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BOUNDEDNESS OF SOME OPERATORS COMPOSED OF FOURIER MULTIPLIERS

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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Introduction and notations. We will consider the transplantation theorems for the operators defined by Fourier multipliers.

We will use the notations and conventions as follows.

$\boldsymbol{R}^n$ denotes the n-dimensional Euclidean space and $\boldsymbol{Q}^n$ the unit cube $\{\theta = (\theta_1, \cdots, \theta_n) \in \boldsymbol{R}^n; -1/2 \leqq \theta_j < 1/2\ (j = 1, \cdots, n)\}$. $\boldsymbol{Q}^n$ is identified with the n-dimensional torus $\boldsymbol{T}^n$. The dual of $\boldsymbol{R}^n$ is denoted by $\hat{\boldsymbol{R}}^n$ and the totality of all lattice points with integral coordinates in $\hat{\boldsymbol{R}}^n$ is denoted by $\boldsymbol{Z}^n$, which is the dual of $\boldsymbol{T}^n$.

The Fourier transform $\hat{f}$ of $f \in L^1(\boldsymbol{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\boldsymbol{R}^n} f(x) e(-x\xi) dx ,$$

where $e(t) = \exp(2\pi i t)$, $x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n$, $\xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n$ and $x\xi = \xi x = \sum_{j=1}^n x_j \xi_j$. $g^\vee$ denotes the inverse Fourier transform of g. The Fourier coefficients $\hat{F}(m)$ $(m \in \boldsymbol{Z}^n)$ of $F \in L^1(\boldsymbol{T}^n)$ are defined by

$$\hat{F}(m) = \int_{Q^n} F(\theta) e(-m\theta) d\theta .$$

For a bounded function λ on $\hat{\boldsymbol{R}}^n$, the operator T_λ is defined as follows. Let $f \in \mathscr{S}(\boldsymbol{R}^n)$, where $\mathscr{S}(\boldsymbol{R}^n)$ denotes the Schwartz class. $T_\lambda f$ is defined by

$$(T_\lambda f)(x) = \int_{\hat{\boldsymbol{R}}^n} \lambda(\xi) \hat{f}(\xi) e(x\xi) d\xi .$$

On the other hand, for an indefinitely differentiable periodic function $F \in C^\infty(\boldsymbol{T}^n)$, $\tilde{T}_\lambda F$ is defined by $(\tilde{T}_\lambda F)(\theta) = \sum_{m \in \boldsymbol{Z}^n} \lambda(m) \hat{F}(m) e(m\theta)$. The operators T_λ and $\tilde{T}_\lambda$ are usually called Fourier multiplier operators defined by λ and the sequence $\{\lambda(m)\}$, respectively. The extensions of T_λ and $\tilde{T}_\lambda$ to $L^p(\boldsymbol{R}^n)$ and $L^p(\boldsymbol{T}^n)$, respectively, will be denoted by the same notations.

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By a theorem of de Leeuw [20], *if* λ *is regulated and* T_λ *is bounded on* $L^p(\boldsymbol{R})$, *then* $\tilde{T}_\lambda$ *is bounded on* $L^p(\boldsymbol{T})$. Conversely, *if* λ *is continuous a.e. and if* $\tilde{T}_{\lambda(\varepsilon\cdot)}$ *is bounded on* $L^p(\boldsymbol{T}^n)$ *for any* $\varepsilon > 0$ *and the operator norms of* $\tilde{T}_{\lambda(\varepsilon\cdot)}$ *are uniformly bounded with respect to* ε, *then* T_λ *is bounded on* $L^p(\boldsymbol{R}^n)$ (Igari [13], Stein and Weiss [23, pp. 260–267]). The last result is extended to the boundedness from L^p to L^q by Jodeit [17]. The former is treated in a more abstract setting by Coifman and Weiss [5]. Replacement of dilations by translations in the above argument is studied by Coifman and Meyer [4], and they treat also Hardy class H^1 there; see also Goldberg [10].

Let T^* and $\tilde{T}^*$ be the maximal operators defined by the families $\{T_{\lambda(\cdot/R)};\, R > 0\}$ and $\{\tilde{T}_{\lambda(\cdot/R)};\, R > 0\}$, respectively. Kenig and Tomas [19] have proved *the equivalence between the boundedness of* T^* *and that of* $\tilde{T}^*$. They have used duality argument in the L^p-theory. We shall try to take a direct approach, which seems to be more fruitful.

Let $(\Gamma_j, \mathscr{M}_j, \mu_j)$ and $(\Gamma_j, \mathscr{N}_j, \nu_j)$ $(j = 1, \cdots, N)$ be sequences of totally σ-finite measure spaces such that $\mathscr{M}_j \subset \mathscr{N}_j$ $(j = 1, \cdots, N)$. Let $(\Gamma, \mathscr{M}, \mu)$ and $(\Gamma, \mathscr{N}, \nu)$ be the product measure spaces of the families $(\Gamma_j, \mathscr{M}_j, \mu_j)$ and $(\Gamma_j, \mathscr{N}_j, \nu_j)$, respectively. Let $P = (p_1, \cdots, p_N)$ and $Q = (q_1, \cdots, q_N)$, $1 \leqq p_j, q_j \leqq \infty$ $(j = 1, \cdots, N)$, be multi-indices. We denote the mixed normed spaces $L^P(\Gamma, \mathscr{M}, \mu)$ and $L^Q(\Gamma, \mathscr{N}, \nu)$ by $\mathscr{A}$ and $\mathscr{B}$, respectively (cf. Benedek and Panzone [1]). For an $\mathscr{M}$-measurable function f, we denote the mixed $L^P(\Gamma, \mathscr{M}, \mu)$-norm of f;

$$\left(\int_{\Gamma_N}\left(\cdots\left(\int_{\Gamma_1}|f(\gamma_1, \cdots, \gamma_N)|^{p_1}d\mu_1(\gamma_1)\right)^{p_2/p_1}\cdots\right)^{p_N/p_{N-1}}d\mu_N(\gamma_N)\right)^{1/p_N}$$

by $\|f\|_{\mathscr{A}}$. The case where $p_j = \infty$ will be modified in an obvious way. Similarly $\|g\|_{\mathscr{B}}$ is defined in the same manner for an $\mathscr{N}$-measurable function g.

We consider the Lebesgue measures on $\boldsymbol{R}^n$ and $\hat{\boldsymbol{R}}^n$, and denote by $\mathscr{L}$ and $\hat{\mathscr{L}}$ the families of all Lebesgue measurable sets on $\boldsymbol{R}^n$ and $\hat{\boldsymbol{R}}^n$, respectively.

For an $(\mathscr{L} \times \mathscr{M})$-measurable function f on $\boldsymbol{R}^n \times \Gamma$, $\|f\|_{L^p(\boldsymbol{R}^n, \mathscr{A})}$, $0 < p < \infty$, is defined by

$$\|f\|_{L^p(\boldsymbol{R}^n, \mathscr{A})} = \left(\int_{\boldsymbol{R}^n}\|f(x, \cdot)\|_{\mathscr{A}}^p dx\right)^{1/p} .$$

The definition for $p = \infty$ will be obvious.

For simplicity, we introduce the class $\mathscr{C}(\Gamma, \mathscr{F})$. For a class $\mathscr{F}$ of scalar valued functions on $\boldsymbol{R}^n$ or $\boldsymbol{T}^n$, $\mathscr{C}(\Gamma, \mathscr{F})$ denotes the class of all $(\mathscr{L} \times \mathscr{M})$-measurable functions f defined on $\boldsymbol{R}^n \times \Gamma$ or $\boldsymbol{T}^n \times \Gamma$ such that $f(\cdot, \gamma) \in \mathscr{F}$ for each $\gamma \in \Gamma$. $f(\cdot, \gamma)$ will be frequently denoted by f_γ.

$C_0^\infty(\boldsymbol{R}^n)$ denotes the class of all infinitely differentiable functions with compact support and $\mathscr{P}$ denotes the class of all trigonometric polynomials.

We will use the letter C for a constant, which may be different in each occurence, but specific constants will be denoted by the letters A and B.

For a measurable set E, $|E|$ denotes the Lebesgue measure of E.

We will discuss the boundedness of some operators from $L^p(\boldsymbol{R}^n, \mathscr{A})$ to $L^p(\boldsymbol{R}^n, \mathscr{B})$ or from $L^p(\boldsymbol{T}^n, \mathscr{A})$ to $L^p(\boldsymbol{T}^n, \mathscr{B})$ and the weak type estimates of such operators in the following sections. In §1, we will show that the boundedness of some operators from $L^p(\boldsymbol{R}^n, \mathscr{A})$ to $L^p(\boldsymbol{R}^n, \mathscr{B})$ induces the boundedness of corresponding operators from $L^p(\boldsymbol{T}^n, \mathscr{A})$ to $L^p(\boldsymbol{T}^n, \mathscr{B})$ and also treat the weak type cases. The converse case will be discussed in §2. In §3, the Littlewood-Paley g^*-functions will be systematically discussed, which have been studied in the unit disc $\boldsymbol{D} = \{z \in \boldsymbol{C}; |z| < 1\}$ and the upper-half plane $\boldsymbol{H} = \{z \in \boldsymbol{C}; \operatorname{Im} z > 0\}$ separately in most cases, where $\boldsymbol{C}$ denotes the complex plane. In the last section, the a.e. convergence of the lacunary partial means of $\hat{f}(\xi)e(x\xi)$ for $f \in H^1(\boldsymbol{R})$ will be discussed.

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The following paper came to my attention after the preparation of this paper: Schmeisser and Sickel, On strong summability of multiple Fourier series and smoothness properties of functions, Anal. Math. 8 (1982), 57–70. They have obtained a theorem for a Fourier multiplier matrix, which is more general than Th. 1 (i), if $\mathscr{A}$ and $\mathscr{B}$ are sequence spaces and $1 < p < \infty$.

