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ON SPHERICAL REPRESENTATION OF AN m-DIMENSIONAL SUBMANIFOLD IN THE EUCLIDEAN n-SPACE

Dedicated to Professor Shigeo Sasaki on his seventieth birthday

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1. Introduction. The spherical representation of a curve in the Euclidean 3-space is a representation on the unit sphere S^2 obtained with the use of tangent vectors. We consider a generalization of the notion of spherical representations to an m-dimensional submanifold in the Euclidean n-space. We denote a submanifold by (i, M) where M is an m-dimensional manifold and i is an immersion $i: M \to R^n$. If the spherical representation of (i, M) is regular, the image is an immersed submanifold of dimension $2m-1$ in the unit hypersphere of R^n. Any submanifold and its infinitesimal deformations we consider are assumed to be C^∞.

Let p be any point of M and $\{O\}_p$ be the origin of $T_p(M)$. To any half line of $T_p(M)$ from $\{O\}_p$ there corresponds a point of the unit hypersphere $S_0^{n-1}(1)$ of R^n. Taking all points p of M and all half lines of $T_p(M)$ from $\{O\}_p$ we get the spherical representation of (i, M).

For our purpose a little more precise description will be preferable. Any immersion i of M induces a Riemannian metric g on M and this determines the unit hypersphere $S_p(M)$ of $T_p(M)$. For any point (i, p) of (i, M) there exists just one m-dimensional tangent plane of (i, M) and in this tangent plane we can take a hypersphere of radius 1 and with center (i, p). Let us denote this hypersphere by $(i', S_p(M))$. Then for any point $q \in S_p(M)$ we have just one point (i', q) of R^n. Let O be the origin of R^n and OX be the oriented segment obtained by a parallel translation of oriented segment joining (i, p) to (i', q). Then X is a point of $S_0^{n-1}(1)$. Thus a mapping $s: S(M) \to S_0^{n-1}(1)$ is obtained such that $s(q) = X$ and we call s the spherical representation of (i, M), or the spherical representation of M induced by the immersion i.

In the present paper we consider only such cases that s is an immersion. Then s is called a regular spherical representation or a regular spherical map and its image a spherical image.

We take a compact orientable manifold M and consider the integral I of the volume element of the spherical image $s(S(M))$. I is a functional of the immersion i. The purpose of the present paper is to get some submanifolds (i, M) such that the functional I is stationary at this immersion i with respect to any infinitesimal deformation of i. Our original aim was to find critical points of I in general cases, but the necessary and sufficient condition for (i, M) to be a critical point of I was not obtained in a clear-cut form. Hence only some special cases are treated in the present paper where (i, M) is an isometric and isotropic immersion of a space form. But the final result is still a little complicated. Hence we assume further that the immersion is constant isotropic. The main results are the following theorems.

THEOREM 1. *Let (M, g) be an m-dimensional space form of constant curvature $c > 0$ and (i, M) be a submanifold of R^n such that the immersion is isometric to (M, g) and the normal curvature vector $\sigma_p(t, t)$ has constant length $\sqrt{h}$, $h \neq c$, independent of the tangent vector t and the point p of M. This submanifold is a critical point of the functional I if and only if every component of the mean curvature vector is an eigenfunction of the Laplacian of (M, g) with an eigenvalue λ where $\lambda = ((m + 2)h + 2(m - 1)c)/3$.*

THEOREM 2. *Let (M, g) be as in Theorem 1. Furthermore we assume that the submanifold lies on the hypersphere $S_O^{n-1}(\rho)$ of R^n where the center is the origin O and the radius is ρ. Let* (i) *and* (ii) *be the following conditions,*

(i) *(i, M) is a minimal submanifold of the hypersphere $S_O^{n-1}(\rho)$,*

(ii) *(i, M) is a critical point of I and ρ satisfies*

$$m\rho^{-2} = ((m + 2)h + 2(m - 1)c)/3 .$$

Then (i) *and* (ii) *are equivalent conditions.*

This theorem shows that a Veronese manifold considered as a submanifold of a Euclidean space is a critical point of I.

