Tôhoku Math. Journ.
35 (1983), 289-301.

UNIFORM INTEGRABILITY OF CONTINUOUS EXPONENTIAL MARTINGALES

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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(Received June 7, 1982)

1. Introduction. Given a local martingale M, the associated exponential local martingale Z is not necessarily uniformly integrable. As is first indicated by Girsanov in [2], to know whether Z is a uniformly integrable martingale is very important in certain questions concerning the absolute continuity of probability laws of stochastic processes. However, it seems to us that the essential feature of the problem appears in the case where M is continuous. For that reason, assuming the sample continuity of M we consider the uniform integrability of Z in this paper: but we have no mind to deny the significance of the extension to right continuous martingales. Historically, in the last ten years many sufficient conditions about this problem have successively been found: for example, see Novikov [9, 10], Kazamaki [4, 5], Lépingle and Mémin [7] and Okada [11].

This paper consists of six sections, and Section 5 contains the main result. Our aim here is to give a new sufficient condition which is an improvement of the above-mentioned criteria. In Sections 2 and 3 we collect some notations and technical results that are used in later sections. In Section 4 we shall deal with a special case in order to explain our idea explicitly. Finally, in Section 6 we shall state some remarks on a BMO-martingale in connection with the problem about the uniform integrability of exponential martingales.

2. Preliminaries. Let (Ω, F, P) be a complete probability space with a non-decreasing right continuous family $(F_t)_{0\leqq t<\infty}$ of sub σ-fields of F such that $F = \bigvee_{t\geqq 0} F_t$ and F_0 contains all null sets. It goes without saying that the martingales here are adapted to this filtration.

Given a continuous local martingale M with $M_0 = 0$, consider the exponential local martingale Z defined by the formula

$$Z_t = \exp\left(M_t - \frac{1}{2}\langle M\rangle_t\right) \quad (0 \leqq t < \infty) \tag{1}$$

This research was partially supported by the Grant-in-Aid for Scientific Research C-56540069 (1981) of the Ministry of Education.

where $\langle M\rangle$ denotes the continuous increasing process associated with M. Clearly, $E[Z_T] \leqq 1$ for any stopping time T, because Z is a supermartingale with $Z_0 = 1$. Therefore, a necessary and sufficient condition for Z to be a uniformly integrable martingale is the validity of the equality $E[Z_\infty] = 1$ where $Z_\infty = \lim_{t\to\infty} Z_t$. But, unfortunately, the direct verification is usually hard to carry out.

For any real number a, let us denote by $Z^{(a)}$ the process given by the formula

$$Z_t^{(a)} = \exp\left(aM_t - \frac{a^2}{2}\langle M\rangle_t\right) \quad (0 \leqq t < \infty), \tag{2}$$

which is also an exponential local martingale. An easy calculation shows that

$$ZZ^{(-1)} = \exp(-\langle M\rangle), \quad Z = \{Z^{(1/2)}\}^2 \exp\left(-\frac{1}{4}\langle M\rangle\right).$$

From these relations we can immediately derive that $\{Z_\infty = 0\} = \{\langle M\rangle_\infty = \infty\}$.

We shall denote for convenience by $\mathscr{M}_u$ the class of all uniformly integrable martingales and by $\mathscr{S}_b$ the class of all bounded stopping times (relative to (F_t)). Let now $\phi: R_+ \to R_+$ be a continuous function such that $\phi(0) = 0$, and for each real number α we set

$$G_\alpha(t, \phi) = \exp\left\{\alpha M_t + \left(\frac{1}{2} - \alpha\right)\langle M\rangle_t - |1 - \alpha|\phi(\langle M\rangle_t)\right\} \quad (0 \leqq t < \infty)$$

$$g(\alpha, \phi) = \sup_{T\in\mathscr{S}_b} E[G_\alpha(T, \phi)].$$

Obviously, $G_1(t, \phi) = Z_t$, and it follows at once from Fatou's lemma that $g(\alpha, \phi) = \sup_T E[G_\alpha(T, \phi)]$, where the supremum is taken over all stopping times.

The reader is assumed to be familiar with the martingale theory as expounded in [8]. Throughout the paper, let us denote by C a positive constant and by C_p a positive constant depending on p only, both letters are not necessarily the same from line to line.

3. Properties of $g(\alpha, \phi)$. Clearly, $G_0(t, 0) = \exp(\langle M\rangle_t/2)$ and $G_{1/2}(t, 0) = \exp(M_t/2)$. Furthermore, it is known that $g(0, 0) < \infty \Rightarrow Z \in \mathscr{M}_u$ (Novikov [9]) and $g(0, 0) < \infty \Rightarrow g(1/2, 0) < \infty \Rightarrow Z \in \mathscr{M}_u$ (Kazamaki [4]).

PROPOSITION 1. *Let $\alpha < \beta < 1$. Then*

$$g(\beta, \phi) \leqq g(\alpha, \phi)^{(1-\beta)/(1-\alpha)}. \tag{3}$$

On the other hand, if $1 < \alpha < \beta$, we have

(4) $$g(\alpha, \phi) \leqq g(\beta, \phi)^{(\alpha-1)/(\beta-1)} .$$

PROOF. We first show (3). For that, let $\alpha < \beta < 1$, and set $p = (1-\alpha)/(1-\beta)$ which is larger than 1. Then, the exponent conjugate q to p being $(1-\alpha)/(\beta-\alpha)$, it is easy to see that $G_\beta(T, \phi) = G_\alpha(T, \phi)^{1/p} Z_T^{1/q}$. So, applying Hölder's inequality to the right hand side we find

(5) $$E[G_\beta(T, \phi)] \leqq E[G_\alpha(T, \phi)]^{1/p} E[Z_T]^{1/q} \leqq E[G_\alpha(T, \phi)]^{1/p} .$$

Consequently, $g(\beta, \phi) \leqq g(\alpha, \phi)^{1/p}$.