1. The transplantation from $\boldsymbol{R}^n$ to $\boldsymbol{T}^n$. For a given $(\hat{\mathscr{L}} \times \mathscr{N})$-measurable function λ, such that $\|\lambda(\cdot, \gamma)\|_\infty < \infty$ for each $\gamma \in \Gamma$, $g = T_\lambda f$ is defined by $g(x, \gamma) = (T_{\lambda(\cdot,\gamma)} f_\gamma)(x)$ for $f \in \mathscr{C}(\Gamma, \mathscr{S}(\boldsymbol{R}^n))$, and $G = \tilde{T}_\lambda F$ is defined by $G(\theta, \gamma) = (\tilde{T}_{\lambda(\cdot,\gamma)} F_\gamma)(\theta)$ for $F \in \mathscr{C}(\Gamma, C^\infty(\boldsymbol{T}^n))$. In this section, we prove the following theorem.

THEOREM 1. *Assume that an $(\hat{\mathscr{L}} \times \mathscr{N})$-measurable function λ on $\hat{\boldsymbol{R}}^n \times \Gamma$ satisfies the following conditions. $\lambda(\cdot, \gamma)$ is bounded for every $\gamma \in \Gamma$, and there exist $\Phi \in L^1(\hat{\boldsymbol{R}}^n)$ and an $(\hat{\mathscr{L}} \times \mathscr{N})$-measurable function ϕ such that $\{(\phi_\gamma)_\varepsilon * \lambda(\cdot, \gamma)\}(m) \to \lambda(m, \gamma)$ as $\varepsilon \to 0$ for all $(m, \gamma) \in \boldsymbol{Z}^n \times \Gamma$ and $|\phi(\xi, \gamma)| \leqq \Phi(\xi)$ for all $(\xi, \gamma) \in \hat{\boldsymbol{R}}^n \times \Gamma$, where $(\phi_\gamma)_\varepsilon(\xi) = \varepsilon^{-n}\phi(\varepsilon^{-1}\xi, \gamma)$. Then we have the following* (i) *and* (ii) *for $T = T_\lambda$ and $\tilde{T} = \tilde{T}_\lambda$*

(i) *Assume $1 \leqq p \leqq \infty$. If $\|Tf\|_{L^p(\boldsymbol{R}^n, \mathscr{B})} \leqq A\|f\|_{L^p(\boldsymbol{R}^n, \mathscr{A})}$ $(f \in \mathscr{C}(\Gamma,$*

$C_0^\infty(\boldsymbol{R}^n))$), *then*

$$\|\tilde{T}F\|_{L^p(T^n,\mathscr{B})} \leqq AB\|F\|_{L^p(T^n,\mathscr{A})} \quad (F \in \mathscr{E}(\Gamma, \mathscr{P}))\ ,$$

where B *is the* $L^1(\hat{\boldsymbol{R}}^n)$-*norm of* Φ.

(ii) *Assume that* $1 < p < \infty$. *If*

$$|\{x \in \boldsymbol{R}^n;\ \|(Tf)(x, \cdot)\|_{\mathscr{B}} > t\}| \leqq [At^{-1}\|f\|_{L^p(\boldsymbol{R}^n,\mathscr{A})}]^p$$

for all $t > 0$ *and* $f \in \mathscr{E}(\Gamma, C_0^\infty(\boldsymbol{R}^n))$, *then*

$$|\{\theta \in \boldsymbol{Q}^n;\ \|(\tilde{T}F)(\theta, \cdot)\|_{\mathscr{B}} > t\}| \leqq [\{p/(p-1)\}ABt^{-1}\|F\|_{L^p(T^n,\mathscr{A})}]^p$$

for all $t > 0$ *and* $F \in \mathscr{E}(\Gamma, \mathscr{P})$.

Our proof proceeds along the line of Calderón [2] and Coifman and Weiss [5].

LEMMA 1. *Suppose that* $k \in C_0^\infty(\boldsymbol{R}^n)$ *has the support in* $B(R_0) = \{x;\ |x| \leqq R_0\}$ *and* $K \in C^\infty(\boldsymbol{T}^n)$ *is defined by*

$$K(\theta) = \sum_{m \in \boldsymbol{Z}^n} k(\theta + m)\ .$$

Let $R > 0$ *and* $\chi = \chi_R$ *be a function on* $\boldsymbol{R}^n$ *such that* $\chi(x) = 1$ $(|x| \leqq R_0 + R)$. *Then*

$$(K*F)(\theta + x) = [k*\{F(\theta + \cdot)\chi\}](x)$$

for $|x| \leqq R$, $\theta \in \boldsymbol{Q}^n$ *and* $F \in L^1(\boldsymbol{T}^n)$.

PROOF. By the definition of K, the left hand side equals

$$\int_{B(R_0)} k(y)F(\theta + x - y)dy\ .$$

Since $\chi(x - y) = 1$ $(|x| \leqq R,\ |y| \leqq R_0)$, the above integral coincides with the right hand side.

LEMMA 2. *Let* $\lambda \in L^\infty(\hat{\boldsymbol{R}}^n)$ *and assume that* ψ *and* h *are in* $C_0^\infty(\boldsymbol{R}^n)$. *If* $k = \{(\lambda*\hat{\psi})\hat{h}\}^\vee$, then $k \in C_0^\infty(\boldsymbol{R}^n)$ *and* $\operatorname{supp} k \subset (\operatorname{supp}\psi) + (\operatorname{supp} h)$.

The proof is obvious.

LEMMA 3. *Assume that* $\lambda \in L^\infty(\hat{\boldsymbol{R}}^n)$, $\phi \in L^1(\hat{\boldsymbol{R}}^n)$ *and* $\psi, h \in C_0^\infty(\boldsymbol{R}^n)$. *Set* $k = \{(\phi*\lambda*\hat{\psi})\hat{h})\}^\vee$. *Then, for any* $f \in \mathscr{S}(\boldsymbol{R}^n)$ *and* $x \in \boldsymbol{R}^n$,

$$(k*f)(x) = \int_{\hat{\boldsymbol{R}}^n} (\phi*\hat{\psi})(\xi)e(x\xi)\{T_\lambda(e(-\cdot\xi)(h*f))\}(x)d\xi\ .$$

PROOF. By the Plancherel theorem and an interchange of the order of integrations, $(k*f)(x)$ equals

$$\int_{\hat{\boldsymbol{R}}^n} (\phi*\hat{\psi})(\xi)d\xi \int_{\hat{\boldsymbol{R}}^n} \lambda(\zeta - \xi)\hat{h}(\zeta)\hat{f}(\zeta)e(x\zeta)d\zeta\ .$$

The inner integral turns out to be

$$e(x\xi)\int_{\hat{R}^n}\lambda(\tau)\hat{h}(\tau+\xi)\hat{f}(\tau+\xi)e(x\tau)d\tau ,$$

which is equal to $e(x\xi)\{T_\lambda(e(-\cdot\xi)(h*f))\}(x)$.

We state briefly the definition of the Lorentz spaces $L(p, q)$ and some of their properties according to Hunt [12], which will be used in the following proof of weak type result. Let (M, m) be a totally σ-finite measure space. Let f^* be the non-increasing rearrangement of an m-measurable function f on M into $(0, \infty)$. Then $\|f\|^*_{p,q}$ is defined by

$$\|f\|^*_{p,q}=\left[(q/p)\int_0^\infty\{t^{1/p}f^*(t)\}^q t^{-1}dt\right]^{1/q}$$

for $0<p<\infty$ and $0<q<\infty$, and $\|f\|^*_{p,\infty}=\sup_{t>0}\{t^{1/p}f^*(t)\}$ for $0<p\leqq\infty$ and $q=\infty$. The Lorentz space $L(p, q)$ is the class of f such that $\|f\|^*_{p,q}<\infty$. On the other hand, f^{**} is defined by

$$f^{**}(t)=\sup m(E)^{-1}\int_E|f(x)|dm(x)$$

for $0<t\leqq m(M)$, where the supremum is taken over all E such that $m(E)\geqq t$, and

$$f^{**}(t)=t^{-1}\int_M|f(x)|dm(x)$$

for $t>m(M)$, and $\|f\|_{p,q}$ is defined by $\|f\|_{p,q}=\|f^{**}\|^*_{p,q}$, where $\|\cdot\|^*_{p,q}$ denotes the norm in the Lorentz space over the measure space $(0, \infty)$. Then we have $f^*(t)\leqq f^{**}(t)$ $(0<t<\infty)$ and

$$\|f\|_{p,q}\leqq\{p/(p-1)\}\|f\|^*_{p,q}\qquad(1<p\leqq\infty,\ 0<q\leqq\infty)$$

(see Hunt [12, p. 258]).

LEMMA 4. *Let (M, m) be a totally σ-finite measure space and (N, n) be a totally finite measure space. If a non-negative measurable function g on $M\times N$ satisfies $\|g(\cdot, y)\|^*_{p,q}\leqq 1$ $(y\in N)$, and if f is given by*

$$f(x)=\int_N g(x, y)dn(y) ,$$

then $\|f\|^*_{p,q}\leqq\{p/(p-1)\}n(N)$, provided that $1<p\leqq\infty$ *and* $1\leqq q\leqq\infty$.

PROOF. First, it is clear that

$$f^*(t)\leqq f^{**}(t)\leqq\int_N\{g(\cdot, y)\}^{**}(t)dn(y) .$$

Multiplying by $t^{1/p}$ and taking $L^q(dt/t)$-norms, we get

$$\|f\|_{p,q}^{*} \leqq \int_{N} \|g(\cdot, y)\|_{p,q} dn(y) .$$

Since $\|g(\cdot, y)\|_{p,q} \leqq \{p/(p-1)\} \|g(\cdot, y)\|_{p,q}^{*} \leqq p/(p-1)$, the right hand side is bounded by $\{p/(p-1)\}n(N)$. This completes the proof.