In §2 we introduce a Riemannian metric to the spherical image $s(S(M))$. From this Riemannian metric we get the formula for the volume element of $s(S(M))$. In §3 the integral I of this volume element and the derivative of I with respect to an infinitesimal deformation of the immersion are calculated. In §4 we consider the special case where (i, M) is isometric to a space form and the immersion is isotropic, namely, $\sigma_p(t, t)$ has constant length $(h(p))^{1/2}$ but $h(p)$ may depend on p. In §5 we consider the case where $h(p)$ is independent of the point p and prove the main theorems. In §6 we prove that a Veronese manifold is a critical point of I. There we also discuss some relation of the present result to some of the results obtained by O'Neill [5] and by Itoh and Ogiue [2], [3].

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2. The Riemannian metric G of a spherical image. We first give a local expression for a spherical map. We use indices

$$a, b, c, \cdots, h, i, j, \cdots = 1, \cdots, m,$$
$$\kappa, \lambda, \mu, \cdots, \rho, \sigma, \tau, \cdots = 1, \cdots, n$$

and adopt usual summation convention with respect to Latin indices. $x^1, \cdots, x^m$ are local coordinates of M so that a point p of M in a coordinate neighborhood is expressed by $p = (x^1, \cdots, x^m)$, and $U^1, \cdots, U^n$ are the rectangular coordinates of a point in R^n. Thus i is expressed locally by

$$U^\kappa = U^\kappa(x^1, \cdots, x^m) . \tag{2.1}$$

We put

$$B_i^\kappa = \partial U^\kappa / \partial x^i = \partial_i U^\kappa , \quad g_{ji} = B_j^\kappa B_i^\kappa \tag{2.2}$$

where the summation symbol $\sum_\kappa$ is omitted for short. g_{ji} are the components of the Riemannian metric induced on the submanifold (i, M) from the natural metric of R^n. Thus we can consider (i, M) as a Riemannian manifold (M, g).

The Christoffel symbols of g_{ji} are denoted by $\left\{ {h \atop j\ i} \right\}$ and the components of the second fundamental form of (i, M) are

$$H_{ji}{}^\kappa = \nabla_j B_i^\kappa = \partial_j B_i^\kappa - \left\{ {h \atop j\ i} \right\} B_h^\kappa \tag{2.3}$$

where ∇ is the Riemannian connection of (M, g).

If t is a unit tangent vector of (M, g) at a point $p \in M$, then $t = t^h \partial_h$ where $(\partial_1, \cdots, \partial_m)$ is the natural frame of $T_p(M)$ and the components t^h satisfy $g_{ji} t^j t^i = 1$. A point q of $S_p(M)$ is nothing but a unit tangent vector of (M, g) at p. If the spherical map s carries q to $s(q) = X$, then the rectangular coordinates X^κ of X are given by

$$X^\kappa = t^i B_i^\kappa , \quad g_{ji} t^j t^i = 1 . \tag{2.4}$$

Since $S_p(M)$ is an $(m-1)$-dimensional sphere, we need $m-1$ numbers $y^1, \cdots, y^{m-1}$ to determine a point of $S_p(M)$ in some open subset. Thus a point X of the spherical image $s(S(M))$, such that $X \in s(U)$ where U is some open subset of $S(M)$, is determined by $2m-1$ numbers

$x^1, \cdots, x^m, y^1, \cdots, y^{m-1}$ and we have n functions $X^\kappa = X^\kappa(x^1, \cdots, x^m; y^1, \cdots, y^{m-1})$.

Now we introduce new indices

$$u, v, w, x, y, z = m + 1, \cdots, 2m - 1 ,$$
$$A, B, C, D, \cdots = 1, \cdots, 2m - 1$$

and put $x^u = y^{u-m}$. A covering of $S(M)$ by suitable neighborhoods U_λ ($\lambda \in \Lambda$) is considered and the spherical image is expressed by

$$X^\kappa = X^\kappa_{(\lambda)}(x^1_{(\lambda)}, \cdots, x^m_{(\lambda)}; y^1_{(\lambda)}, \cdots, y^{m-1}_{(\lambda)})$$

for the part $s(U_\lambda)$. The spherical map s is regular if and only if the rank of the $(n, 2m-1)$-matrix $[\partial X^\kappa_{(\lambda)}/\partial x^A_{(\lambda)}]$ is $2m-1$ for all $\lambda \in \Lambda$. This is assumed throughout the paper.