Secondly, we show (4). If $1 < \alpha < \beta$, then

$$G_\alpha(T, \phi) = G_\beta(T, \phi)^{(\alpha-1)/(\beta-1)} Z_T^{(\beta-\alpha)/(\beta-1)}$$

and we apply Hölder's inequality with exponent $(\beta-1)/(\alpha-1)$ and $(\beta-1)/(\beta-\alpha)$ to the right hand side:

(6) $$E[G_\alpha(T, \phi)] \leqq E[G_\beta(T, \phi)]^{(\alpha-1)/(\beta-1)} E[Z_T]^{(\beta-\alpha)/(\beta-1)} .$$

Thus, (4) is obtained. This completes the proof.

We close this section with two examples which are used in Section 5. In the following $B = (B_t)$ denotes a one dimensional Brownian motion starting at 0.

EXAMPLE 1. For $0 < a < \infty$, we define the stopping time

(7) $$\tau_a = \inf\{t \geqq 0; B_t \leqq t - \phi(t) - a\} .$$

Let now $M = B^{\tau_a}$ where $B_t^{\tau_a} = B_{t \wedge \tau_a}$ ($x \wedge y$ denotes the minimum of x and y). Then $\langle M \rangle_T/2 - \phi(\langle M \rangle_T) \leqq M_T - \langle M \rangle_T/2 + a$ $(T \in \mathscr{S}_b)$ by the definition of τ_a, so that $g(0, \phi) \leqq e^a \sup_{T \in \mathscr{S}_b} E[Z_T] \leqq e^a$. Furthermore, combining this fact with (3) we have $g(\alpha, \phi) < \infty$ for all α with $0 \leqq \alpha < 1$.

EXAMPLE 2. For $0 < a < \infty$, let ν_a be the stopping time given by

(8) $$\nu_a = \inf\{t \geqq 0; B_t \geqq t + \phi(t) + a\} ,$$

and consider the martingale $M = B^{\nu_a}$. Then it follows from the definition of ν_a that for any $T \in \mathscr{S}_b$

$$2M_T + \left(\frac{1}{2} - 2\right)\langle M \rangle_T - \phi(\langle M \rangle_T) \leqq M_T - \frac{1}{2}\langle M \rangle_T + a .$$

Thus $g(2, \phi) \leqq e^a$. Combining this with (4) we have $g(\beta, \phi) < \infty$ for all β with $1 < \beta \leqq 2$.

4. An application of a simple criterion to the case $\phi = 0$. Recently, Lépingle and Mémin have proved in [7] that if for some α with $0 \leqq \alpha < 1$ the family $\{G_\alpha(T, 0)\}_{T \in \mathscr{S}_b}$ is uniformly integrable, then $Z \in \mathscr{M}_u$. But the verification of the uniform integrability is often hard to carry out. The purpose of this section is to improve their result by applying the following simple criterion.

LEMMA 1. *Let* $a \neq 0$ *and* $T_\lambda = \inf\{t \geqq 0; \langle M\rangle_t > \lambda\}$ *for* $\lambda > 0$. *Then* $Z^{(a)} \in \mathscr{M}_u$ *if and only if* $\liminf_{\lambda\to\infty} E[Z^{(a)}_{T_\lambda}; T_\lambda < \infty] = 0$.

PROOF. For each $\lambda > 0$ it follows from the definition of T_λ that $\langle M\rangle_{T_\lambda} \leqq \lambda$ and $\{T_\lambda = \infty\} = \{\langle M\rangle_\infty \leqq \lambda\}$. Therefore, we have

$$E[Z^{(a)}_{T_\lambda}; T_\lambda < \infty] = E[Z^{(a)}_{T_\lambda}] - E[Z^{(a)}_{T_\lambda}; T_\lambda = \infty] = 1 - E[Z^{(a)}_\infty; \langle M\rangle_\infty \leqq \lambda],$$

completing the proof.

LEMMA 2. *Let* $\alpha \neq 1$. *If* $g(\alpha, 0) < \infty$, *then* $Z^{(\alpha)} \in \mathscr{M}_u$.

PROOF. Since $G_\alpha(t, 0) = Z^{(\alpha)}_t \exp\{(1-\alpha)^2\langle M\rangle_t/2\}$ and $\langle M\rangle_{T_\lambda} = \lambda$ on $\{T_\lambda < \infty\}$, we find

$$\begin{aligned} E[Z^{(\alpha)}_{T_\lambda}; T_\lambda < \infty] &= E[G_\alpha(T_\lambda, 0); T_\lambda < \infty] \exp\left\{-\frac{1}{2}\lambda(1-\alpha)^2\right\} \\ &\leqq g(\alpha, 0) \exp\left\{-\frac{\lambda}{2}(1-\alpha)^2\right\}, \end{aligned}$$

which converges to 0 as $\lambda \to \infty$. Thus $Z^{(\alpha)} \in \mathscr{M}_u$ by Lemma 1.