PROOF OF THEOREM 1. Assume $\psi, h \in C_0^\infty(\boldsymbol{R}^n)$ and $\psi(0) = \hat{h}(0) = 1$, and, further, assume $\hat{\psi} \geqq 0$ and $h \geqq 0$. For positive constants ε, δ and η, define $\lambda_\varepsilon^{\delta,\eta}$ by

$$\lambda_\varepsilon^{\delta,\eta}(\xi, \gamma) = [\{(\phi_\gamma)_\varepsilon * \lambda(\cdot, \gamma)\} * (\psi^\delta)^\wedge](\xi)\hat{h}_\eta(\xi) ,$$

where $\psi^\delta(x) = \psi(\delta x)$. Let $F \in \mathscr{E}(\Gamma, \mathscr{P})$ be given. Define $G_\varepsilon^{\delta,\eta}$ by

$$G_\varepsilon^{\delta,\eta}(\theta, \gamma) = \sum_{m \in Z^n} \lambda_\varepsilon^{\delta,\eta}(m, \gamma)\hat{F}_\gamma(m)e(m\theta) .$$

Since $\lambda(m, \gamma)$ is the iterated limit of $\lambda_\varepsilon^{\delta,\eta}(m, \gamma)$ as $\eta \to 0$, $\delta \to 0$ and then $\varepsilon \to 0$, $(\tilde{T}F)(\theta, \gamma)$ is equal to the iterated limit of $G_\varepsilon^{\delta,\eta}(\theta, \gamma)$ in the same order. Therefore, we have

$$(1) \qquad \|\tilde{T}F\|_{L^p(T^n, \mathscr{B})} \leqq \liminf_{\varepsilon\to 0} \liminf_{\delta\to 0} \liminf_{\eta\to 0} \|G_\varepsilon^{\delta,\eta}\|_{L^p(T^n, \mathscr{B})}$$

and

$$(2) \qquad |\{\theta \in \boldsymbol{Q}^n;\ \|(\tilde{T}F)(\theta, \cdot)\|_{\mathscr{B}} > t\}| \leqq \liminf_{\varepsilon\to 0} \liminf_{\delta\to 0} \liminf_{\eta\to 0} |\{\theta \in \boldsymbol{Q}^n;\ \|G_\varepsilon^{\delta,\eta}(\theta, \cdot)\|_{\mathscr{B}} > t\}|$$

for all $t > 0$. Fix ε, δ and $\eta > 0$, and put $G = G_\varepsilon^{\delta,\eta}$. Our next task is to estimate G. Define k_γ, $\gamma \in \Gamma$, by

$$k_\gamma(x) = \int_{\hat{R}^n} \lambda_\varepsilon^{\delta,\eta}(\xi, \gamma)e(x\xi)d\xi .$$

Then, by Lemma 2, $k_\gamma \in C_0^\infty(\boldsymbol{R}^n)$ and $\operatorname{supp} k_\gamma \subset (\operatorname{supp} \psi^\delta) + (\operatorname{supp} h_\eta)$. We remark that the set on the right hand side is independent of γ. Define $K_\gamma \in C^\infty(\boldsymbol{T}^n)$ by $K_\gamma(\theta) = \sum_{m \in Z^n} k_\gamma(\theta + m)$. Take $R_0 > 0$ such that $(\operatorname{supp} \psi^\delta) + (\operatorname{supp} h_\eta) \subset B(R_0)$. Then $\operatorname{supp} k_\gamma \subset B(R_0)$ $(\gamma \in \Gamma)$. For an $R > 0$, take a function $\chi = \chi_R$ such that $\chi \in C_0^\infty(\boldsymbol{R}^n)$, $0 \leqq \chi \leqq 1$, $\chi(x) = 1$ $(|x| \leqq R_0 + R)$ and $\chi(x) = 0$ $(|x| \geqq R_0 + R + 1)$. Since $G = K_\gamma * F_\gamma$, Lemmas 1 and 3 imply

$$G(\theta + x, \gamma) = \int_{\hat{R}^n} \{(\phi_\gamma)_\varepsilon * (\psi^\delta)^\wedge\}(\xi)e(x\xi)(Tf_{\theta,\xi})(x, \gamma)d\xi$$

for $|x| \leqq R$ and $\theta \in \boldsymbol{Q}^n$, where

$$f_{\theta,\xi}(x, \gamma) = e(-x\xi)[h_\eta * \{F_\gamma(\theta + \cdot)\chi\}](x) .$$

Taking the $\mathscr{B}$- and $\mathscr{A}$-norms with respect to γ, we have

$$(3) \qquad \|G(\theta + x, \cdot)\|_{\mathscr{B}} \leqq \int_{\hat{R}_n} \{\Phi_\varepsilon * (\psi^\delta)^\wedge\}(\xi) \|(Tf_{\theta,\xi})(x, \cdot)\|_{\mathscr{B}} d\xi$$

for all $\theta \in \boldsymbol{Q}^n$ and all x such that $|x| \leqq R$, and

$$(4) \qquad \|f_{\theta,\xi}(x, \cdot)\|_{\mathscr{A}} \leqq \int_{R^n} h_\eta(y) \chi(x - y) \|F(\theta + x - y, \cdot)\|_{\mathscr{A}} dy$$

for all $x \in \boldsymbol{R}^n$. Now we divide the proof into two cases.

Proof of (i). First, assume $p \neq \infty$. Using the periodicity and applying Jensen's inequality to (3), we have

$$\begin{aligned} \|G\|^p_{L^p(T^n,\mathscr{B})} &= \int_{Q^n} \|G(\theta + x, \cdot)\|^p_{\mathscr{B}} d\theta \\ &\leqq B^{p-1} \int_{\hat{R}^n} \{\Phi_\varepsilon * (\psi^\delta)^\wedge\}(\xi) d\xi \int_{Q^n} \|(Tf_{\theta,\xi})(x, \cdot)\|^p_{\mathscr{B}} d\theta \end{aligned}$$

for $|x| \leqq R$. Integrating each term over $B(R)$ with respect to x, we have

$$(5) \qquad |B(R)| \, \|G\|^p_{L^p(T^n,\mathscr{B})} \leqq B^{p-1} \int_{\hat{R}^n} \{\Phi_\varepsilon * (\psi^\delta)^\wedge\}(\xi) d\xi \int_{Q^n} \|Tf_{\theta,\xi}\|^p_{L^p(R^n,\mathscr{B})} d\theta \, .$$

By the hypothesis of (i),

$$(6) \qquad \|Tf_{\theta,\xi}\|^p_{L^p(R^n,\mathscr{B})} \leqq A^p \|f_{\theta,\xi}\|^p_{L^p(R^n,\mathscr{A})} \, .$$

Applying Jensen's inequality to (4) and using the properties of h and χ, we have

$$(7) \qquad \|f_{\theta,\xi}\|^p_{L^p(R^n,\mathscr{A})} \leqq \int_{B(R_0+R+1)} \|F(\theta + x, \cdot)\|^p_{\mathscr{A}} dx \, .$$

Applying (7) to the right hand side of (6) and integrating both sides over $\boldsymbol{Q}^n$ with respect to θ, we have

$$\int_{Q^n} \|Tf_{\theta,\xi}\|^p_{L^p(R^n,\mathscr{B})} d\theta \leqq A^p |B(R_0 + R + 1)| \, \|F\|^p_{L^p(T^n,\mathscr{A})} \, .$$

In the last inequality, the periodicity of F has been used. Using this relation along with (5), we get

$$(8) \qquad \|G\|_{L^p(T^n,\mathscr{B})} \leqq AB \|F\|_{L^p(T^n,\mathscr{A})} \{|B(R_0 + R + 1)|/|B(R)|\}^{1/p} \, .$$

In the case $p = \infty$, (8) is directly obtained from (3) and (4). Letting R tend to ∞ in (8), we have

$$\|G^{\delta,\eta}_\varepsilon\|_{L^p(T^n,\mathscr{B})} = \|G\|_{L^p(T^n,\mathscr{B})} \leqq AB \|F\|_{L^p(T^n,\mathscr{A})} \, .$$

From the last inequality and (1), we have the conclusion of (i).

Proof of (ii). For a given $t > 0$, put

$$E_0 = \{\theta \in \boldsymbol{Q}^n; \, \|G(\theta, \cdot)\|_{\mathscr{B}} > t\} \, .$$

By the periodicity of G,

$$|E_0| = |\{\theta \in \boldsymbol{Q}^n;\ \|G(\theta + x, \cdot)\|_{\mathscr{B}} > t\}| \tag{9}$$

for all $x \in \boldsymbol{R}^n$. Therefore, if we put

$$E = \{(\theta, x) \in \boldsymbol{Q}^n \times B(R);\ \|G(\theta + x, \cdot)\|_{\mathscr{B}} > t\}\ ,$$

$$E(\theta) = \{x \in B(R);\ (\theta, x) \in E\} \quad \text{and} \quad E(x) = \{\theta \in \boldsymbol{Q}^n;\ (\theta, x) \in E\}\ ,$$

then (9) is equivalent to $|E_0| = |E(x)|$, and we have

$$|B(R)|\,|E_0| = |E| = \int_{Q^n} |E(\theta)|\,d\theta\ . \tag{10}$$

For a fixed θ, put

$$g'(x, \xi) = \|(Tf_{\theta,\varepsilon})(x, \cdot)\|_{\mathscr{B}} \quad \text{and} \quad f'(x) = \int_{\hat{R}^n} g'(x, \xi)\{\Phi_\varepsilon * (\psi^\delta)^\wedge\}(\xi)d\xi\ .$$

Since $\|G(\theta + x, \cdot)\|_{\mathscr{B}} \leqq f'(x)$ by (3),

$$E(\theta) \subset \{x \in \boldsymbol{R}^n;\ f'(x) > t\}\ . \tag{11}$$

The hypothesis of (ii) implies $\|g'(\cdot, \xi)\|^*_{p,\infty} \leqq A\|f_{\theta,\xi}\|_{L^p(R^n,\mathscr{A})}$. The last term is bounded by the right hand side of (7). Applying Lemma 4, we have

$$\|f'\|^*_{p,\infty} \leqq \{p/(p-1)\}AB\left(\int_{B(R_0+R+1)} \|F(\theta + x, \cdot)\|^p_{\mathscr{A}}dx\right)^{1/p}\ .$$

By (11), we have

$$|E(\theta)| \leqq [\{p/(p-1)\}ABt^{-1}]^p \int_{B(R_0+R+1)} \|F(\theta + x, \cdot)\|^p_{\mathscr{A}}dx\ .$$

Integrating both sides of the last inequality and using (10), we have

$$|E_0| \leqq [\{p/(p-1)\}ABt^{-1}\|F\|_{L^p(T^n,\mathscr{A})}]^p\{|B(R_0 + R + 1)|/|B(R)|\}\ ,$$

by the periodicity of F. Letting $R \to \infty$ and then applying (2), we obtain (ii).