We define G_{CB} by

$$G_{CB} = \partial_C X^\kappa \partial_B X^\kappa \tag{2.5}$$

where $\partial_C = \partial/\partial x^C$. That s is regular is equivalent to that G_{CB} are the coefficients of a positive quadratic form and our assumption assures that the spherical image becomes a Riemannian manifold with the Riemannian metric G whose components are G_{CB}. As we have

$$\partial_j X^\kappa = t^i H_{ji}{}^\kappa + \nabla_j t^i B_i^\kappa , \quad \partial_u X^\kappa = \partial_u t^i B_i^\kappa , \tag{2.6}$$

we get

$$\begin{aligned} G_{ji} &= H_{jc}{}^\kappa H_{ib}{}^\kappa t^c t^b + g_{cb} \nabla_j t^c \nabla_i t^b , \\ G_{ju} &= g_{cb} \nabla_j t^c \partial_u t^b , \\ G_{vu} &= g_{cb} \partial_v t^c \partial_u t^b . \end{aligned} \tag{2.7}$$

DEFINITION. We define D_{ji}, γ_{vu}, $u_i{}^u$ by

$$D_{ji} = H_{jc}{}^\kappa H_{ib}{}^\kappa t^c t^b , \quad \gamma_{vu} = g_{cb} \partial_v t^c \partial_u t^b , \tag{2.8}$$

$$\nabla_i t^h = u_i{}^v \partial_v t^h . \tag{2.9}$$

We prove that $u_i{}^u$ are uniquely determined by (2.9). As the vector field t satisfies $g_{ji} t^j t^i = 1$, we get

$$t_i \nabla_k t^i = 0 , \quad t_i \partial_u t^i = 0$$

where $t_i = g_{ij} t^j$. As the rank of the $(m, m-1)$-matrix $[\partial_u t^i]$ is $m-1$, there exists one and only one $(m-1, m)$-matrix $[u_i{}^u]$ satisfying (2.9).

As s is regular, $\operatorname{rank}[D_{ji}] = m$ and (2.8) shows that D_{ji} are the coefficients of a positive quadratic form. We get from (2.7), (2.8) and (2.9)

(2.10) $$G_{ji} = D_{ji} + \gamma_{vu}u_j{}^v u_i{}^u\ , \quad G_{ju} = \gamma_{vu}u_j{}^v\ , \quad G_{vu} = \gamma_{vu}\ .$$

This implies

$$G_{CB}P^C P^B = D_{ji}P^j P^i + \gamma_{vu}(u_j{}^v P^j + P^v)(u_i{}^u P^i + P^u)\ .$$

REMARK. We denote the normal curvature vector of (i, M) at (i, p) by $\sigma_p(t, t)$ where t is a unit tangent vector. The components of $\sigma_p(t, t)$ are $H_{ji}{}^\kappa t^j t^i$. The normal curvature vector at (i, p) associated with a pair of unit tangent vectors u and v is denoted by $\sigma_p(u, v)$. Its components are $H_{ji}{}^\kappa u^j v^i$. Suppose that $\sigma_p(u, v) = 0$ for some p, u and v. As we can choose $(y^1, \cdots, y^{m-1})$ in such a way that $t(x^1, \cdots, x^m; y^1, \cdots, y^{m-1}) = v$, we get $H_{ji}{}^\kappa u^j t^i = 0$ and consequently $D_{ji}u^j u^i = 0$ for this $(y^1, \cdots, y^{m-1})$. This proves that $\|\sigma_p(u, v)\| > 0$ for every p, u and v.

DEFINITION. We define D^{ji} and γ^{vu} by

(2.11) $$D_{bj}D^{bi} = \delta_j^i\ , \quad \gamma_{xv}\gamma^{xu} = \delta_v^u\ .$$

Then the contravariant components of the Riemannian metric G of $s(S(M))$ are

(2.12) $$G^{ji} = D^{ji}\ , \quad G^{vi} = -u_c{}^v D^{ci}\ , \quad G^{vu} = D^{cb}u_c{}^v u_b{}^u + \gamma^{vu}\ .$$

From (2.10) we get

(2.13) $$\det[G_{BA}] = (\det[D_{ji}])(\det[\gamma_{vu}])\ ,$$

or, in short, $\det G = (\det D)(\det \gamma)$.