PROPOSITION 2. *If* $g(\alpha, 0) < \infty$ *for some* $\alpha \neq 1$, *then* $Z \in \mathscr{M}_u$.

PROOF. Suppose first that $g(\alpha, 0) < \infty$ for some α with $-\infty < \alpha < 1$. If $\alpha \leqq \beta < 1$, then $g(\beta, 0) < \infty$ by (3) and so $Z^{(\beta)} \in \mathscr{M}_u$ by Lemma 2. Moreover, since $Z^{(\beta)}_T \leqq G_\beta(T, 0)$ for any stopping time T, it follows from (5) that

$$1 = E[Z^{(\beta)}_\infty] \leqq g(\alpha, 0)^{(1-\beta)/(1-\alpha)} E[Z_\infty]^{(\beta-\alpha)/(1-\alpha)}.$$

The last expression converging to $E[Z_\infty]$ as $\beta \uparrow 1$, we find $1 \leqq E[Z_\infty]$. This implies that $Z \in M_u$. The proof for the case where $g(\alpha, 0) < \infty$ for some α with $1 < \alpha < \infty$ is completely analogous except that (6), instead of (5), is used.

It is necessary to compare the two conditions (a) $g(\alpha, 0) < \infty$ for some $\alpha < 1$ and (b) $g(\beta, 0) < \infty$ for some $\beta > 1$, but we shall explain it in the next section. In conclusion, one of these conditions does not always imply the other. However, as is seen in Proposition 4, if $g(\alpha, 0) < \infty$ for some α with $\alpha < 1$ (resp. $1 < \alpha$), then the family $\{G_\beta(T, 0)\}_{T\in\mathscr{S}_b}$ is uniformly integrable for all β with $\alpha < \beta < 1$ (resp. $1 < \beta < \alpha$). At all events, for the purpose of verifying the uniform integrability of Z, Proposition 2 is more suitable than the criterion of Lépingle and Mémin.

5. A new criterion. Recall that ϕ is said to be a lower function if

$P\{B_t < \phi(t),\ t \to \infty\} = 0$; as is well-known, by Blumenthal's zero-one law this probability is equal to 0 or 1.

Our main result is the following.

THEOREM 3. *Let* ϕ *be a lower function. If* $g(\alpha, \phi) < \infty$ *for some* $\alpha \neq 1$, *then* $Z \in \mathscr{M}_u$.

Of course, Proposition 2 corresponds to the special case $\phi = 0$. Moreover, letting $\alpha = 1/2$ we obtain the criterion given by Novikov in [10]. Quite recently, Okada has proved in [11] that if for some α with $0 \leqq \alpha < 1$ the family $\{G_\alpha(T, C\sqrt{t})\}_{T \in \mathscr{S}_b}$ is uniformly integrable, then $Z \in \mathscr{M}_u$. On the other hand, by Kolmogorov's criterion, for any positive continuous function ϕ satisfying $\phi(t)/t \downarrow$ and $\phi(t)/\sqrt{t} \uparrow$ as $t \to \infty$, $P\{B_t < \phi(t),\ t \to \infty\} = 0$ or 1 according as

$$\int^{+\infty} t^{-3/2}\phi(t) \exp\left\{-\frac{1}{2t}\phi(t)^2\right\} dt \gtreqless \infty$$

(see Section 1.8 in [3]). For example, $C\sqrt{t}$ and $\sqrt{2t \log\log t}$ are lower functions. Therefore, as a special case the following corollary contains the result which is an improvement of Okada's criterion.

COROLLARY. *If* $g(\alpha, C\sqrt{t}) < \infty$ *for some* $\alpha \neq 1$, *then* $Z \in \mathscr{M}_u$.

In order to prove Theorem 3, we need four lemmas.

LEMMA 3 (Shepp [12]). *Let* τ_a *and* ν_a *be the stopping times given by* (7) *and* (8) *respectively. Then we have*

$$E\left[\exp\left(B_{\tau_a} - \frac{1}{2}\tau_a\right);\ \tau_a < \infty\right] = P(\tilde{\tau}_a < \infty) \tag{9}$$

$$E\left[\exp\left(B_{\nu_a} - \frac{1}{2}\nu_a\right);\ \nu_a < \infty\right] = P(\tilde{\nu}_a < \infty) \tag{10}$$

where $\tilde{\tau}_a = \inf\{t \geqq 0;\ B_t \leqq -\phi(t) - a\}$ *and* $\tilde{\nu}_a = \inf\{t \geqq 0;\ B_t \geqq \phi(t) + a\}$.

PROOF. We show only (9), because the proof of (10) is similar. Let $N = B^{\tau_a}$ and $W = \exp(N - \langle N\rangle/2)$. Note that W is a martingale, because $\langle N\rangle_t \leqq t$. Thus $E[W_t] = 1$ for every t. Then, from Girsanov's theorem it follows that for each t the process $\{B_s - s \wedge t \wedge \tau_a\}$ is a Brownian motion relative to the now probability measure $W_t dP$. Let now $\Delta_j\colon 0 < t_1^{(j)} < t_2^{(j)} < \cdots < t_{n_j}^{(j)} < t$ $(j = 1, 2, \cdots)$ be a sequence of refining partitions which become dense in $[0, t]$. Then, noticing the theorem of Girsanov we find