If $\mathscr{C}(\Gamma, \mathscr{P})$ is dense in $L^p(\boldsymbol{T}^n, \mathscr{A})$, $(\tilde{T}_\lambda F)(\theta, \gamma)$ is defined for all $F \in L^p(\boldsymbol{T}^n, \mathscr{A})$ at $(d\theta \times d\nu(\gamma))$-a.e. point (θ, γ), and if $F_j \to F$ in $L^p(\boldsymbol{T}^n, \mathscr{A})$ as $j \to \infty$ implies $(\tilde{T}_\lambda F_j)(\theta, \gamma) \to (\tilde{T}_\lambda F)(\theta, \gamma)$ $(d\theta \times d\nu(\gamma))$-a.e. as $j \to \infty$, then the conclusions of (i) and (ii) in Theorem 1 are true for all $F \in L^p(\boldsymbol{T}^n, \mathscr{A})$.

Now we give some applications of Theorem 1.

The Riesz-Bochner means $S^\sigma_R f$ and $\tilde{S}^\sigma_R F$ are defined by the following formulae:

$$(S^\sigma_R f)(x) = \int_{|\xi|<R} (1 - |\xi|^2R^{-2})^\sigma \hat{f}(\xi)e(x\xi)d\xi$$

and

$$(\tilde{S}^\sigma_R F(\theta) = \sum_{|m|<R} (1 - |m|^2 R^{-2})^\sigma \hat{F}(m) e(m\theta) .$$

Assume that a sequence $\{R(\gamma); \gamma = 1, 2, \cdots\}$ of positive real numbers is given. Let $\Gamma = \{1, 2, \cdots\}$ and $(\Gamma, \mathscr{N}, \nu)$ be the discrete measure space on Γ. Define $\lambda(\xi, \gamma)$ by

$$\lambda(\xi, \gamma) = (1 - |\xi|^2 R(\gamma)^{-2})^\sigma_+ .$$

Then we have

$$(T_\lambda f)(x, \gamma) = (S^\sigma_{R(\gamma)} f_\gamma)(x) \quad \text{and} \quad (\tilde{T}_\lambda F)(\theta, \gamma) = (\tilde{S}^\sigma_{R(\gamma)} F_\gamma)(\theta) . \tag{12}$$

Let $\mathscr{A} = \mathscr{B} = L^2(\Gamma, \mathscr{N}, \nu) = \ell^2$. Then

$$\| (\sum | S^\sigma_{R(\gamma)} f_\gamma |^2)^{1/2} \|_{L^p(R^2)} \leqq A \| (\sum | f_\gamma |^2)^{1/2} \|_{L^p(R^2)} \tag{13}$$

implies

$$\| (\sum | \tilde{S}^\sigma_{R(\gamma)} F_\gamma |^2)^{1/2} \|_{L^p(T^2)} \leqq A \| (\sum | F_\gamma |^2)^{1/2} \|_{L^p(T^2)}$$

by (12) and (i) of Theorem 1. It has been proved, by Igari [16], that, *if* $\sigma > 0$, $4/3 \leqq p \leqq 4$ *and the sequence* $\{R(\gamma)\}$ *satisfies the lacunary condition* $R(\gamma + 1)/R(\gamma) \geqq \alpha > 1$ $(\gamma = 1, 2, \cdots)$, *then* (13) *holds*. Recently Córdoba and López-Melero [6] have obtained the same theorem without the lacunary condition for $\{R(\gamma)\}$.

Now let $\{R(\gamma)\}$ be lacunary and $\mathscr{B} = L^\infty(\Gamma, \mathscr{N}, \nu)$, where $(\Gamma, \mathscr{N}, \nu)$ is the same as above, and let $\mathscr{M} = \{\emptyset, \Gamma\}$ and $\mu(\Gamma) = 1$. Then, in this case, $\mathscr{A}$ may be identified with the set of all scalars. In [16], [15] and [6], it has also been proved that, *if* $\sigma > 0$ *and* $4/3 \leqq p \leqq 4$, *then*

$$\| \sup_{\gamma \in \Gamma} | S^\sigma_{R(\gamma)} f | \|_{L^p(R^2)} \leqq A \| f \|_{L^p(R^2)} .$$

Applying Theorem 1 to this relation, we have

$$\| \sup_{\gamma \in \Gamma} | \tilde{S}^\sigma_{R(\gamma)} F | \|_{L^p(T^2)} \leqq A \| F \|_{L^p(T^2)}$$

for $\sigma > 0$ and $4/3 \leqq p \leqq 4$. This has been stated in [15] with $R(\gamma) = 2^\gamma$.

Igari [16] has proved the following result (14) which is a decomposition theorem of the Littlewood-Paley type for weak annular truncations. *Let* $0 < \tau < 1$ *and set* $\hat{D}_0(\xi) = 1$, *if* $|\xi| \leqq 2$, $= \{(2 + \tau - |\xi|)/\tau\}^\sigma$, *if* $2 \leqq |\xi| \leqq 2 + \tau$, *and* $= 0$, *if* $2 + \tau \leqq |\xi| < \infty$. *Further set* $\Delta_\gamma(x) = D_\gamma(x) - D_{\gamma-1}(x)$, $\hat{D}_\gamma(\xi) = \hat{D}_0(2^{-\gamma}\xi)$ $(\gamma \in \boldsymbol{Z})$. *Then*

$$A' \| f \|_{L^p(R^2)} \leqq \| (\sum | \Delta_\gamma * f |^2)^{1/2} \|_{L^p(R^2)} \leqq B' \| f \|_{L^p(R^2)} \tag{14}$$

for $\sigma > 0$ *and* $4/3 \leqq p \leqq 4$. If we set $\mathscr{A} = \boldsymbol{C}$, $\mathscr{B} = L^2(\Gamma, \mathscr{N}, \nu) = \ell^2$, $\Gamma = \boldsymbol{Z}$ and $(\tilde{\Delta}_\gamma F)(\theta) = \sum \hat{\Delta}_\gamma(m) \hat{F}(m) e(m\theta)$, then we have

$$\| (\sum | \tilde{\Delta}_\gamma F |^2)^{1/2} \|_{L^p(T^2)} \leqq B' \| F \|_{L^p(T^2)} \tag{15}$$

for $\sigma > 0$ and $4/3 \leqq p \leqq 4$ by Theorem 1. On the other hand, if $\hat{F}(0) = 0$, then there exists a sequence $\{\tilde{K}_\gamma\}$ of operators such that

$$\int_{T^2} F(\theta)G(\theta)d\theta = \sum \int_{T^2} (\tilde{\Delta}_\gamma F)(\theta)(\tilde{K}_\gamma G)(\theta)d\theta$$

and

$$\|(\sum |\tilde{K}_\gamma G|^2)^{1/2}\|_{L^r(T^2)} \leqq C_r \|G\|_{L^r(T^2)}$$

for $1 < r < \infty$ (see [16]). Therefore, we have

$$\|F\|_{L^p(T^2)} \leqq C_p \|(\sum |\tilde{\Delta}_\gamma F|^2)^{1/2}\|_{L^p(T^2)}$$

for $\sigma > 0$ and $4/3 \leqq p \leqq 4$, if $\hat{F}(0) = 0$. Combining this with (15), we obtain the transplantation of (14) to the periodic case.

2. The transplantation from T^n to R^n. Let λ be an $(\hat{\mathscr{L}} \times \mathscr{N})$-measurable function such that $\lambda(\cdot, \gamma)$ is bounded for all $\gamma \in \Gamma$. Then we have defined the operators $T = T_\lambda$ and $\tilde{T} = \tilde{T}_\lambda$ in §1. At the same time, we may consider the operators defined by the dilations of λ. We define $\tilde{T}_\varepsilon$, $\varepsilon > 0$, by $(\tilde{T}_\varepsilon F)(\theta, \gamma) = (\tilde{T}_{\lambda(\varepsilon\cdot,\gamma)} F_\gamma)(\theta)$ for $F \in \mathscr{C}(\Gamma, C^\infty(T^n))$. Our aim in this section is to prove the following theorems.

THEOREM 2. *Assume that λ is an $(\hat{\mathscr{L}} \times \mathscr{N})$-measurable function defined on $\hat{R}^n \times \Gamma$, and that $\lambda(\cdot, \gamma)$ is bounded and continuous a.e. in $\hat{R}^n$ for every $\gamma \in \Gamma$. Then we have* (i) *and* (ii) *for $T = T_\lambda$ and $\tilde{T}_\varepsilon$.*

(i) *Assume that $0 < p, q \leqq \infty$. If*

$$\|\tilde{T}_\varepsilon F\|_{L^q(T^n, \mathscr{B})} \leqq A_\varepsilon \|F\|_{L^p(T^n, \mathscr{A})}$$

for all $\varepsilon > 0$ and $F \in \mathscr{C}(\Gamma, C^\infty(T^n))$, and if

$$A = \liminf_{\varepsilon \to 0} \varepsilon^{n\{(1/p)\} - (1/q)\}} A_\varepsilon < \infty , \tag{1}$$

then

$$\|Tf\|_{L^q(R^n, \mathscr{B})} \leqq A \|f\|_{L^p(R^n, \mathscr{A})}$$

for $f \in \mathscr{C}_0(\Gamma, C_0^\infty(R^n))$, where $\mathscr{C}_0(\Gamma, C_0^\infty(R^n))$ is the class of all $f \in \mathscr{C}(\Gamma, C_0^\infty(R^n))$ such that $\bigcup \operatorname{supp} f(\cdot, \gamma)$ is bounded, where the union runs over all $\gamma \in \Gamma$.