3. The functional I and its derivative. As the regular spherical image $s(S(M))$ is endowed with the Riemannian metric G, we can consider its volume element. Dividing $S(M)$ into a number of parts $S(M)_\lambda$, $\lambda \in \Lambda$, so that each part is contained in some coordinate neighborhood of $S(M)$, we can express the volume element in the form

$$((\det D)(\det \gamma))^{1/2}dx^1 \cdots dx^m dy^1 \cdots dy^{m-1}\ ,$$

or in the form $((\det D)(\det \gamma))^{1/2}dxdy$, for short. We define I by

$$I = \sum_\lambda I_\lambda\ , \qquad I_\lambda = \iint_{S(M)_\lambda} ((\det D)(\det \gamma))^{1/2}dxdy$$

which we write, for convenience, as

(3.1) $$I = \iint_{S(M)} ((\det D)(\det \gamma))^{1/2}dxdy\ .$$

I is a functional of immersion i.

Let us consider an infinitesimal deformation of i.

If the immersion i of M into R^n depends on a parameter α, the position vector of (i, p), $p \in M$, is written locally as

$$U^\kappa = U^\kappa(x^1, \cdots, x^m; \alpha) .$$

We consider only the case where U^κ are C^∞ functions of $x^1, \cdots, x^m$ and α. As the tangent vector $t = t^h \partial_h$ also depends on α we have in general

$$t^h = t^h(x^1, \cdots, x^m, y^1, \cdots, y^{m-1}; \alpha)$$

in each suitable coordinate neighborhood. But we can consider without loss of generality that, at each point $p \in M$, the ratio $t^1 : t^2 : \cdots : t^m$ does not depend on α. Thus there exists a function φ satisfying $\partial t^h/\partial\alpha = \varphi t^h$. As t is a unit tangent vector, we get

$$\varphi = -2^{-1}(\partial g_{ji}/\partial\alpha)t^j t^i . \tag{3.2}$$

DEFINITION. We define the vector field V of deformation as the vector field whose components are given by $V^\kappa = \partial U^\kappa/\partial\alpha$.

Then we have $\partial(\partial_i U^\kappa)/\partial\alpha = \partial_i V^\kappa$ and

$$\partial g_{ji}/\partial\alpha = \partial_j V^\kappa B_i^\kappa + B_j^\kappa \partial_i V^\kappa . \tag{3.3}$$

From (3.2) we get

$$\varphi = -t^j \partial_j V^\kappa t^i B_i^\kappa , \tag{3.4}$$

$$\partial t^h/\partial\alpha = -t^j \partial_j V^\kappa t^i B_i^\kappa t^h . \tag{3.5}$$

As we have the general formula

$$\partial \left\{ {h \atop j\ i} \right\} \Big/ \partial\alpha = (1/2) g^{ha}[\nabla_j(\partial g_{ia}/\partial\alpha) + \nabla_i(\partial g_{ja}/\partial\alpha) - \nabla_a(\partial g_{ji}/\partial\alpha)] ,$$

we get, by substituting (3.3) into the second member,

$$\partial \left\{ {h \atop j\ i} \right\} \Big/ \partial\alpha = g^{ha}(\nabla_j \nabla_i V^\kappa B_a^\kappa + \partial_a V^\kappa H_{ji}{}^\kappa) . \tag{3.6}$$

For the second fundamental form we have

$$\partial H_{ji}{}^\kappa/\partial\alpha = \nabla_j \nabla_i V^\kappa - g^{cb}(\nabla_j \nabla_i V^\lambda B_c^\lambda + \partial_c V^\lambda H_{ji}{}^\lambda) B_b^\kappa . \tag{3.7}$$

As V^κ and U^κ are independent of $y^1, \cdots, y^{m-1}$, we get from (3.5)

$$\partial(\partial_u t^h)/\partial\alpha = \partial_u(\partial t^h/\partial\alpha) = -(t^j \partial_j V^\kappa t^i B_i^\kappa)\partial_u t^h - (\partial_j V^\kappa B_i^\kappa)\partial_u(t^j t^i) t^h , \tag{3.8}$$

$$\partial \gamma_{vu}/\partial\alpha = (\partial_c V^\kappa B_b^\kappa + \partial_b V^\kappa B_c^\kappa)\partial_v t^c \partial_u t^b - 2\gamma_{vu}\partial_c V^\kappa B_b^\kappa t^c t^b . \tag{3.9}$$