$$E[W_t;\ t \leqq \tau_a] = \lim_{j\to\infty} E[W_t;\ B_{t_k^{(j)}} > t_k^{(j)} - \phi(t_k^{(j)}) - a,\ 1 \leqq k \leqq n_j,\ t \leqq \tau_a]$$

$$= \lim_{j\to\infty} P\{B_{t_k^{(j)}} > -\phi(t_k^{(j)}) - a,\ 1 \leqq k \leqq n_j\}$$
$$= P\{B_s > -\phi(s) - a,\ 0 \leqq s \leqq t\},$$

from which $\lim_{t\to\infty} E[W_t; t \leqq \tau_a] = P(\tilde{\tau}_a = \infty)$. Therefore,

$$P(\tilde{\tau}_a < \infty) = \lim_{t\to\infty} E[W_t; \tau_a < t] = E\Big[\exp\Big(B_{\tau_a} - \frac{1}{2}\tau_a\Big); \tau_a < \infty\Big],$$

completing the proof.

LEMMA 4. *Let* $0 < a < \infty$. *If* ϕ *is a lower function, so is* $\phi + a$.

PROOF. Consider the stopping times

$$\mu = \inf\{t > 0; B_t \geqq t\phi(1/t) + a(1 \wedge t)\},$$
$$\sigma = \inf\{t > 0; B_t \geqq t\phi(1/t) + at\}.$$

Clearly, $\{\mu = 0\} = \{\sigma = 0\}$. Recall that $\{tB_{1/t}\}$ is also a Brownian motion. Then

$$P\{B_t < \phi(t) + a, t \to \infty\} = P\{B_t < t\phi(1/t) + at, t \to 0\}$$
$$= P(\sigma > 0) = P(\mu > 0).$$

On the other hand, by the theorem of Girsanov the process $\hat{B}$ defined by the formula $\hat{B}_t = B_t - a(1 \wedge t)$ $(0 \leqq t)$ is a Brownian motion under $d\hat{P}$ where $d\hat{P} = \exp(aB_1 - a^2/2)dP$. Now, let us assume that ϕ is a lower function. Then $\hat{P}(\mu > 0) = \hat{P}\{\hat{B}_t < t\phi(1/t), t \to 0\} = 0$ and, since $\hat{P}$ is equivalent to P, we have $P(\mu > 0) = 0$. This completes the proof.

LEMMA 5. ϕ *is a lower function if and only if* $P(\tilde{\tau}_a < \infty) = 1$ *for all* $a > 0$.

PPOOF. Suppose first that ϕ is a lower function. Then $\phi + a$ is also a lower function by Lemma 4. So, we have

$$P(\tilde{\tau}_a = \infty) = P\{B_t < \phi(t) + a,\ 0 \leqq t < \infty\} = 0.$$

Conversely, let us assume that $P(\tilde{\tau}_a < \infty) = 1$ for all $a > 0$. Then, since $\{tB_{1/t}\}$ is a Brownian motion, we find

(11) $$P\{B_t < t\phi(1/t) + at,\ 0 < t < \infty\} = P(\tilde{\tau}_a = \infty) = 0.$$

Suppose now that ϕ is not a lower function. Then $P\{B_t < t\phi(1/t), t \to 0\} = 1$ by Blumenthal's zero-one law. In other words, $P(\tau > 0) = 1$ where $\tau = \inf\{t > 0; B_t \geqq t\phi(1/t)\}$. Let u be a positive number such that $P(\tau > u) > 1/2$; namely, $P\{B_s < s\phi(1/s),\ 0 < s \leqq u\} > 1/2$. On the other hand, $P(B_s < a, s \to 0) = 1$ and so $P(\sigma_a > 0) = 1$, where $\tau_a = \inf\{t \geqq 0; B_t \geqq a\}$. Furthermore, $P(\sigma_a > 1/u) > 1/2$ for large a, because $\sigma_a \to \infty$ as

$a \to \infty$ from the the definition of σ_a. That is,

$$P(B_s < as,\ u \leqq s < \infty) = P(B_t < a,\ 0 < t \leqq 1/u) = P(\sigma_a > 1/u) > 1/2\,.$$

Then, combining these facts we find

$$\begin{aligned} &P\{B_t < t\phi(1/t) + at,\ 0 < t < \infty\} \\ &\qquad \geqq P\{B_t < t\phi(1/t) \text{ for all } t \leqq u \text{ and } B_t < at \text{ for all } t \geqq u\} > 0\,, \end{aligned}$$

which is inconsistent with (11). Consequently, ϕ is a lower function.

By using the same argument we obtain the following.

LEMMA 6. *ϕ is a lower function if and only if $P(\tilde{\nu}_a < \infty) = 1$ for all $a > 0$.*

PROOF OF THEOREM 3. As is well-known, any continuous local martingale can be reduced to stopped Brownian motion by means of a continuous change of time. Therefore, it suffices essentially to verify this theorem in the case where $M = B^{\zeta}$ for a certain stopping time ζ. We begin with the case where $g(\alpha, \phi) < \infty$ for some α with $-\infty < \alpha < 1$. Let $\tau_j = \inf\{t \geqq 0; B_t \leqq t - \phi(t) - j\}$ $(j \geqq 1)$, which is nothing but the stopping time obtained by letting $a = j$ in (7). Then, since ϕ is a lower function by the assumption, combining Lemmas 3 and 5 we have

$$E\Big[\exp\Big(B_{\tau_j} - \frac{1}{2}\tau_j\Big)\Big] = 1 \quad (j \geqq 1)$$

and so

$$1 = E\Big[\exp\Big\{B_{\zeta\wedge\tau_j} - \frac{1}{2}(\zeta \wedge \tau_j)\Big\}\Big] \leqq E[Z_\infty] + E\Big[\exp\Big(B_{\tau_j} - \frac{1}{2}\tau_j\Big);\ \tau_j < \zeta\Big].$$

