begin{aligned}\begin{pmatrix}\xi^*_+(L^*; m/2-s)\\ \xi^*_-(L^*; m/2-s)\end{pmatrix} &= (-1)^m |\det Y|^{(m-1)/2} v(L)^{-1} \pi^{-2(m-1)s+(m-1)(m-2)/2}\\ &\quad \times \prod_{i=0}^{m-2} \Gamma(s-i/2)^2 \prod_{i=1}^{m-2} \sin(2s-i-1)\pi/2\\ &\quad \times \begin{pmatrix}-\sin(2s-m+q)\pi/2 & \sin q\pi/2\\ \sin p\pi/2 & -\sin(2s-m+p)\pi/2\end{pmatrix}\begin{pmatrix}\xi_+(L; s)\\ \xi_-(L; s)\end{pmatrix} .\end{aligned}$$

REMARK 1. In his lecture at RIMS, Kyoto University in the autumn of 1974, T. Shintani gave a general formula relating the functional equation satisfied by complex powers of relative imvariants of a p.v. to that of its casting transform under the assumptions that G_0 is reductive and the singular set is an irreducible hypersurface.

REMARK 2. In [17], the following condition, which assures the convergence of zeta functions and is checked by the Weil-Igusa criterion ([25, p. 20], [4, § 2]), is imposed on p.v.'s ([17, p. 146]):

(3-3) *For every* $f \in \mathscr{S}(V_{\boldsymbol{R}})$, *the integral*

$$I(f) = \int_{G^1_{\boldsymbol{R}}/G^1_{\boldsymbol{Z}}} \sum_{x \in V_{\boldsymbol{Z}}} f(\rho(g)x) d^1 g$$

converges absolutely and the mapping $f \mapsto I(f)$ *defines a tempered distribution on* $V_{\boldsymbol{R}}$ *(where* $G^1 = G_x[G, G]$ *for a generic point* x *and* $d^1 g$ *is a Haar measure on* $G^1_{\boldsymbol{R}}$*).*

This condition is however much stronger than what is needed to ensure the convergence of zeta functions (cf. [17, p. 169, Additional Remark 2]). For example, if $i \geqq 2$, the p.v. $(G^{(i)}, \rho^{(i)}, V^{(i)})$ does not satisfy (3-3). Though our assumptions (S), (H) and (W) are fairly restrictive, the class of p.v.'s treated in this paper contains several interesting examples which do not satisfy the condition (3-3).

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DEPARTMENT OF MATHEMATICS
RIKKYO UNIVERSITY
NISHI-IKEBUKURO, TOKYO 171
JAPAN