From (3.5) and (3.7) we get

$$\partial D_{ji}/\partial\alpha = 2\varphi D_{ji} + (\nabla_j \nabla_c V^\kappa H_{ib}{}^\kappa + \nabla_i \nabla_c V^\kappa H_{jb}{}^\kappa) t^c t^b . \tag{3.10}$$

From (2.13) we get

$$\partial(\det G)^{1/2}/\partial\alpha = (1/2)(D^{ji}\partial D_{ji}/\partial\alpha + \gamma^{vu}\partial\gamma_{vu}/\partial\alpha)(\det G)^{1/2} .$$

Now we have

$$(1/2)(D^{ji}\partial D_{ji}/\partial\alpha + \gamma^{vu}\partial\gamma_{vu}/\partial\alpha) = D^{ji}\nabla_j\nabla_c V^\kappa H_{ib}{}^\kappa t^c t^b + m\varphi + \gamma^{vu}\partial_v t^j\partial_u t^i\nabla_j V^\kappa B_i^\kappa - (m-1)\nabla_j V^\kappa B_i^\kappa t^j t^i$$

in view of (3.9), (3.10) and $D^{ji}D_{ji} = m$, $\gamma^{vu}\gamma_{vu} = m-1$. On the other hand we have

$$\gamma^{vu}\partial_v t^j\partial_u t^i = g^{ji} - t^j t^i$$

from

$$(\gamma^{vu}\partial_v t^j\partial_u t^i - g^{ji} + t^j t^i)g_{ai}t^a = 0 ,$$
$$(\gamma^{vu}\partial_v t^j\partial_u t^i - g^{ji} + t^j t^i)g_{ai}\partial_x t^a = 0 .$$

Thus we get

$$(3.11)\qquad (1/2)(D^{ji}\partial D_{ji}/\partial\alpha + \gamma^{vu}\partial\gamma_{vu}/\partial\alpha) = D^{ji}\nabla_j\nabla_c V^\kappa H_{ib}{}^\kappa t^c t^b + g^{ji}\nabla_j V^\kappa B_i^\kappa - 2m\nabla_j V^\kappa B_i^\kappa t^j t^i .$$

Substituting this result into

$$\frac{dI}{d\alpha} = \iint_{S(M)} \frac{\partial(\det G)^{1/2}}{\partial\alpha} dx dy = \int_M \left[\int_{S_p(M)} \frac{\partial(\det G)^{1/2}}{\partial\alpha} dy\right] dx ,$$

we get

$$(3.12)\quad \frac{dI}{d\alpha} = \int_M \left[\int_{S_p(M)} (D^{ji}\nabla_j\nabla_c V^\kappa H_{ib}{}^\kappa t^c t^b + g^{ji}\nabla_j V^\kappa B_i^\kappa - 2m\nabla_j V^\kappa B_i^\kappa t^j t^i)(\det\gamma)^{1/2} dy\right](\det D)^{1/2} dx .$$

4. The differential coefficient of I in some special cases. Assume M is compact orientable. That the submanifold (i, M) is a critical point of I means that for any infinitesimal deformation from (i, M) the second member of (3.12) vanishes. The vector field V of deformation is defined on M but the domain of integration in (3.12) is $S(M)$. In order to get a clear-cut formula for a critical point we must first compute the integral over each $S_p(M)$, but as D^{ji} are not polynomials in $t^1, \cdots, t^m$ in general, the computation is practically difficult. Thus we consider only some special cases satisfying the following:

ASSUMPTION. *(i, M) is an isometric and isotropic immersion of a space form of constant curvature $c > 0$.*

Then we have

$$H_{kh}{}^{\kappa}H_{ji}{}^{\kappa} - H_{jh}{}^{\kappa}H_{ki}{}^{\kappa} = c(g_{kh}g_{ji} - g_{jh}g_{ki}) , \tag{4.1}$$

$$H_{kj}{}^{\kappa}H_{ih}{}^{\kappa} + H_{ki}{}^{\kappa}H_{jh}{}^{\kappa} + H_{kh}{}^{\kappa}H_{ji}{}^{\kappa} = h(g_{kj}g_{ih} + g_{ki}g_{jh} + g_{kh}g_{ji}) \tag{4.2}$$

where h is a function on M.