Noticing $B_{\tau_j} = \tau_j - \phi(\tau_j) - j$ on $\{\tau_j < \infty\}$, we find that the expectation in the last expression is smaller than

$$\begin{aligned} &E[G_\alpha(\tau_j, \phi) \exp\{(1-\alpha)(B_{\tau_j} - \tau_j + \phi(\tau_j))\};\ \tau_j < \infty] \\ &\qquad \leqq g(\alpha, \phi) \exp\{-(1-\alpha)j\}\,, \end{aligned}$$

which tends to 0 as $j \to \infty$, because $g(\alpha, \phi) < \infty$ by the assumption. Thus $1 \leqq E[Z_\infty]$, from which $Z \in \mathscr{M}_u$.

Secondly, we deal with the case where $g(\beta, \phi) < \infty$ for some β with $1 < \beta < \infty$. Instead of τ_j, consider this time the stopping time $\nu_j = \inf\{t \geqq 0; B_t \geqq t + \phi(t) + j\}$. Then $P(\tilde{\nu}_j < \infty) = 1$ $(j \geqq 1)$ by Lemma 6, because ϕ is a lower function. Therefore, combining this fact with Lemma 3 we have

$$1 \leqq E[Z_\infty] + E\Big[\exp\Big(B_{\nu_j} - \frac{1}{2}\nu_j\Big);\ \nu_j < \zeta\Big].$$

Since $B_{\nu_j} = \nu_j + \phi(\nu_j) + j$ on $\{\nu_j < \infty\}$, the second term on the right hand side is smaller than

$$E[G_\beta(\nu_j, \phi) \exp\{-(\beta - 1)(B_{\nu_j} - \nu_j - \phi(\nu_j))\}; \nu_j < \infty] \leqq g(\beta, \phi) \exp\{-(\beta - 1)j\} \to 0 \quad (j \to \infty) .$$

Consequently $1 \leqq E[Z_\infty]$. This completes the proof.

PROPOSITION 4. *Let ϕ be a lower function. If $g(\alpha, \phi) < \infty$ for some α with $-\infty < \alpha < 1$ (resp. $1 < \alpha < \infty$), then the family $\{G_\beta(T, \phi)\}_{T \in \mathscr{S}_b}$ is uniformly integrable for every β with $\alpha < \beta < 1$ (resp. $1 < \beta < \alpha$).*

PROOF. Let $\alpha < \beta < 1$. Then for $\lambda > 0$ and $T \in \mathscr{S}_b$ we can obtain

$$E[G_\beta(T, \phi); G_\beta(T, \phi) > \lambda] \leqq g(\alpha, \phi)^{(1-\beta)/(1-\alpha)} E[Z_T; G_\beta(T, \phi) > \lambda]^{(\beta-\alpha)/(1-\alpha)}$$

by modifying slightly the proof of (5). Assume now $g(\alpha, \phi) < \infty$. Then $Z \in \mathscr{M}_u$ by Theorem 3, so that

$$E[Z_T; G_\beta(T, \phi) > \lambda] = E[Z_\infty; G_\beta(T, \phi) > \lambda] .$$

Furthermore, by using Chebyshev's inequality and then the inequality (3) in Proposition 1 we find

$$\lambda P\{G_\beta(T, \phi) > \lambda\} \leqq g(\alpha, \Phi)^{(1-\beta)/(1-\alpha)} \quad (\lambda > 0) .$$

From these estimates the uniform integrability of the family $\{G_\beta(T, \phi)\}_{T \in \mathscr{S}_b}$ follows immediately.

On the other hand, in order to give the proof for the case where $g(\alpha, \phi) < \infty$ for some α with $1 < \alpha < \infty$ it suffices to apply (4) and (6) instead of (3) and (5).

Now, let us consider the class Φ of positive continuous functions ϕ satisfying $\liminf_{t\to\infty} \phi(t)/t = 0$. Obviously, it is larger than the class of all lower functions. But the reverse inclusion fails. For example, $\phi(t) = (1 + \varepsilon)\sqrt{2t \log\log t}$ $(\varepsilon > 0)$ belongs to the class Φ, but it is not a lower function by Kolmogorov's criterion.

PROPOSITION 5. *Let $\phi \in \Phi$. If $g(\alpha, \phi) < \infty$ for $\alpha \neq 1$, then $Z^{(\alpha)} \in \mathscr{M}_u$.*

PROOF. For $\lambda > 0$, let $T_\lambda = \inf\{t \geqq 0; \langle M\rangle_t > \lambda\}$ as before. Since $G_\alpha(t, \phi) = Z_t^{(\alpha)} \exp\{(1 - \alpha)^2\langle M\rangle_t/2 - |1 - \alpha|\phi(\langle M\rangle_t)\}$, we find

$$E[Z_{T_\lambda}^{(\alpha)}; T_\lambda < \infty] = E[G_\alpha(T_\lambda, \phi); T_\lambda < \infty] \exp\left\{|1 - \alpha|\phi(\lambda) - \frac{1}{2}\lambda(1 - \alpha)^2\right\}$$
$$\leqq g(\alpha, \phi) \exp\left\{-\lambda|1 - \alpha|\left(\frac{1}{2}|1 - \alpha| - \frac{\phi(\lambda)}{\lambda}\right)\right\} .$$

Moreover, $\phi(\lambda_n)/\lambda_n$ converges to 0 for some sequence $\lambda_n \uparrow \infty$ by the assumption. Therefore, we have $\liminf_{\lambda\to\infty} E[Z^{(\alpha)}_{T_\lambda}; T_\lambda < \infty] = 0$. Then $Z^{(\alpha)} \in \mathscr{M}_u$ by Lemma 1. Thus the proof is complete.

