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REMARKS ON CONFORMALITY AT THE BOUNDARY

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1. Let f be a regular and univalent function on the strip domain $S = \{z \mid |\operatorname{Im} z| < \pi/2\}$ such that

$$\lim_{R \ni x \to +\infty} \operatorname{Re} f(x) = +\infty . \tag{1}$$

If the finite real value

$$\lim_{\substack{z \to +\infty \\ \text{Stolz}}} (z - f(z)) = \alpha \tag{2}$$

exists, then we say that f is *conformal at* $+\infty$ and α is the *angular derivative* of f at $+\infty$. Here the notation means that z tends to $+\infty$ within Stolz domains; more precisely, the left-hand side of (2) is, if $z = x + iy$, the uniform limit of $(z - f(z))$ as $x \to +\infty$ for $|y| < \pi/2 - \delta$, being independent of δ $(0 < \delta < \pi/2)$.

Denote by D the image domain: $D = f(S)$. It is a well-known fact that the conformality of f at $+\infty$ implies the following:

(3) For every δ $(0 < \delta < \pi/2)$ there exists u_0 such that $\{w \mid u_0 < \operatorname{Re} w, |\operatorname{Im} w| < \pi/2 - \delta\} \subset D$. The accessible boundary point of D over $w = \infty$ determined by the positive real-axis corresponds under f to the boundary point $+\infty$ of S, namely the accessible boundary point of S over $z = \infty$ determined by the positive real-axis.

We are interested in finding a condition for D so that f with (1) and (3) be conformal at $+\infty$.

Several years ago Rodin and Warschawski [4] and Jenkins and Oikawa [2] independently discovered a necessary and sufficient condition expressed by module (= reciprocal of extremal length) of families of curves. In terms of elementary quantities, necessary and sufficient conditions have not been obtained yet. Many necessary conditions and sufficient conditions are known.

2. If D is of special kind, necessary and sufficient conditions in elementary terms are available.

For the case where $D \supset S$, Ferrand [1] and Rodin and Warschawski [5] have obtained a number of criteria of this nature. To state their result, given a real value a, consider the connected component of the set $\{w \mid \operatorname{Re} w = a\} \cap D$ which meets S, and denote its upper and lower end points by $a + iv''(a)$ and $a + iv'(a)$, respectively. Next, given a sequence $u_n \uparrow \infty$, consider the quantities

$$v_n'' = \min\{v''(a) \mid u_n \leqq a \leqq u_{n+1}\}, \qquad v_n' = \max\{v'(a) \mid u_n \leqq a \leqq u_{n+1}\}.$$

FERRAND-RODIN-WARSCHAWSKI CRITERIA. *If $D \supset S$ the following conditions are equivalent:*

(i) *A regular and univalent function f on S satisfying* (1), (3), *and $f(S) = D$ is conformal at $+\infty$;*

(ii) *If R_n, $n = 1, 2, \cdots$, are mutually disjoint similar rectangles in $D - S$ each of which has one side on ∂S, then the sum of their areas is finite;*

(iii) *There exists a sequence $u_n \uparrow \infty$ such that*

$$\sum (u_{n+1} - u_n)^2 < \infty, \qquad \sum (v_n'' - \pi/2)^2 < \infty, \qquad \sum (v_n' + \pi/2)^2 < \infty;$$

(iv) *There exists a sequence $u_n \uparrow \infty$ such that*

$$\sum (u_{n+1} - u_n)^2 < \infty,$$
$$\sum (v_n'' - \pi/2)(u_{n+1} - u_n) < \infty, \qquad \sum -(v_n' + \pi/2)(u_{n+1} - u_n) < \infty.$$

The condition (ii) is due to Rodin and Warschawski [5]. The conditions (iii) and (iv) are due to Ferrand [1] and verified in a different method by Rodin and Warschawski [5].

3. The purpose of the present paper is to add one more to this list by introducing the quantities

$$\delta^+(u) = \operatorname{dist}(u + i\pi/2, \partial D) \quad \text{and} \quad \delta^-(u) = \operatorname{dist}(u - i\pi/2, \partial D)$$

for the real value u.