From (4.1) and (4.2) we get

$$H_{kj}{}^{\kappa}H_{ih}{}^{\kappa} = (1/3)((h + 2c)g_{kj}g_{ih} + (h - c)(g_{ki}g_{jh} + g_{kh}g_{ji})) \tag{4.3}$$

and from (2.8)

$$D_{ji} = (1/3)((h - c)g_{ji} + (2h + c)t_j t_i) , \tag{4.4}$$

$$D^{ji} = \frac{3}{h - c}g^{ji} - \frac{2h + c}{h(h - c)}t^j t^i , \tag{4.5}$$

$$\det D = ((h - c)/3)^{m-1}h \det g . \tag{4.6}$$

As we have assumed that the spherical map s is regular, $h - c > 0$ everywhere on M.

Now $d\omega = (\det \gamma)^{1/2}dy^1 \cdots dy^{m-1}$ is the volume element of the sphere $S_p(M)$ which is isometric to the standard $(m - 1)$-sphere $S^{m-1}(1)$. Hence we have at p

$$\int_{S_p(M)} t^j t^i d\omega = \frac{1}{m}c_{m-1}g^{ji} \tag{4.7}$$

where c_{m-1} is the volume of $S^{m-1}(1)$.

Let us consider $S^{m-1}(1)$ as the unit hypersphere of R^m given by $(u^1)^2 + \cdots + (u^m)^2 = 1$ where $u^1, \cdots, u^m$ are the rectangular coordinates of R^m. Then we get

$$\int u^k u^j u^i u^h d\omega = (c_{m-1}/(m(m + 2)))(\delta^{kj}\delta^{ih} + \delta^{ki}\delta^{jh} + \delta^{kh}\delta^{ji})$$

where the domain of integration is $S^{m-1}(1)$. Applying this result to $S_p(M)$ we get

$$\int t^k t^j t^i t^h d\omega = (c_{m-1}/(m(m + 2)))(g^{kj}g^{ih} + g^{ki}g^{jh} + g^{kh}g^{ji}) . \tag{4.8}$$

From (4.5), (4.7) and (4.8) we get

$$\int D^{ji}t^c t^b (\det \gamma)^{1/2}dy$$
$$= \left[\frac{3}{m(h - c)}g^{ji}g^{cb} - \frac{2h + c}{m(m + 2)h(h - c)}(g^{ji}g^{cb} + g^{jc}g^{ib} + g^{jb}g^{ic})\right]c_{m-1} .$$

Then, as $\nabla_j\nabla_c V^{\kappa}$, $H_{ib}{}^{\kappa}$, $\nabla_j V^{\kappa}$, B_i^{κ} are independent of the unit tangent vector

t, we get from (3.12)

(4.9) $$\frac{dI}{d\alpha} = c_{m-1}\int_M\Big[\Big(\frac{3}{m(h-c)} - \frac{2(2h+c)}{m(m+2)h(h-c)}\Big)\nabla_j\nabla_i V^\kappa H^{ji\kappa} - \frac{2h+c}{m(m+2)h(h-c)}\nabla_j\nabla^j V^\kappa H_i^{\ j\kappa} - g^{ji}\nabla_j V^\kappa B_i^\kappa\Big] \times((h-c)/3)^{(m-1)/2}(h \det g)^{1/2}dx\ .$$

5. Some critical points of the functional I. Hereafter we assume h is constant. This means that the normal curvature vector $\sigma_p(t, t)$ of (i, M) has constant length $\sqrt{h}$ independent of p and t. In this case $dI/d\alpha$ vanishes for every infinitesimal deformation if and only if the following equation is satisfied,

(5.1) $$\Big(\frac{3}{m(h-c)} - \frac{4h+2c}{m(m+2)h(h-c)}\Big)\nabla_j\nabla_i H^{ji\kappa} - \frac{2h+c}{m(m+2)h(h-c)}\nabla_j\nabla^j H_i^{\ i\kappa} + H_i^{\ i\kappa} = 0\ .$$

This is a direct consequence of Green's theorem. On the other hand we have

$$\nabla_j\nabla_i H^{ji\kappa} = \nabla_j\nabla^j H_i^{\ i\kappa} + \nabla_j(K^{jk}B_k^\kappa) = \nabla_j\nabla^j H_i^{\ i\kappa} + (m-1)cH_i^{\ i\kappa}\ ,$$

where K^{jk} are the contravariant components of the Ricci tensor. Hence (5.1) becomes