As an illustration, consider the case where $g(\alpha, \phi) < \infty$ for some α with $-\infty < \alpha < 1$. If $\alpha \leqq \beta < 1$, then $g(\beta, \phi) < \infty$ by (4) and so $Z^{(\beta)} \in \mathscr{M}_u$ by Proposition 5. However, there are some cases where $Z \notin \mathscr{M}_u$ in addition to that. For example, for $\varepsilon > 0$ let $\phi(t) = (1+\varepsilon)\sqrt{2t \log\log t}$ and let τ_a be the corresponding stopping time defined by (7). Consider now $M = B^{\tau_a}$. Then $g(0, \phi) < \infty$ by Example 1, and so $Z^{(\beta)} \in \mathscr{M}_u$ for all β with $0 \leqq \beta < 1$ by Proposition 5. On the other hand, since ϕ is not a lower function, it follows from Lemmas 3 and 5 that $E[Z_\infty] < 1$. Namely, $Z \notin \mathscr{M}_u$. By contrast, considering the martingale $M = B^{\nu_a}$ where ν_a denotes the stopping time defined by (8), we can obtain an example such that $Z^{(\beta)} \in \mathscr{M}_u$ for all β with $1 < \beta \leqq 2$ and $Z \notin \mathscr{M}_u$. We close this section with three further counterexamples.

EXAMPLE 3. Let $\tau = \inf\{t \geqq 0; B_t \leqq t-1\}$, which is nothing but the stopping time defined by setting $\phi = 0$ and $a = 1$ in (7). Consider now the martingale $M = B^\tau$. Then $g(0, 0) \leqq e$ by Example 1, so that $Z^{(\beta)} \in \mathscr{M}_u$ for all β with $0 \leqq \beta \leqq 1$ according to Theorem 3 and Proposition 5. On the other hand, since $Z^{(a)}_\infty = \exp(aB_\tau - a^2\tau/2) \leqq e^{-a}\exp(\tau/2)$ on $\{\tau < \infty\}$, we find that for $a > 1$

$$E[Z^{(a)}_\infty] \leqq e^{-a} g(0, 0) \leqq e^{1-a} < 1\,.$$

Namely, $Z^{(a)} \notin \mathscr{M}_u$ for $a > 1$. This implies that $g(\beta, \psi) = \infty$ for any $\psi \in \Phi$ and any number $\beta > 1$.

EXAMPLE 4. Let now $M = B^\nu$, where $\nu = \inf\{t \geqq 0; B_t \geqq t+1\}$. Obviously, ν is the stopping time obtained by setting $\phi = 0$ and $a = 1$ in (8). Then $g(2, 0) < \infty$ by Example 2. Therefore, from Theorem 3 and Proposition 5 it follows that $Z^{(\beta)} \in \mathscr{M}_u$ for all β with $1 \leqq \beta \leqq 2$. On the other hand, since $Z_\infty = \exp(1 + \nu/2)$ on $\{\nu < \infty\}$, we find

$$\begin{aligned} E[Z^{(a)}_\infty; \nu < \infty] &= E\left[\exp\left\{\frac{1}{2}a(2-a)\nu + a\right\}; \nu < \infty\right] \\ &\leqq e^a E[\exp(\nu/2); \nu < \infty] \leqq e^{a-1}\,. \end{aligned}$$

Thus $Z^{(a)} \notin \mathscr{M}_u$ if $a < 1$. This implies that $g(\alpha, \psi) = \infty$ for any $\psi \in \Phi$ and any number $\alpha < 1$. By this example it comes out that Theorem 3 properly contains the criteria of Okada and Novikov.

EXAMPLE 5. Let $B = (B_t)$ and $B = (\tilde{B}_t)$ be two independent Brownian motions, and define the stopping times $\tau = \inf\{t \geqq 0; B_t \leqq t-1\}$ and $\nu = \inf\{t \geqq 0; \tilde{B}_t \geqq t+1\}$ as before. We have already seen in Examples

3 and 4 that the exponential processes $X = \exp(B^\tau - \langle B^\tau\rangle/2)$ and $Y = \exp(\tilde{B}^\nu - \langle \tilde{B}^\nu\rangle/2)$ are uniformly integrable martingales. Consider now the continuous martingale $M \equiv B^\tau + \tilde{B}^\nu$. From the independence of B and $\tilde{B}$ it follows immediately that $\langle M\rangle_t = t\wedge\tau + t\wedge\nu$. Therefore, the exponential martingale Z associated with M is equal to XY, and then we have $E[Z_\infty] = E[X_\infty]E[Y_\infty] = 1$ by the independence. Namely, $Z \in \mathscr{M}_u$. However, for $a > 1$, $E[\exp(aB_\tau - a^2\tau/2)] < 1$ by Example 3. On the other hand, for $a < 1$, $E[\exp(a\tilde{B}_\nu - a^2\nu/2)] < 1$ by Example 4. Therefore, $E[Z_\infty^{(a)}] < 1$ for all $a \neq 1$, which implies that $g(\alpha, \phi) = \infty$ for any $\phi \in \Phi$ and any number $\alpha \neq 1$. Thus the converse of Theorem 3 does not hold.