(ii) *Assume that $0 < p \leqq \infty$ and $0 < q < \infty$. If*

$$|\{\theta \in Q^n;\ \|(\tilde{T}_\varepsilon F)(\theta, \cdot)\|_{\mathscr{B}} > t\}| \leqq \{A_\varepsilon t^{-1} \|F\|_{L^p(T^n, \mathscr{A})}\}^q$$

for $t > 0, \varepsilon > 0$ and $F \in \mathscr{C}(\Gamma, C^\infty(T^n))$, and if the constants A_ε satisfy (1), *then*

$$|\{x \in R^n;\ \|(Tf)(x, \cdot)\|_{\mathscr{B}} > t\}| \leqq \{At^{-1} \|f\|_{L^p(R^n, \mathscr{A})}\}^q$$

for all $t>0$ *and* $f\in\mathscr{C}_0(\Gamma, C_0^\infty(\boldsymbol{R}^n))$.

If $\mathscr{C}_0(\Gamma, C_0^\infty(\boldsymbol{R}^n))$ is dense in $L^p(\boldsymbol{R}^n, \mathscr{A})$, and if $f_j\to f$ in $L^p(\boldsymbol{R}^n, \mathscr{A})$ as $j\to\infty$ implies $(Tf_j)(x, \gamma)\to(Tf)(x, \gamma)$ $(dx\times d\nu(\gamma))$-a.e. as $j\to\infty$, then the conclusions of Theorem 2 are true for every $f\in L^p(\boldsymbol{R}^n, \mathscr{A})$.

We introduce some notations to state the next theorem.

For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ with non-negative integers α_j, we define the operators R_α and $\tilde{R}_\alpha$ as follows. First, for scalar valued functions $f\in\mathscr{S}(\boldsymbol{R}^n)$ and $F\in C^\infty(\boldsymbol{T}^n)$, define

$$(R_\alpha f)(x)=\int_{\hat{\boldsymbol{R}}^n}(-i|\xi|^{-1}\xi)^\alpha\hat{f}(\xi)e(x\xi)d\xi$$

and

$$(\tilde{R}_\alpha F)(\theta)=\sum_{m\in\boldsymbol{Z}^n}(-i|m|^{-1}m)^\alpha\hat{F}(m)e(m\theta)\ .$$

For $f\in\mathscr{C}(\Gamma, \mathscr{S}(\boldsymbol{R}^n))$, we define $R_\alpha f$ by $(R_\alpha f)(x, \gamma)=(R_\alpha f_\gamma)(x)$. In the same manner, for $F\in\mathscr{C}(\Gamma, C^\infty(\boldsymbol{T}^n))$, we define $\tilde{R}_\alpha F$ by $(\tilde{R}_\alpha F)(\theta, \gamma)=(\tilde{R}_\alpha F_\gamma)(\theta)$.

We denote by $\mathscr{S}_0(\boldsymbol{R}^n)$ the space of all $f\in\mathscr{S}(\boldsymbol{R}^n)$ such that $0\notin\operatorname{supp}\hat{f}$, which is dense in the Hardy class $H^p(\boldsymbol{R}^n)$ of Fefferman and Stein [8].

THEOREM 3. *Let* λ *be an* $(\hat{\mathscr{L}}\times\mathscr{N})$*-measurable function on* $\hat{\boldsymbol{R}}^n\times\Gamma$ *such that* $\lambda(\cdot, \gamma)$ *is bounded and continuous a.e. for every* $\gamma\in\Gamma$. *Then we have* (i) *and* (ii) *for* $T=T_\lambda$ *and* $\tilde{T}_\varepsilon$.

(i) Assume that $0<p\leqq 1$ and $0<q\leqq\infty$. If

$$\|\tilde{T}_\varepsilon F\|_{L^q(\boldsymbol{T}^n, \mathscr{B})}\leqq A_\varepsilon\sum_{|\alpha|\leqq K}\|\tilde{R}_\alpha F\|_{L^p(\boldsymbol{T}^n, \mathscr{A})}$$

for all $\varepsilon>0$ *and all* $F\in\mathscr{C}(\Gamma, C^\infty(\boldsymbol{T}^n))$, *and if the constants* A_ε *satisfy* (1), *then*

$$\|Tf\|_{L^q(\boldsymbol{R}^n, \mathscr{B})}\leqq A\sum_{|\alpha|\leqq K}\|R_\alpha f\|_{L^p(\boldsymbol{R}^n, \mathscr{A})}$$

for $f\in\mathscr{C}(\Gamma, \mathscr{S}_0(\boldsymbol{R}^n))$.

(ii) *Assume that* $0<p\leqq 1$ *and* $0<q<\infty$. If

$$|\{\theta\in\boldsymbol{Q}^n;\ \|(\tilde{T}_\varepsilon F)(\theta, \cdot)\|_{\mathscr{B}}>t\}|\leqq\Big\{A_\varepsilon t^{-1}\sum_{|\alpha|\leqq K}\|\tilde{R}_\alpha F\|_{L^p(\boldsymbol{T}^n, \mathscr{A})}\Big\}^q$$

for $t>0, \varepsilon>0$ *and* $F\in\mathscr{C}(\Gamma, C^\infty(\boldsymbol{T}^n))$, *and if the constants* A_ε *satisfy the condition* (1), *then*

$$|\{x\in\boldsymbol{R}^n;\ \|(Tf)(x, \cdot)\|_{\mathscr{B}}>t\}|\leqq\Big\{At^{-1}\sum_{|\alpha|\leqq K}\|R_\alpha f\|_{L^p(\boldsymbol{R}^n, \mathscr{A})}\Big\}^q$$

for $t>0$ *and* $f\in\mathscr{C}(\Gamma, \mathscr{S}_0(\boldsymbol{R}^n))$.

In the above theorem, the constant K may be an arbitrary integer, but K should be larger than $(n-1)\{(1/p)-1\}$, if $H^p(\boldsymbol{R}^n)$ is under consideration.

Our proofs depend strongly upon the following simple lemma, which we obtain by representing $(T_\lambda f)(x)$ as the limit of the Riemann sums of the integrand. The proof is found in [13, p.p. 154–155], [14] and [23, p. 266].

LEMMA 1. *Assume that* λ *is a scalar valued function on* $\hat{\boldsymbol{R}}^n$, *which is bounded and continuous a.e. Let* $f \in \mathscr{S}(\boldsymbol{R}^n)$. *If* F_ε *is defined by* $F_\varepsilon(\theta) = \sum f_\varepsilon(\theta + m)$, *where the summation is taken over all* $m \in \boldsymbol{Z}^n$ *and* $f_\varepsilon(x) = \varepsilon^{-n} f(\varepsilon^{-1} x)$, *then*

$$(T_\lambda f)(x) = \lim_{\varepsilon \to 0} \{\tilde{T}_{\lambda(\varepsilon \cdot)}(\varepsilon^n F_\varepsilon)\}(\varepsilon x) .$$

LEMMA 2. *Let* λ *be an* $(\hat{\mathscr{L}} \times \mathscr{N})$*-measurable function defined on* $\hat{\boldsymbol{R}}^n \times \Gamma$ *such that* $\lambda(\cdot, \gamma)$ *is bounded and continuous a.e. for all* $\gamma \in \Gamma$. *If* $f \in \mathscr{C}(\Gamma, \mathscr{S}(\boldsymbol{R}^n))$ *and if* F_ε *is defined by* $F_\varepsilon(\theta, \gamma) = \sum (f_\gamma)_\varepsilon(\theta + m) = \sum \varepsilon^{-n} f((\theta + m)/\varepsilon, \gamma)$, *then we have*

$$\| Tf \|_{L^q(\boldsymbol{R}^n, \mathscr{B})} \leqq \liminf_{\varepsilon \to 0} \varepsilon^{-n/q} \| \tilde{T}_\varepsilon(\varepsilon^n F_\varepsilon) \|_{L^q(T^n, \mathscr{B})} \tag{2}$$

and

$$\begin{aligned} &|\{x \in \boldsymbol{R}^n;\ \|(Tf)(x, \cdot)\|_{\mathscr{B}} > t\}| \\ &\qquad \leqq \liminf_{\varepsilon \to 0} \varepsilon^{-n} |\{\theta \in \boldsymbol{Q}^n;\ \|\{\tilde{T}_\varepsilon(\varepsilon^n F_\varepsilon)\}(\theta, \cdot)\|_{\mathscr{B}} > t\}| \end{aligned} \tag{3}$$

for all $t > 0$.