(5.2) $$(mh-c)[3\Delta H^\kappa - ((m+2)h + 2(m-1)c)H^\kappa] = 0$$

where Δ is the Laplacian, $\Delta = -\nabla_i\nabla^i$, and H^κ are the components of the mean curvature vector defined by $mH^\kappa = H_i^{\ i\kappa}$. As we have $h - c > 0$, the case $mh - c = 0$ is excluded. Hence we get from (5.2)

(5.3) $$\Delta H^\kappa = \lambda H^\kappa$$

where

(5.4) $$\lambda = ((m+2)h + 2(m-1)c)/3\ .$$

Thus we have proved Theorem 1.

Now suppose that (i, M) lies on the hypersphere $S_0^{n-1}(\rho)$, namely the hypersphere of radius ρ and with center at the origin of R^n. Then we have $U^\kappa U^\kappa = \rho^2$, $U^\kappa B_i^\kappa = 0$, $g_{ji} + U^\kappa H_{ji}^{\ \ \kappa} = 0$, hence

(5.5) $$U^\kappa H^\kappa = -1\ .$$

If (i, M) is a minimal submanifold of $S_0^{n-1}(\rho)$, then we get

(5.6) $$mH^\kappa = -\Delta U^\kappa = -m\rho^{-2}U^\kappa$$

as in [6]. On the other hand we have from (4.3)

$$H^{\kappa}H^{\kappa} = ((m+2)h + 2(m-1)c)/(3m)\,. \tag{5.7}$$

Hence we get

$$m\rho^{-2} = ((m+2)h + 2(m-1)c)/3 \tag{5.8}$$

which proves that $\Delta H^{\kappa} = \lambda H^{\kappa}$ holds with λ satisfying (5.4). Thus (i, M) is a critical point of I.

Conversely, suppose (i, M) is a critical point of I and ρ satisfies (5.8). Then we get, in view of (5.5),

$$U^{\kappa}(mH^{\kappa} + \lambda U^{\kappa}) = -m + \lambda\rho^2$$

which vanishes because of (5.4) and (5.8). On the other hand we have

$$\begin{aligned}\int_M (\Delta U^{\kappa} - \lambda U^{\kappa})(\Delta U^{\kappa} - \lambda U^{\kappa})d\omega &= \int_M U^{\kappa}(\Delta\Delta U^{\kappa} - 2\lambda\Delta U^{\kappa} + \lambda^2 U^{\kappa})d\omega \\ &= \lambda\int_M U^{\kappa}(mH^{k} + \lambda U^{\kappa})d\omega\,,\end{aligned}$$

hence $\Delta U^{\kappa} - \lambda U^{\kappa} = 0$. Thus we have proved Theorem 2.

6. A space form immersed isometrically as an isotropic submanifold in a hypersphere of R^n.

REMARK. In §6 an immersed submanifold is denoted by M. The notation (i, M) is not used.

In a paper of O'Neill [5] it is stated that, if M is an m-dimensional space form of constant curvature c and at the same time M is an isotropic submanifold of an $(m + m(m+1)/2 - 1)$-dimensional space form $\tilde{M}$ of constant curvature $\tilde{c}$, with $c < \tilde{c}$, then M is a minimal submanifold of M and $\|\sigma(t, t)\|^2 = (2(m-1)/(m+2))(\tilde{c} - c)$. On the other hand we find in a paper [2] by Itoh and Ogiue the following theorems.

THEOREM A. *Let M be an m-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(m + m(m+1)/2 - 1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$ with parallel second fundamental form, then $c = (m/2(m+1))\tilde{c}$, and the immersion is rigid.*

THEOREM B. *Let M be an m-dimensional space form of constant curvature c, and $\tilde{M}$ be an $(m+m(m+1)/2-1)$-dimensional space form of constant curvature $\tilde{c}$. If $c < \tilde{c}$, and M is an isotropic submanifold of $\tilde{M}$, then $c = (m/2(m+1))\tilde{c}$, and the immersion is rigid provided that $m \leqq 4$.*

It seems that such results have some relation to some of the results of the present paper. In the present paper the dimension n of the ambient space is undecided since the immersion may not be full.

