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ON THE REVERSE HÖLDER INEQUALITIES FOR CERTAIN EXPONENTIAL PROCESSES

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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Given a continuous local martingale M with $M_0 = 0$, we denote by $\langle M\rangle$ the associated increasing process. The purpose of this note is to establish the reverse Hölder inequality for the process defined by the formula

$$G_\alpha(t) = \exp\left\{\alpha M_t + \left(\frac{1}{2} - \alpha\right)\langle M\rangle_t\right\} \qquad (0 \leqq t < \infty),$$

where α is a real number. This exponential process plays an important role in connection with the problem of finding out sufficient conditions for the uniform integrability of the exponential martingale $Z = \exp(M - \langle M\rangle/2)$ (see [4] and [5]). We remark in passing that it is the solution of the stochastic integral equation $Y_t = 1 + \int_0^t Y_s dX_s$, where $X = \alpha M + (1-\alpha)^2\langle M\rangle/2$.

Let now (Ω, F, P) be a fixed probability space with a right filtration (F_t), where $F_\infty = F$, and we assume that F_0 contains all the null sets. Every martingale here is adapted to this filtration. For simplicity, we denote by $\mathscr{S}$ the class of all stopping times. Recall that a continuous local martingale N is said to be in the class BMO if $E[\langle N\rangle_\infty - \langle N\rangle_T | F_T] \leqq C$ for every $T \in \mathscr{S}$, where C is an absolute constant. It is well-known that the space BMO is a Banach space with the norm $\|N\|_{\mathrm{BMO}} = \sup_{T\in\mathscr{S}} \|E[\langle N\rangle_\infty - \langle N\rangle_T | F_T]^{1/2}\|_\infty$.

LEMMA. *If* $\|N\|_{\mathrm{BMO}} < 1$, *then we have*

$$E[\exp(\langle N\rangle_\infty - \langle N\rangle_T) | F_T] \leqq (1 - \|N\|^2_{\mathrm{BMO}})^{-1} \qquad (T \in \mathscr{S}).$$

In [2] Garsia has first established this inequality for discrete parameter martingales, and it is of fundamental importance in our investigation. For the proof, see [3].

Our first result is the following.

PROPOSITION 1. *If* $M \in \mathrm{BMO}$, *then there exist* $p > 1$ *and* $\delta > 0$ *such* *that the reverse Hölder inequality*

$$E[G_\alpha(\infty)^p \mid F_T] \leqq C_\alpha G_\alpha(T)^p$$

holds for every $T \in \mathscr{S}$ *and every* α *with* $|\alpha - 1| < \delta$, *where* C_α *is a constant depending on* α.

The result for the case $\alpha = 1$ is obtained in [1] by C. Doléans-Dade and P. A. Meyer.

PROOF. Let $M \in \mathrm{BMO}$. Then it is easy to see that for any α the process $Z^{(\alpha)} = \exp(\alpha M - \alpha^2 \langle M \rangle /2)$ is a uniformly integrable martingale. Moreover, for $|\alpha| \leqq 2$ the reverse Hölder inequality

$$(1) \qquad E[\{Z_\infty^{(\alpha)}\}^r \mid F_T] \leqq C_r \{Z_T^{(\alpha)}\}^r$$

holds for every $T \in \mathscr{S}$ and some $r > 1$, where C_r depends only on r (see the proof of Lemma 9 in [4]). On the other hand, by a simple calculation we have

$$(2) \qquad G_\alpha(T) = Z_T^{(\alpha)} \exp\left\{\frac{1}{2}(1-\alpha)^2 \langle M \rangle_T\right\} .$$

Let now $1 < p < r$, and we set $u = r/p$ and $v = r/(r-p)$. Applying Hölder's inequality to the right hand side of (2) we find

$$(3) \quad E[\{G_\alpha(\infty)/G_\alpha(T)\}^p \mid F_T] \leqq E[\{Z_\infty^{(\alpha)}/Z_T^{(\alpha)}\} \mid F_T]^{1/u} \times E\left[\exp\left\{\frac{1}{2}(1-\alpha)^2 pv(\langle M \rangle_\infty - \langle M \rangle_T)\right\} \middle| F_T\right]^{1/v} .$$

By (1) the first term on the right hand side is smaller than $C_r^{1/v}$. In proving our claim, we may assume that $0 < \|M\|_{\mathrm{BMO}}$, and so we let $\delta = (\sqrt{pv}\ \|M\|_{\mathrm{BMO}})^{-1}$. Then $(1-\alpha)^2 pv \|M\|_{\mathrm{BMO}}^2 < 1$ for any α with $|\alpha - 1| < \delta$, and thus from the lemma it follows at once that the second term on the right hand side of (3) is bounded by $2^{1/v}$. Combining these estimates, we find that the inequality

$$E[G_\alpha(\infty)^p \mid F_T] \leqq 2^{1/v} C_r^{1/u} G_\alpha(T)^p \qquad (T \in \mathscr{S})$$

is valid for any α with $|\alpha - 1| < \delta$. This completes the proof.

Furthermore, in the above setting we have $E[\sup_t G_\alpha(t)^p] < \infty$ for any α with $|\alpha - 1| < \delta$, because $\{G_\alpha(t), F_t\}$ is an L^p-bounded submartingale. But it should be noted that the condition $M \in \mathrm{BMO}$ does not always imply the integrability of $G_\alpha(\infty)$ for *all* α (see Example 4 in [3]).

Finally, we prove the following converse of Proposition 1.

PROPOSITION 2. *Let* $\alpha \neq 1$. *Suppose that* $Z^{(\alpha)}$ *is a uniformly integrable martingale and that the inequality*

$$E[G_\alpha(\infty)\,|\,F_T] \leqq C_\alpha G_\alpha(T) \tag{4}$$

is valid for every $T \in \mathscr{S}$, *with some constant* $C_\alpha > 0$. *Then M belongs to the class* BMO.

PROOF. We begin with the case $\alpha = 0$. Since $G_0(t) = \exp(\langle M\rangle_t/2)$, it follows from (4) that

$$E\left[\exp\left\{\frac{1}{2}(\langle M\rangle_\infty - \langle M\rangle_T)\right\}\,\middle|\,F_T\right] \leqq C_0\,.$$

Then we get $\|M\|^2_{\mathrm{BMO}} \leqq 2C_0$.

Secondly, we deal with the case $\alpha \neq 0$. By the assumption, $Z^{(\alpha)}$ is a uniformly integrable martingale, and so $d\hat{P} = Z^{(\alpha)}_\infty dP$ is also a probability measure. Then, according to the theorem of Girsanov-Schuppen-Wong, for any continuous local matingale X the process $\hat{X}$ defined by the formula $\hat{X}_t = \alpha\langle X, M\rangle_t - X_t$ is a continuous local martingale relative to $\hat{P}$ such that $\langle\hat{X}\rangle = \langle X\rangle$ under either probability measure (see [6] for example). Applying this result to the local martingale αM, we can derive from (2) and (4) that

$$\hat{E}\left[\exp\left\{\frac{1}{2}\left(1 - \frac{1}{\alpha}\right)(\langle\widehat{\alpha M}\rangle_\infty - \langle\widehat{\alpha M}\rangle_T)\right\}\,\middle|\,F_T\right] \leqq C_\alpha \qquad (T \in \mathscr{S})\,.$$

This implies that $\widehat{\alpha M}$ is a BMO-martingale relative to $\hat{P}$. So, we have $M \in \mathrm{BMO}$ by Theorem 2 in [3]. Thus the proof is complete.

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