6. Remarks on a BMO-martingale. The purpose of this section is to explain an interesting role played by a BMO-martingale in the problem about the uniform integrability of exponential martingales. Recall that a continuous local martingale M is said to be in the class BMO if $E[\langle M\rangle_\infty - \langle M\rangle_T | F_T] \leqq C$ for any stopping time T, where C is an absolute constant. As is well-known, the space BMO is a Banach space if we let $\|M\|_{\mathrm{BMO}} = \sup_T \|E[\langle M\rangle_\infty - \langle M\rangle_T | F_T]^{1/2}\|_\infty$ be its norm.

We remark first that the exponential process Z associated with a BMO-martingale M is a uniformly integrable martingale. In fact, $Z_T > 0$ for any stopping time T and, since M is obviously in the class $\mathscr{M}_u$, we have by Jensen's inequality

$$E[Z_\infty/Z_T | F_T] \geqq \exp\left\{E\left[M_\infty - M_T - \frac{1}{2}(\langle M\rangle_\infty - \langle M\rangle_T)\Big| F_T\right]\right\}$$
$$\geqq \exp\left\{-\frac{1}{2}\|M\|_{\mathrm{BMO}}^2\right\},$$

which implies that $Z \in \mathscr{M}_u$. We claim here that the following remarkable result holds further.

THEOREM 6. *If $M \in \mathrm{BMO}$, then $g(\alpha, 0) < \infty$ for some $\alpha \neq 1$.*

The proof which follows is rather long: it must be possible to give a shorter proof, but regretfully we did not succeed in it.

LEMMA 7. *If $\|X\|_{\mathrm{BMO}} < 1$, then for any stopping time T we have*

$$E[\exp(\langle X\rangle_\infty - \langle X\rangle_T) | F_T] \leqq (1 - \|X\|_{\mathrm{BMO}}^2)^{-1}.$$

This inequality was first established by Garsia for discrete parameter martingales. For the proof, see [1].

LEMMA 8. *Let $M \in \mathrm{BMO}$ and $p = (1 + 2\|M\|_{\mathrm{BMO}})^2$. Then for any α with $|\alpha| \leqq 2$ we have*

$$\sup_T E[\{Z_T^{(\alpha)}/Z_\infty^{(\alpha)}\}^{1/(p-1)} \mid F_T] \leqq 2^{\sqrt{p}/(\sqrt{p}+1)} .$$

PROOF. This is not necessarily a well-known fact, though it was essentially proved in [5] or [6]. For that reason, we shall sketch its proof. We may assume that $\|M\|_{\text{BMO}} > 0$. Let now $r = \sqrt{p}+1$, $s = (\sqrt{p}+1)/\sqrt{p}$ and $b = -\alpha r/(p-1)$. If $|\alpha| \leqq 2$, then $\alpha^2\|M\|^2_{\text{BMO}}/\{2(\sqrt{p}-1)^2\} \leqq 1/2$. On the other hand, by a simple calculation we have $1/\{s(\sqrt{p}-1)^2\} - r/(p-1)^2 = 1/(p-1)$. So, by using Hölder's inequality

$$\begin{aligned} &E[\{Z_T^{(\alpha)}/Z_\infty^{(\alpha)}\}^{1/(p-1)} \mid F_T] \\ &\quad = E\left[\{Z_\infty^{(b)}/Z_T^{(b)}\}^{1/r} \exp\left\{\frac{\alpha^2}{2s(\sqrt{p}-1)^2}(\langle M\rangle_\infty - \langle M\rangle_T)\right\} \middle| F_T\right] \\ &\quad \leqq E[Z_\infty^{(b)}/Z_T^{(b)} \mid F_T]^{1/r} E\left[\exp\left\{\frac{\alpha^2}{2(\sqrt{p}-1)^2}(\langle M\rangle_\infty - \langle M\rangle_T)\right\} \middle| F_T\right]^{1/s} . \end{aligned}$$

Noticing $Z^{(b)} \in \mathcal{M}_u$ and applying Lemma 7 we find that the last expression is smaller than

$$\left\{1 - \frac{\alpha^2}{2(\sqrt{p}-1)^2}\|M\|^2_{\text{BMO}}\right\}^{-1/s} \leqq 2^{\sqrt{p}/(\sqrt{p}+1)} ,$$

which completes the proof.

In the next place, let $K_\varepsilon = 2\varepsilon(1+\varepsilon)^{-1}2^{(1+\varepsilon)(2p-\sqrt{p})}$ where $p = (1 + 2\|M\|_{\text{BMO}})^2$ as above, and choose $\varepsilon > 0$ such that $K_\varepsilon < 1$. Then we have the following.