As we are considering the case where the immersed submanifold M lies on $S_0^{n-1}(\rho)$, we express the latter locally by

$$U^\kappa = U^\kappa(u^1, \cdots, u^{n-1})$$

where $u^1, \cdots, u^{n-1}$ are the local coordinates of $S_0^{n-1}(\rho)$. We use indices

$$\alpha, \beta, \gamma, \delta = 1, \cdots, n-1$$

and the immersion of M into $S_0^{n-1}(\rho)$ is given locally by

$$u^\alpha = u^\alpha(x^1, \cdots, x^m) .$$

We also use the notations,

$$B_\alpha^\kappa = \partial U^\kappa/\partial u^\alpha , \quad B_i^\alpha = \partial u^\alpha/\partial x^i ,$$

and get

$$\partial U^\kappa/\partial x^i = B_i^\kappa = B_\alpha^\kappa B_i^\alpha .$$

Then the natural Riemannian metric on $S_0^{n-1}(\rho)$ has components $g_{\beta\alpha}$ such that

$$g_{ji} = B_j^\kappa B_i^\kappa = g_{\beta\alpha} B_j^\beta B_i^\alpha , \quad g_{\beta\alpha} = B_\beta^\kappa B_\alpha^\kappa$$

and the components $H_{\beta\alpha}{}^\kappa$ of the second fundamental form of $S_0^{n-1}(\rho)$ in R^n and the components $H_{ji}{}^\alpha$ of the second fundamental form of M in $S_0^{n-1}(\rho)$ satisfy [1]

$$\begin{aligned} H_{\beta\alpha}{}^\kappa &= -\rho^{-2} g_{\beta\alpha} U^\kappa , \\ K_{\delta\gamma\beta\alpha} &= H_{\delta\alpha}{}^\kappa H_{\gamma\beta}{}^\kappa - H_{\gamma\alpha}{}^\kappa H_{\delta\beta}{}^\kappa = \tilde{c}(g_{\delta\alpha} g_{\gamma\beta} - g_{\gamma\alpha} g_{\delta\beta}) , \\ H_{ji}{}^\kappa &= H_{ji}{}^\alpha B_\alpha^\kappa + B_j^\beta B_i^\alpha H_{\beta\alpha}{}^\kappa = H_{ji}{}^\alpha B_\alpha^\kappa - \rho^{-2} g_{ji} U^\kappa \end{aligned}$$

where $K_{\delta\gamma\beta\alpha}$ are the covariant components of the curvature tensor of $S_0^{n-1}(\rho)$ and $\tilde{c} = \rho^{-2}$. Thus we get

$$H_{kj}{}^\kappa H_{ih}{}^\kappa = H_{kj}{}^\beta H_{ih}{}^\alpha g_{\beta\alpha} + \tilde{c} g_{kj} g_{ih} . \tag{6.1}$$

This shows that M is isotropic in R^n if and only if M is isotropic in $S_0^{n-1}(\rho)$. If we denote by $\sigma_R(t, t)$ the normal curvature vector of M in R^n and by $\sigma_S(t, t)$ the normal curvature vector of M in $S_0^{n-1}(\rho)$, then we get from (6.1)

$$\|\sigma_S(t, t)\|^2 = \|\sigma_R(t, t)\|^2 - \tilde{c} = h - \tilde{c} . \tag{6.2}$$

On the other hand we get from (5.7), where we now put $\rho^{-2} = \tilde{c}$,

$$h = (3m\tilde{c} - 2(m-1)c)/(m+2) . \tag{6.3}$$

Hence we have the formula

$$\|\sigma_S(t,t)\|^2 = (2(m-1)/(m+2))(\tilde{c}-c) \tag{6.4}$$

which has been obtained by O'Neill [5].

As we can see in [2], [3] and [4], a Veronese manifold satisfies the equations

$$c = 1\,, \quad c = (m/2(m+1))\tilde{c}\,, \quad \tilde{c} = \rho^{-2} = 2(m+1)/m\,, \quad \lambda = \lambda_2 = 2(m+1)\,.$$

Since a Veronese manifold is an isotropic submanifold (see [2]), we get $h=4$ from (6.2) and (6.4), which is valid as a result of O'Neill's paper [5]. Hence (5.8) is satisfied and a Veronese manifold is a critical point of I.

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