THEOREM 1. *In the case where $D \supset S$, a regular and univalent function f on S satisfying* (1), (3), *and $f(S) = D$ is conformal at $+\infty$ if and only if*

$$\text{(v)} \qquad \int^\infty \delta^+(u)du < \infty \quad \text{and} \quad \int^\infty \delta^-(u)du < \infty.$$

To prove this theorem we need certain generalization of a part of Ferrand-Rodin-Warschawski Criteria. Instead of a single sequence $\{u_n\}$ we consider two $\{u_n^+\}$ and $\{u_n^-\}$, and

$$v_n^+ = \min\{v''(a) \mid u_n^+ \leqq a \leqq u_{n+1}^+\}, \qquad v_n^- = \max\{v'(a) \mid u_n^- \leqq a \leqq u_{n+1}^-\}.$$

THEOREM 2. *If $D \supset S$ the following conditions are equivalent:*

(i) *The same as in Ferrand-Rodin-Warschawski criteria;*
(iii′) *There exist sequences $u_n^+ \uparrow \infty$ and $u_n^- \uparrow \infty$ such that*

$$\sum (u_{n+1}^+ - u_n^+)^2 < \infty , \qquad \sum (u_{n+1}^- - u_n^-)^2 < \infty ,$$
$$\sum (v_n^+ - \pi/2)^2 < \infty , \qquad \sum (v_n^- + \pi/2)^2 < \infty ;$$

(iv′) *There exist sequences $u_n^+ \uparrow \infty$ and $u_n^- \uparrow \infty$ such that*

$$\sum (u_{n+1}^+ - u_n^+)^2 < \infty , \qquad \sum (u_{n+1}^- - u_n^-)^2 < \infty ,$$
$$\sum (v_n^+ - \pi/2)(u_{n+1}^+ - u_n^+) < \infty , \qquad \sum -(v_n^- + \pi/2)(u_{n+1}^- - u_n^-) < \infty .$$

We shall prove Theorems 1 and 2 by verifying (iv′) $\Rightarrow$ (v) $\Rightarrow$ (iii′) $\Rightarrow$ (iii), for (iv) $\Rightarrow$ (iv′) is trivial.

It would be interesting to compare the condition (v) with Ostrowski's semi-conformality condition. A regular and univalent function f on S with (1) is said to be *semi-conformal* at $+\infty$ if

$$\lim_{\substack{z \to +\infty \\ \text{Stolz}}} \operatorname{Im}(z - f(z)) = 0 .$$

The necessary and sufficient condition for semi-conformality obtained by Ostrowski [3] (see also Warschawski [6, Theorem 1a, 1b], Jenkins and Oikawa [2, Theorem 3]) is, if $D \supset S$, readily seen to be equivalent to the following: *A regular and univalent function S with* (1), (3), *and* $f(S) = D \supset S$ *is semi-conformal if and only if* $\lim_{u \to +\infty} \delta^+(u) = \lim_{u \to +\infty} \delta^-(u) = 0$.

4. **Proof of** (iv′) $\Rightarrow$ (v). For $u_n^+ \leqq u \leqq u_{n+1}^+$,

$$\delta^+(u) \leqq (u_{n+1}^+ - u_n^+) + (v_n^+ - \pi/2) .$$

Accordingly

$$\int^\infty \delta^+(u)du \leqq \sum \{(u_{n+1}^+ - u_n^+) + (v_n^+ - \pi/2)\}(u_{n+1}^+ - u_n^+) < \infty ,$$

and, similarly, $\int^\infty \delta^-(u)du < \infty$.

5. As a preparation for the proof of (v) $\Rightarrow$ (iii′), we list some properties of the function δ^+ for the case $D \supset S$.

(a) $D \supset \{u + iv \mid 0 \leqq v < \pi/2 + \delta^+(u)\}$.

(b) $\delta^+(a) = 0$ if and only if $a + i\pi/2 \in \partial D$.

(c) If $\delta^+(a) > 0$, then D contains the open disk with center at $a + i\pi/2$ and radius $\delta^+(a)$, a point on the circumference of which is on ∂D.

(d) $|\delta^+(u) - \delta^+(u')| \leqq |u - u'|$ for every u, u'.

(e) If $\delta^+(a) > 0$, then for $I = [a - \delta^+(a), a + \delta^+(a)]$

$$|I|^2/4 \leqq \int_I \delta^+(u)du \, .$$

Here $|I|$ stands for the length of I. The proofs of these are easy and may be omitted. Needless to say, δ^- has the corresponding properties.

6. We now suppose that the condition (v) is satisfied, and construct inductively a sequence $u_n^+ \uparrow \infty$. First set $u_1^+ = 0$.