PROOF. Let $\{\varepsilon(j)\}$ be an arbitrary sequence of positive numbers such that $\varepsilon(j) \to 0$ as $j \to \infty$. By the definition of $(Tf)(x, \gamma)$ and Lemma 1,

$$(Tf)(x, \gamma) = \lim_{j \to \infty} \{\tilde{T}_{\varepsilon(j)}(\varepsilon(j)^n F_{\varepsilon(j)})\}(\varepsilon(j)x, \gamma) .$$

Therefore, $\|(Tf)(x, \cdot)\|_{\mathscr{B}}$ is bounded by the inferior limit of $\|\{\tilde{T}_{\varepsilon(j)}(\varepsilon(j)^n F_{\varepsilon(j)})\}(\varepsilon(j)x, \cdot)\|_{\mathscr{B}}$ as $j \to \infty$. Let χ be the characteristic function of $\boldsymbol{Q}^n$. Since $\chi(\varepsilon(j)x) \to 1$ as $j \to \infty$,

$$\|(Tf)(x, \cdot)\|_{\mathscr{B}} \leqq \liminf_{j \to \infty} \|\{\tilde{T}_{\varepsilon(j)}(\varepsilon(j)^n F_{\varepsilon(j)})\}(\varepsilon(j)x, \cdot)\|_{\mathscr{B}} \chi(\varepsilon(j)x) . \tag{4}$$

When $q \neq \infty$, integrating the q-th powers of both sides of (4), using Fatou's lemma and then changing the variables on the right hand side, we see that $\| Tf \|^q_{L^q(\boldsymbol{Q}^n, \mathscr{B})}$ is bounded by

$$\liminf_{j \to \infty} \varepsilon(j)^{-n} \{\| \tilde{T}_{\varepsilon(j)}(\varepsilon(j)^n F_{\varepsilon(j)}) \|_{L^q(\boldsymbol{Q}^n, \mathscr{B})}\}^q .$$

Therefore, (2) is obtained. When $q = \infty$, it is evident from (4). Further-more, (4) implies

$$\{x \in \boldsymbol{R}^n;\ \|(Tf)(x,\ \cdot)\|_{\mathscr{B}} > t\}$$
$$\subset \liminf_{j\to\infty} \{x \in \boldsymbol{R}^n;\ \|\{\tilde{T}_{\varepsilon(j)}(\varepsilon(j)^n F_{\varepsilon(j)})\}(\varepsilon(j)x,\ \cdot)\|_{\mathscr{B}} \chi(\varepsilon(j)x) > t\}$$

for all $t > 0$. The last set is equal to

$$\varepsilon(j)^{-1}\{\theta \in \boldsymbol{Q}^n;\ \|\{\tilde{T}_{\varepsilon(j)}(\varepsilon(j)^n F_{\varepsilon(j)})(\theta,\ \cdot)\|_{\mathscr{B}} > t\}\ .$$

Therefore, (3) is obtained.

PROOF OF THEOREM 2. Let $f \in \mathscr{C}_0(\Gamma, C_0^\infty(\boldsymbol{R}^n))$ and $F_\varepsilon(\theta, \gamma) = \sum \varepsilon^{-n} f((\theta + m)/\varepsilon, \gamma)$. Assume the hypothesis of (i). Then

$$\varepsilon^{-n/q} \|\tilde{T}_\varepsilon(\varepsilon^n F_\varepsilon)\|_{L^q(T^n, \mathscr{B})} \leqq \varepsilon^{-n/q} A_\varepsilon \|\varepsilon^n F_\varepsilon\|_{L^p(T^n, \mathscr{A})} \tag{5}$$

for all $\varepsilon > 0$. Since $\varepsilon^n F_\varepsilon(\theta, \gamma) = f(\theta/\varepsilon, \gamma)$ for sufficiently small $\varepsilon > 0$ and $\theta \in \boldsymbol{Q}^n$, $\|\varepsilon^n F_\varepsilon\|_{L^p(T^n, \mathscr{A})} = \varepsilon^{n/p} \|f\|_{L^p(R^n, \mathscr{A})}$ for such ε. Therefore, the right hand side of (5) equals $\varepsilon^{n\{(1/p)-(1/q)\}} A_\varepsilon \|f\|_{L^p(R^n, \mathscr{A})}$. Applying this estimate to (2), we obtain the conclusion of (i). Now assume the hypothesis of (ii). Then

$$\varepsilon^{-n} |\{\theta \in \boldsymbol{Q}^n;\ \|\{\tilde{T}_\varepsilon(\varepsilon^n F_\varepsilon)\}(\theta,\ \cdot)\|_{\mathscr{B}} > t\}| \leqq \varepsilon^{-n}\{A_\varepsilon t^{-1} \|\varepsilon^n F_\varepsilon\|_{L^p(T^n, \mathscr{A})}\}^q$$

for $t > 0$. Since $\|\varepsilon^n F_\varepsilon\|_{L^p(T^n, \mathscr{A})} = \varepsilon^{n/p} \|f\|_{L^p(R^n, \mathscr{A})}$ for sufficiently small ε, the right hand side of the above inequality is bounded by

$$[\varepsilon^{n\{(1/p)-(1/q)\}} A_\varepsilon t^{-1} \|f\|_{L^p(R^n, \mathscr{A})}]^q\ .$$

This, together with (3), implies the conclusion of (ii).

PROOF OF THEOREM 3. Let $f \in \mathscr{C}(\Gamma, \mathscr{S}_0(\boldsymbol{R}^n))$ and define F_ε as in Lemma 2. Since $R_\alpha f \in \mathscr{C}(\Gamma, \mathscr{S}_0(\boldsymbol{R}^n))$, $F_\varepsilon^\alpha \in \mathscr{C}(\Gamma, C^\infty(\boldsymbol{T}^n))$ may be defined by $F_\varepsilon^\alpha(\theta, \gamma) = \sum \varepsilon^{-n}(R_\alpha f)((\theta + m)/\varepsilon, \gamma)$. Comparing the Fourier coefficients of both sides, we easily find that

$$F_\varepsilon^\alpha(\theta, \gamma) = (\tilde{R}_\alpha F_\varepsilon)(\theta, \gamma)\ . \tag{6}$$

Since $\|\varepsilon^n F_\varepsilon^\alpha(\theta,\ \cdot)\|_{\mathscr{A}} \leqq \sum \|(R_\alpha f)((\theta + m)/\varepsilon,\ \cdot)\|_{\mathscr{A}}$ and $0 < p \leqq 1$,

$$\|\varepsilon^n F_\varepsilon^\alpha\|_{L^p(T^n, \mathscr{A})}^p \leqq \sum \int_{Q^n} \|(R_\alpha f)((\theta + m)/\varepsilon,\ \cdot)\|_{\mathscr{A}}^p d\theta = \varepsilon^n \|R_\alpha f\|_{L^p(R^n, \mathscr{A})}^p\ . \tag{7}$$

Now, assume the conditions of (i). Then

$$\varepsilon^{-n/q} \|\tilde{T}_\varepsilon(\varepsilon^n F_\varepsilon)\|_{L^q(T^n, \mathscr{B})} \leqq \varepsilon^{-n/q} A_\varepsilon \sum_{|\alpha| \leqq K} \|\tilde{R}_\alpha(\varepsilon^n F_\varepsilon)\|_{L^p(T^n, \mathscr{A})}\ .$$

By (6) and (7), the right hand side of the last inequality is bounded by $\varepsilon^{n\{(1/p)-(1/q)\}} A_\varepsilon \sum_{|\alpha| \leqq K} \|R_\alpha f\|_{L^p(R^n, \mathscr{A})}$. Applying the inequality just obtained to (2), we get the conclusion of (i). Next assume the conditions of (ii). Then, for a given $t > 0$,

$$\varepsilon^{-n}|\{\theta \in \boldsymbol{Q}^n;\ \|\{\tilde{T}_\varepsilon(\varepsilon^n F_\varepsilon)\}(\theta,\ \cdot)\|_{\mathscr{B}} > t\}| \leqq \varepsilon^{-n}\Big\{A_\varepsilon t^{-1} \sum_{|\alpha|\leqq K} \|\tilde{R}_\alpha(\varepsilon^n F_\varepsilon)\|_{L^p(T^n,\mathscr{A})}\Big\}^q .$$

By (6) and (7), the last term is bounded by

$$\Big[\varepsilon^{n\{(1/p)-(1/q)\}} A_\varepsilon t^{-1} \sum_{|\alpha|\leqq K} \|R_\alpha f\|_{L^p(R^n,\mathscr{A})}\Big]^q .$$

This estimate and (3) imply (ii).

3. The Littlewood-Paley g^*-function. We discuss two types of the classical Littlewood-Paley g^*-functions, one of which is defined in the upper-half plane $\boldsymbol{H}$ and the other in the unit disc $\boldsymbol{D}$.

We use the following notations. Let ϕ and Φ be analytic in $\boldsymbol{H}$ and $\boldsymbol{D}$, respectively. Assume $2 \leqq q < \infty$ and $\alpha > 1 - (1/q) = 1/q'$, and define

$$(g^*_{\alpha,q}\phi)(x) = \left[\int_0^\infty \left\{\int_{-\infty}^\infty (y/(|s| + y))^{\alpha q'} |\phi'(x - s + iy)|^{q'} ds\right\}^{q/q'} dy\right]^{1/q}$$

and

$$(G^*_{\alpha,q}\Phi)(\theta) = \left[\int_0^1 \left\{\int_{-1/2}^{1/2} ((1 - r)/|1 - re(\tau)|)^{\alpha q'} |\Phi'(re(\theta - \tau))|^{q'} d\tau\right\}^{q/q'} dr\right]^{1/q} .$$

The norms of ϕ and Φ are defined by

$$\|\phi\|_p = \sup_{y>0} \left\{\int_{-\infty}^\infty |\phi(x + iy)|^p dx\right\}^{1/p}$$

and

$$\|\Phi\|_p = \sup_{0\leqq r<1} \left\{\int_{-1/2}^{1/2} |\Phi(re(\theta))|^p d\theta\right\}^{1/p} ,$$

respectively. The set of all ϕ such that $\|\phi\|_p < \infty$ is denoted by $H^p(\boldsymbol{H})$, and $H^p(\boldsymbol{D})$ is also defined in the same manner.

The following theorem on $G^*_{\alpha,q}\Phi$ is known (Sunouchi [24], Zygmund [27], Flett [9] and Kaneko [18]).

THEOREM A. *If* $0 < p < \infty$ *and* $\alpha > \max\{1/p, 1/q'\}$, *then*

(1) $$\|G^*_{\alpha,q}\Phi\|_{L^p(T)} \leqq A_{\alpha,p,q}\|\Phi\|_p .$$

If $0 < p < 2$, $(1/p) + (1/q) > 1$ *and* $\alpha = 1/p$, *then*

(2) $$|\{\theta \in \boldsymbol{Q};\ (G^*_{\alpha,q}\Phi)(\theta) > t\}| \leqq (A_{p,q}t^{-1}\|\Phi\|_p)^p$$

for all $t > 0$.