LEMMA 9.

$$\sup_{|\alpha| \leqq 2} E[\{Z_\infty^{(\alpha)}\}^{1+\varepsilon}] \leqq \frac{2}{1-K_\varepsilon} .$$

PROOF. Let us assume that $0 < \|M\|_{\text{BMO}} < \infty$. Noticing $Z^{(\alpha)} \in \mathcal{M}_u$ and using Jensen's inequality we have

$$\{Z_S^{(\alpha)}\}^{-1/(p-1)} \leqq E[\{Z_\infty^{(\alpha)}\}^{-1/(p-1)} \mid F_S]$$

for any stopping time S. Then we apply Lemma 7 to the right hand side

$$E[\{Z_{S\wedge T}^{(\alpha)}/Z_S^{(\alpha)}\}^{1/(p-1)} \mid F_{S\wedge T}] \leqq 2^{\sqrt{p}/(\sqrt{p}+1)} \quad (|\alpha| \leqq 2) .$$

Thus the usual stopping argument enables us to assume the boundedness of $Z^{(\alpha)}$ in advance. First of all we shall show the basic inequality

$$E[Z_\infty^{(\alpha)}; Z_\infty^{(\alpha)} > \lambda] \leqq 2\lambda P\left\{Z_\infty^{(\alpha)} > \frac{\lambda}{a}\right\} \qquad (\lambda > 0) , \tag{12}$$

where $a = 2^{2p-\sqrt{p}} > 1$. For that, let $T = \inf\{t \geqq 0; Z_t^{(\alpha)} > \lambda\}$, and consider the bounded martingale X defined by $X_t = P\{Z_\infty^{(\alpha)} \leqq Z_T^{(\alpha)}/a \mid F_t\}$. Then by using Hölder's inequality and Lemma 8

$$\begin{aligned} X_T^p &= E[\{Z_T^{(\alpha)}/Z_\infty^{(\alpha)}\}^{1/p}\{Z_\infty^{(\alpha)}/Z_T^{(\alpha)}\}^{1/p}X_\infty \mid F_T]^p \\ &\leqq E[\{Z_T^{(\alpha)}/Z_\infty^{(\alpha)}\}^{1/(p-1)} \mid F_T]^{p-1}E[I_{\{Z_\infty^{(\alpha)} \leqq Z_T^{(\alpha)}/a\}}Z_\infty^{(\alpha)}/Z_T^{(\alpha)} \mid F_T] \\ &\leqq 2^{\sqrt{p}(p-1)/(\sqrt{p}+1)}a^{-1} = 2^{-p}, \end{aligned}$$

from which $X_T \leqq 1/2$. Therefore, we have

$$\frac{1}{2}I_{\{T<\infty\}} \leqq P\left\{Z_\infty^{(\alpha)} > \frac{1}{a}Z_T^{(\alpha)},\ T < \infty \mid F_T\right\}$$

and so

$$E[Z_\infty^{(\alpha)}; Z_\infty^{(\alpha)} > \lambda] = E[Z_T^{(\alpha)}; T < \infty] \leqq \lambda P(T < \infty) \leqq 2\lambda P\{Z_\infty^{(\alpha)} > \lambda/a\}.$$

Finally, multiplying both sides of (12) by $\varepsilon\lambda^{\varepsilon-1}$ and integrating with respect to λ on the interval $[1, \infty[$, we find

$$E[\{Z_\infty^{(\alpha)}\}^{1+\varepsilon} - Z_\infty^{(\alpha)}; Z_\infty^{(\alpha)} > 1] \leqq K_\varepsilon E[\{Z_\infty^{(\alpha)}\}^{1+\varepsilon}; Z_\infty^{(\alpha)} > 1] + K_\varepsilon.$$

That is, $(1 - K_\varepsilon)E[\{Z_\infty^{(\alpha)}\}^{1+\varepsilon}; Z_\infty^{(\alpha)} > 1] \leqq K_\varepsilon + E[Z_\infty^{(\alpha)}; Z_\infty^{(\alpha)} > 1] \leqq K_\varepsilon + 1$, which yields

$$E[\{Z_\infty^{(\alpha)}\}^{1+\varepsilon}] \leqq \frac{2}{1 - K_\varepsilon} \quad (|\alpha| \leqq 2).$$

This completes the proof.

PROOF OF THEOREM 6. We may assume that $\|M\|_{\text{BMO}} > 0$. Let $|\alpha| \leqq 2$. Then $\|Z_\infty^{(\alpha)}\|_{1+\varepsilon} \leqq C_\varepsilon$ for some $\varepsilon > 0$ by Lemma 9. Moreover, we can easily see that $G_\alpha(T, 0) = Z_T^{(\alpha)} \exp\{(1 - \alpha)^2\langle M\rangle_T/2\}$ for every stopping time T, and so we apply Hölder's inequality with exponents $1 + \varepsilon$ and $(1 + \varepsilon)/\varepsilon$ to the right hand side:

$$E[G_\alpha(T, 0)] \leqq \|Z_T^{(\alpha)}\|_{1+\varepsilon}E\left[\exp\left\{\frac{1 + \varepsilon}{2\varepsilon}(1 - \alpha)^2\langle M\rangle_T\right\}\right]^{\varepsilon/(1+\varepsilon)}.$$

Let now $0 < \delta < \min\{1, \sqrt{\varepsilon}/(\sqrt{1 + \varepsilon}\|M\|_{\text{BMO}})\}$. Then, using Lemma 7 we find that for any α with $|1-\alpha|<\delta$ the expectation on the right hand side is bounded by 2, because $(1 + \varepsilon)(1 - \alpha)^2\|M\|_{\text{BMO}}^2 \leqq \varepsilon$. Consequently, $g(\alpha, 0) < \infty$ for all α with $|1 - \alpha| < \delta$.

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