If $\delta^+(u_1^+) > 0$, consider a point of intersection of the graph of $v = \pi/2 + \delta^+(u)$ with the ray $(u_1^+ + t) + i(\pi/2 + t)$, $0 \leqq t < \infty$. Such a point exists because, by the property (d), the assumption (v) implies $\lim_{u\to+\infty} \delta^+(u) = 0$. Denote by $u_1^* + i(\pi/2 + \delta^+(u_1^*))$ the point of intersection nearest to the point $u_1^+ + i\pi/2$, and set $u_2^+ = u_1^+ + 2(u_1^* - u_1^+)$. Observe that

$$\delta^+((u_1^+ + u_2^+)/2) = (u_2^+ - u_1^+)/2$$

and, therefore, the isosceles triangle with vertices $u_1^+ + i\pi/2$, $u_2^+ + i\pi/2$, and $(u_1^+ + u_2^+)/2 + i(\pi/2 + \delta^+((u_1^+ + u_2^+)/2))$ is contained in the domain D.

If $\delta^+(u_1^+) = 0$, then we set $u_2^+ = u_1^+ + 1$ and call $[u_1^+, u_2^+]$ the first *zero interval*. The name suggests that δ^+ vanishes at a point of the interval.

Suppose $u_1^+, \cdots, u_n^+$ have been obtained. If $\delta^+(u_n^+) > 0$, then we define u_{n+1}^+ exactly in the same way as above, so that

$$\delta^+((u_n^+ + u_{n+1}^+)/2) = (u_{n+1}^+ - u_n^+)/2 \, .$$

If $\delta^+(u_n^+) = 0$, on denoting by q the number of zero intervals among $[u_1^+, u_2^+], \cdots, [u_{n-1}^+, u_n^+]$, we set

$$u_{n+1}^+ = u_n^+ + (q+1)^{-1}$$

and call $[u_n^+, u_{n+1}^+]$ the $(q+1)$-st zero interval.

If $\lim_{n\to\infty} u_n^+ = +\infty$ then we are through, and we go to the next section.

In the case where $\lim_{n\to\infty} u_n^+ = r_1 < \infty$, we have to reconstruct the sequence. If this is the case, the total number of zero intervals must be finite, which we denote by p_1. Observe that

$$\delta^+(r_1) = \lim_{n\to\infty} \delta^+((u_n^+ + u_{n+1}^+)/2) = \lim_{n\to\infty} (u_{n+1}^+ - u_n^+)/2 = 0 \, .$$

Now, take a number $n(1)$ such that $r_1 - (1 + p_1)^{-1} < u_{n(1)-1}$ and that there is no zero interval contained in $[u_{n(1)}^+, r_1)$. Eliminate u_k^+, $k \geqq n(1)$ obtained so far. Then set newly

$$u_{n(1)}^+ = u_{n(1)-1}^+ + (p_1 + 1)^{-1}$$

and call $[u^+_{n(1)-1}, u^+_{n(1)}]$ the (p_1+1)-st zero interval. Note that it contains the point r_1, where δ^+ vanishes. From this new $u^+_{n(1)}$ we make a fresh start as we did at the beginning of this section, and construct $u^+_{n(1)+1}$, $u^+_{n(1)+2}, \cdots$.

If $\lim u^+_n = +\infty$, we are through.

If $\lim u^+_n = r_2 < +\infty$, we do a similar thing as before. Namely, on letting $p_2\ (> p_1)$ be the total number of zero intervals, we eliminate u^+_k, $k \geqq n(2)\ (> n(1))$ obtained so far, set newly

$$u^+_{n(2)} = u^+_{n(2)-1} + (p_2+1)^{-1}$$

so that

$$r_2 \in [u^+_{n(2)-1}, u^+_{n(2)}] \quad \text{and} \quad \delta^+(r_2) = 0\ ,$$

and call $[u^+_{n(2)-1}, u^+_{n(2)}]$ the (p_2+1)-st zero interval.

We repeat this reconstruction if need be.

If we get $\lim u^+_n = +\infty$ after a finite number of steps, we are through.

If we get $\lim u^+_n = r_k < +\infty$ after every k-th step, then $p_1 < p_2 < \cdots \uparrow \infty$ so that

$$u^+_{n(k)} \geqq 1 + \frac{1}{2} + \cdots + \frac{1}{p_k+1} \to \infty \qquad (k \to \infty)\ .$$

Accordingly the totality of $\{u^+_1, \cdots, u^+_{n(1)-1}\}$ and $\{u^+_{n(k)}, u^+_{n(k)+1}, \cdots, u^+_{n(k+1)-1}\}$, $k = 1, 2, \cdots$, constitutes a sequence $u^+_n \uparrow \infty$.