We will show that the following theorem on $g^*_{\alpha,q}\phi$ can be directly obtained from (1) and (2) by applying the theorems in §2.

THEOREM 4. *Assume* $\phi \in H^p(\boldsymbol{H})$.

(i) *If* $0 < p < \infty$ *and* $\alpha > \max\{1/p, 1/q'\}$, *then*

$$\| g^*_{\alpha,q}\phi \|_{L^p(R)} \leqq A'_{\alpha,p,q} \| \phi \|_p .$$

(ii) *If* $0 < p < 2$, $(1/p) + (1/q) > 1$ *and* $\alpha = 1/p$, *then*

$$|\{x \in \boldsymbol{R};\ (g^*_{\alpha,q}\phi)(x) > t\}| \leqq (A'_{p,q} t^{-1} \| \phi \|_p)^p$$

for all $t > 0$.

This theorem is partially established by Waterman [26], Sunouchi [25], Stein [21] and Fefferman [7].

To investigate the relation between $g^*_{\alpha,q}\phi$ and $G^*_{\alpha,q}\Phi$, we define $\mathscr{G}\Phi$ by

$$(\mathscr{G}\Phi)(\theta) = \left[\int_0^\infty \left\{\int_{-\infty}^\infty 2\pi r (y/(|s| + y))^{\alpha q'} |\Phi'(re(\theta - s))|^{q'} ds\right\}^{q/q'} 2\pi r dy\right]^{1/q},$$

where $r = \exp(-2\pi y)$ and Φ is analytic in $\boldsymbol{D}$. Then we have

$$C_1(G^*_{\alpha,q}\Phi)(\theta) \leqq (\mathscr{G}\Phi)(\theta) \leqq C_2(G^*_{\alpha,q}\Phi)(\theta) , \tag{3}$$

where the constants C_1 and C_2 are independent of Φ and θ. We shall postpone the proof of (3) until the end of this section.

Let $\Gamma_1 = \{1, 2\}$, $\Gamma_2 = (-\infty, \infty)$ and $\Gamma_3 = (0, \infty)$, and let $(\Gamma, \mathscr{N}, \nu)$ be the product measure space of $(\Gamma_j, \mathscr{N}_j, \nu_j)$ $(j = 1, 2, 3)$, where ν_1 is the counting measure and ν_2 and ν_3 are the Lebesgue measures on Γ_2 and Γ_3, respectively. Set $\mathscr{B} = L^{2,q',q}(\Gamma, \mathscr{N}, \nu)$. On the other hand, let $\mathscr{M}_j = \{\varnothing, \Gamma_j\}$ $(j = 1, 2, 3)$ and each μ_j be the probability measure on each Γ_j. In this case, $\mathscr{A}$ coincides with all the scalars, so that $L^p(\boldsymbol{R}, \mathscr{A})$- and $L^p(\boldsymbol{T}, \mathscr{A})$-norms are the usual $L^p(\boldsymbol{R})$- and $L^p(\boldsymbol{T})$-norms, respectively. We define λ by

$$\lambda(\xi, \gamma) = \begin{cases} 2\pi i\xi \exp(-2\pi i\xi s - 2\pi|\xi|y)\{y/(|s| + y)\}^\alpha & (j = 1) , \\ -2\pi|\xi| \exp(-2\pi i\xi s - 2\pi|\xi|y)\{y/(|s| + y)\}^\alpha & (j = 2) , \end{cases}$$

where $\gamma = (j, s, y)$ and we denote T_λ by T.

For a real valued function $f \in \mathscr{S}(\boldsymbol{R})$, $\tilde{f}$ denotes the Hilbert transform of f and we denote the Poisson integrals of f and $\tilde{f}$ over $\boldsymbol{H}$ by u and v, respectively. If we set $\phi = u + iv$, then ϕ is analytic in $\boldsymbol{H}$ and $|\phi'(x + iy)| = |\nabla u(x, y)|$, and further $(Tf)(x, j, s, y)$ is equal to $\{y/(|s| + y)\}^\alpha \partial u(x - s, y)/\partial x$, if $j = 1$, and to $\{y/(|s| + y)\}^\alpha \partial u(x - s, y)/\partial y$, if $j = 2$. Therefore,

$$(g^*_{\alpha,q}\phi)(x) = \|(Tf)(x, \cdot)\|_{\mathscr{B}} . \tag{4}$$

For a real valued periodic function $F \in C^\infty(\boldsymbol{T})$, let $\tilde{F}$ denote the conjugate function of F, and U and V the Poisson integrals of F and $\tilde{F}$ over $\boldsymbol{D}$, respectively. If we set $\Phi = U + iV$ and write $z = \rho e(\tau) \in \boldsymbol{D}$, then

$$\begin{aligned}\{|U_\tau(\tau, \rho)|^2 + 4\pi^2\rho^2|U_\rho(\tau, \rho)|^2\} + \{|V_\tau(\tau, \rho)|^2 + 4\pi^2\rho^2|V_\rho(\tau, \rho)|^2\} \\ = 8\pi^2\rho^2|\Phi'(\rho e(\tau))|^2 ,\end{aligned}$$

where $U_\tau = \partial U/\partial\tau$, $U_\rho = \partial U/\partial\rho$, $V_\tau = \partial V/\partial\tau$ and $V_\rho = \partial V/\partial\rho$. By the definition of $\tilde{T}_\varepsilon$, $(\tilde{T}_\varepsilon F)(\theta, j, s, y)$ is equal to $\{y/(|s| + y)\}^\alpha \varepsilon U_\tau(\theta - \varepsilon s, r^\varepsilon)$, if $j = 1$, and to $\{y/(|s| + y)\}^\alpha(-2\pi r^\varepsilon)\varepsilon U_\rho(\theta - \varepsilon s, r^\varepsilon)$, if $j = 2$, where $r = \exp(-2\pi y)$. Replacing U by V in the above argument, then we obtain a similar relation for $(\tilde{T}_\varepsilon\tilde{F})(\theta, j, s, y)$. Therefore,

$$(5)\qquad \|(\tilde{T}_\varepsilon F)(\theta, \cdot)\|_{\mathscr{B}} \leqq 2^{1/2}(\mathscr{G}\Phi)(\theta) \leqq \|(\tilde{T}_\varepsilon F)(\theta, \cdot)\|_{\mathscr{B}} + \|(\tilde{T}_\varepsilon\tilde{F})(\theta, \cdot)\|_{\mathscr{B}}$$

for all $\varepsilon > 0$. We remark

$$(6)\qquad \|\Phi\|_p \leqq \begin{cases} 2^{(1/p)-1}(\|F\|_{L^p(T)} + \|\tilde{F}\|_{L^p(T)}) & (0 < p \leqq 1)\,, \\ C_p\|F\|_{L^p(T)} & (1 < p \leqq \infty)\,. \end{cases}$$

PROOF OF THEOREM 4. For $F \in C^\infty(\boldsymbol{T})$, $\tilde{T}_\varepsilon F$ ($\varepsilon > 0$) are estimated by (1), (2), (3), (5) and (6). Applying Theorems 2 and 3 to these estimates, we obtain those for Tf, where $f \in \mathscr{S}_0(\boldsymbol{R})$, if $0 < p \leqq 1$, and $f \in C_0^\infty(\boldsymbol{R})$, if $1 < p < \infty$. If ϕ is the Poisson integral of $f + i\tilde{f}$, then the above estimates together with (4) give those of $g^*_{\alpha,q}\phi$ in terms of the $L^p(\boldsymbol{R})$-norms of f and $\tilde{f}$, which are bounded by $C\|\phi\|_p$. If $\phi_j \to \phi$ in $H^p(\boldsymbol{H})$ as $j \to \infty$, then $\phi_j'(x + iy) \to \phi'(x + iy)$ for all $x + iy \in \boldsymbol{H}$ as $j \to \infty$, and then $(g^*_{\alpha,q}\phi)(x)$ is bounded by the inferior limit of $(g^*_{\alpha,q}\phi_j)(x)$ as $j \to \infty$. Therefore, the conclusions of Theorem 4 hold for all $\phi \in H^p(\boldsymbol{H})$.

Fefferman [7] was the first to succeed in proving the critical case $\alpha = 1/p$. His result is that, *if* $1 < p < 2$ *and* $\alpha = 1/p$, *then*

$$(7)\qquad |\{x \in \boldsymbol{R};\ (g^*_\alpha f)(x) > t\}| \leqq (At^{-1}\|f\|_{L^p(R)})^p$$

for any $t > 0$, *where*

$$(g^*_\alpha f)(x) = \left\{\int_0^\infty\int_R (y/(|s| + y))^{2\alpha}|\nabla u(x - s, y)|^2 ds dy\right\}^{1/2}$$

and u *is the Poisson integral of* f. He has considered this in the n-dimensional case.

We now consider the converse transplantation of (7). Let $T = T_\lambda$ be the same as above and $f \in C_0^\infty(\boldsymbol{R})$. If ϕ is the Poisson integral of $f + i\tilde{f}$, then

$$(8)\qquad (g^*_\alpha f)(x) = (g^*_{\alpha,2}\phi)(x) = \|(Tf)(x, \cdot)\|_{\mathscr{B}}$$

by (4), but, in this case, we have $\mathscr{B} = L^{2,2,2}(\Gamma, \mathscr{N}, \nu)$. Therefore, the weak type estimate for Tf is obtained from (7) and (8). Applying Theorem 1 to this estimate, we obtain that for $\tilde{T}F = \tilde{T}_1F$ for $F \in \mathscr{P}$. Let Φ be an algebraic polynomial such that $\Phi(0) = 0$. Put $F(\theta) = \operatorname{Re}\Phi(e(\theta))$ and $\tilde{F}(\theta) = \operatorname{Im}\Phi(e(\theta))$. Then $(G^*_{\alpha,2}\Phi)(\theta)$ is bounded by a constant multiple of $\{\|(\tilde{T}F)(\theta, \cdot)\|_{\mathscr{B}} + \|(\tilde{T}\tilde{F})(\theta, \cdot)\|_{\mathscr{B}}\}$ by (3) and (5). Therefore, we have

$$(9)\qquad |\{\theta \in \boldsymbol{Q};\ (G^*_{\alpha,2}\Phi)(\theta) > t\}| \leqq (A'_p t^{-1}\|\Phi\|_p)^p$$

for all $t > 0$, if $1 < p < 2$ and $\alpha = 1/p$. If we define $\Phi_j(z)$ as the j-th partial sum of $\Phi(z) = \sum c_\nu z^\nu \in H^p(\boldsymbol{D})$, then $\Phi = \lim \Phi_j$ in $H^p(\boldsymbol{D})$, $\Phi'(z) = \lim \Phi'_j(z)$ for $z \in \boldsymbol{D}$ and $|\Phi(0)| \leqq \|\Phi\|_p$. Therefore, (9) holds for $\Phi \in H^p(\boldsymbol{D})$. This is just (2) in the case of $q = 2$ and $1 < p < 2$.