7. **Proof of** (v) $\Rightarrow$ (iii$'$). It suffices to verify that the sequence $\{u^+_n\}$ constructed above has the property stated in (iii$'$); in completely the same way we obtain a similar sequence $\{u^-_n\}$ on using the function δ^-. We set

$$\sum_{n=1}^{\infty} = \sum{}' + \sum{}''$$

where $\sum'$ is the sum over the n such that $[u^+_n, u^+_{n+1}]$ is a zero interval and $\sum''$ is the one over the rest.

If $I^+_n = [u^+_n, u^+_{n+1}]$ is a zero interval, then δ^+ vanishes at a point of I^+_n, so that $v^+_n - \pi/2 = 0$. If it is the l-th zero interval, then $|I^+_n| = 1/l$ accordingly $\sum' |I^+_n|^2 < \infty$.

If I^+_n is not a zero interval, then the value of δ^+ at the middle point is equal to $|I^+_n|/2$. We have

$$v^+_n - \pi/2 \leqq |I^+_n|/2 \quad \text{and} \quad |I^+_n|^2/4 \leqq \int_{I^+_n} \delta^+(u)du\ .$$

The assumption (v) implies

$$\sum{}'' (v_n^+ - \pi/2)^2 < \infty \quad \text{and} \quad \sum{}'' |I_n^+|^2 < \infty .$$

Consequently

$$\sum_{n=1}^{\infty} (v_n^+ - \pi/2)^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (u_{n+1}^+ - u_n^+)^2 < \infty .$$

8. **Proof of** (iii′) $\Rightarrow$ (iii). Let $\{u_n^+\}$ and $\{u_n^-\}$ be sequences having the properties stated in (iii′). For the sake of simplicity introduce the following notations:

$$I_n^+ = [u_n^+, u_{n+1}^+] , \qquad I_n^- = [u_n^-, u_{n+1}^-] , \qquad |I_n^+| = u_{n+1}^+ - u_n^+ ,$$
$$|I_n^-| = u_{n+1}^- - u_n^- , \qquad \theta_n^+ = v_n^+ - \pi/2 , \qquad \theta_n^- = -v_n^- - \pi/2 .$$

We construct a sequence $u_1 < u_2 < \cdots \uparrow \infty$ consisting of elements of the set $M = \{u_n^+ | n = 1, 2, \cdots\} \cup \{u_n^- | n = 1, 2, \cdots\}$ inductively as follows: First set $u_1 = \min(u_1^+, u_1^-)$; after getting $u_1, \cdots, u_n$, let u_{n+1} be the minimum $a \in M$ such that $I_k^+ \cup I_{k'}^- \subset [u_n, a]$ for some k and k'.

Denote by $k^+(n)$ and $k^-(n)$ the minimum index k such that $I_k^+ \subset [u_n, u_{n+1}]$ and $I_k^- \subset [u_n, u_{n+1}]$, respectively. Similarly denote by $l^{\pm}(n)$ the maximum of k with $I_k^{\pm} \subset [u_n, u_{n+1}]$. If

$$k^+(n) = l^+(n) , \tag{4}$$

we have

$$u_{n+1} - u_n \leqq |I_{k^+(n)-1}^+| + |I_{k^+(n)}^+| + |I_{k^+(n)+1}^+|$$

so that

$$(u_{n+1} - u_n)^2 \leqq 3(|I_{k^+(n)-1}^+|^2 + |I_{k^+(n)}^+|^2 + |I_{k^+(n)+1}^+|^2) .$$

If

$$k^-(n) = l^-(n) \tag{5}$$

we have this inequality obtained by replacing the superscript $+$ by $-$. Since either (4) or (5) holds for every n, we conclude

$$\sum^{\infty} (u_{n+1} - u_n)^2 \leqq 9(\sum^{\infty} |I_k^+|^2 + \sum^{\infty} |I_k^-|^2) < \infty .$$

Next, observe that the quantity v_n'' with respect to the sequence $\{u_n\}$ satisfies

$$v_n'' - \pi/2 \leqq \min\{\theta_k^+ | k = k^+(n) - 1, \cdots, l^+(n) + 1\} ,$$

so that

$$(v_n'' - \pi/2)^2 \leqq \sum_{k=k^+(n)-1}^{l^+(n)+1} (\theta_k^+)^2 .$$

Therefore

$$\sum^{\infty} (v_n'' - \pi/2)^2 \leqq 3 \sum^{\infty} (\theta_k^+)^2 < \infty$$

and, for the same reason,

$$\sum^{\infty} (v_n' + \pi/2)^2 \leqq 3 \sum^{\infty} (\theta_k^-)^2 < \infty \ .$$

Consequently $\{u_n\}$ has the property stated in (iii).

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