We now return to the proof of (3). By simple computations, we have

$$(10)\qquad (1-r)/|1-re(\tau)| \geqq (1-r)/(1+r) \geqq 1/7 \quad (0 < r \leqq 3/4)\ ,$$

$$(11)\qquad (1-r) + 2\pi|\tau| \geqq |1-re(\tau)| \geqq \{(1-r) + 2\pi|\tau|\}/(2\pi)$$

and

$$(12)\qquad 1 - r \geqq \pi y \qquad (1/2 \leqq r < 1,\ r = \exp(-2\pi y))\ .$$

Divide the integral in the definition of $(G^*_{\alpha,q}\Phi)(\theta)$ with respect to r into two integrals one of which is the integral over $(0, 1/4)$ and the other is that over $(1/4, 1)$. We prove that the former is bounded by the latter. Since $1 - r \leqq |1 - re(\tau)|$,

$$\int_0^{1/4}\left\{\int_{-1/2}^{1/2}((1-r)/|1-re(\tau)|)^{\alpha q'}|\Phi'(re(\theta-\tau))|^{q'}d\tau\right\}^{q/q'}dr \leqq \int_0^{1/4}\left\{\int_{-1/2}^{1/2}|\Phi'(re(\theta-\tau))|^{q'}d\tau\right\}^{q/q'}dr\ .$$

Since the inner integral increases as $r \uparrow 1$, the last term does not exceed a constant multiple of

$$\int_{1/4}^{1/2}\left\{\int_{-1/2}^{1/2}((1-r)/|1-re(\tau)|)^{\alpha q'}|\Phi'(re(\theta-\tau))|^{q'}d\tau\right\}^{q/q'}dr\ ,$$

where (10) has been used. Therefore,

$$(13)\quad (G^*_{\alpha,q}\Phi)^q(\theta) \leqq C\int_{1/4}^{1}\left\{\int_{-1/2}^{1/2}((1-r)/|1-re(\tau)|)^{\alpha q'}|\Phi'(re(\theta-\tau))|^{q'}d\tau\right\}^{q/q'}dr\ .$$

On the other hand, restricting the domains of integration with respect to s and y in $(\mathscr{G}\Phi)(\theta)$ to $(-1/2, 1/2)$ and $(0, (\log 2)/\pi)$, respectively, and putting $r = \exp(-2\pi y)$, we have

$$(14)\qquad (\mathscr{G}\Phi)^q(\theta) \geqq \int_{1/4}^{1}\left\{\int_{-1/2}^{1/2}2\pi r(y/(|\tau|+y))^{\alpha q'}|\Phi'(re(\theta-\tau))|^{q'}d\tau\right\}^{q/q'}dr\ .$$

Using the fact that $1 - r \leqq 2\pi y$ and the second inequality in (11), we easily prove that $y/(|\tau| + y) \geqq (1-r)/(2\pi|1-re(\tau)|)$. Therefore, the right hand side of (14) is bounded from below by a constant multiple of

$$\int_{1/4}^{1}\left\{\int_{-1/2}^{1/2}((1-r)/|1-re(\tau)|)^{\alpha q'}|\Phi'(re(\theta-\tau))|^{q'}d\tau\right\}^{q/q'}dr .$$

This and (13) imply $(G^*_{\alpha,q}\Phi)(\theta) \leqq C(\mathscr{G}\Phi)(\theta)$.

Now we prove the second part of (3). We write the inner integral in the definition of $(\mathscr{G}\Phi)(\theta)$ by I. Divide I into the integrals over $(m-1/2, m+1/2)$ $(m \in \boldsymbol{Z})$ and denote them by I_m, respectively. Then

$$I_m = \int_{-1/2}^{1/2} 2\pi r(y/(|m+\tau|+y))^{\alpha q'}|\Phi'(re(\theta-\tau))|^{q'}d\tau . \tag{15}$$

Since $|m+\tau| \geqq |m|/2 \quad (m \neq 0, |\tau| \leqq 1/2)$,

$$I_m \leqq 2\pi r(2y/(|m|+2y))^{\alpha q'}\int_{-1/2}^{1/2}|\Phi'(re(\theta-\tau))|^{q'}d\tau \quad (m \neq 0) .$$

Since $\alpha q' > 1$ and $\sum_{m\neq 0}\{2y/(|m|+2y)\}^{\alpha q'}$ is bounded by both $\sum_{m\neq 0}(2y/|m|)^{\alpha q'}$ and twice the integral of $\{2y/(s+2y)\}^{\alpha q'}$ over $(0, \infty)$ with respect to s, $\sum_{m\neq 0}\{2y/(|m|+2y)\}^{\alpha q'}$ is bounded by a constant multiple of $\min\{y^{\alpha q'}, y\} = \psi(y)$, say. Therefore,

$$\sum_{m\neq 0} I_m \leqq C\int_{-1/2}^{1/2} r\psi(y)|\Phi'(re(\theta-\tau))|^{q'}d\tau . \tag{16}$$

If we consider the two cases $0 < y \leqq 1$ and $1 < y < \infty$ separately, then $r^{q'/3}\psi(y) \leqq C\{y/(|\tau|+y)\}^{\alpha q'}$ is easily obtained. Therefore, the right hand side of (16) is bounded by a constant multiple of

$$\int_{-1/2}^{1/2} r^{1-(q'/3)}(y/(|\tau|+y))^{\alpha q'}|\Phi'(re(\theta-\tau))|^{q'}d\tau . \tag{17}$$

When $m = 0$, it is evident from (15) that I_0 is bounded by a constant multiple of (17). Therefore, $I = \sum I_m$ does not exceed a constant multiple of (17). $(\mathscr{G}\Phi)^q(\theta)$ is the integral of $I^{q/q'}$ over $(0, 1)$ with respect to r. Divide it into the integrals over $(1/2^{n+1}, 1/2^n)$ $(n = 0, 1, \cdots)$ and denote them by J_n, respectively. By (12) and (11), $y/(|\tau|+y) \leqq 2(1-r)/|1-re(\tau)|$ for $1/2 < r < 1$. Since $r^{1-q'/3} \leqq 1$, J_0 is bounded by a constant multiple of

$$\int_{1/2}^{1}\left\{\int_{-1/2}^{1/2}((1-r)/|1-re(\tau)|)^{\alpha q'}|\Phi'(re(\theta-\tau))|^{q'}d\tau\right\}^{q/q'}dr ,$$

and so $J_0 \leqq C(G^*_{\alpha,q}\Phi)^q(\theta)$. Now consider the case $n \neq 0$. Applying the inequality $r^{1-q'/3}\{y/(|\tau|+y)\}^{\alpha q'} \leqq 2^{-n(1-q'/3)}$ $(1/2^{n+1} \leqq r \leqq 1/2^n)$ to (17), we get

$$J_n \leqq C2^{-nq(1/q'-1/3)}\int_{1/2^{n+1}}^{1/2^n}\left\{\int_{-1/2}^{1/2}|\Phi'(re(\theta-\tau))|^{q'}d\tau\right\}^{q/q'}dr .$$

Since the inner integral is an increasing function of r, the right hand side increases, when the domain of the integration with respect to r is replaced by (1/2, 3/4). Since (10) holds for $1/2 < r < 3/4$, J_n is bounded by $C2^{-nq(1/q'-1/3)}$ times

$$\int_{1/2}^{3/4}\left\{\int_{-1/2}^{1/2}((1-r)/|1-re(\tau)|)^{\alpha q'}|\Phi'(re(\theta-\tau))|^{q'}d\tau\right\}^{q/q'}dr ,$$

and so $J_n \leqq C2^{-nq(1/q'-1/3)}(G^*_{\alpha,q}\Phi)^q(\theta)$. Therefore,

$$(\mathscr{G}\Phi)^q(\theta) = \sum_{n=0}^{\infty} J_n \leqq C(G^*_{\alpha,q}\Phi)^q(\theta) .$$

This completes the proof of (3).

4. The lacunary partial means of the integral of $\hat{f}(\xi)e(x\xi)$. Let $H^1(\boldsymbol{R})$ be the set of the real parts of the functions which are the boundary values of functions in $H^1(\boldsymbol{H})$. This is identified with the Hardy class H^1 discussed in [8]. For $f \in H^1(\boldsymbol{R})$, the norm $\|f\|_{H^1(\boldsymbol{R})}$ of f is defined as $\|f\|_{L^1(\boldsymbol{R})} + \|\tilde{f}\|_{L^1(\boldsymbol{R})}$, where $\tilde{f}$ is the Hilbert transform of f. In this section, we will prove the following theorem by using (ii) of Theorem 3.

THEOREM 5. *Let $R(k) > 0$ and $R(k+1)/R(k) \geqq \alpha_0 > 1$ $(k = 1, 2, \cdots)$, and define S^*f for $f \in H^1(\boldsymbol{R})$ by*

$$(S^*f)(x) = \sup_k \left|\int_{|\xi|\leqq R(k)} \hat{f}(\xi)e(x\xi)d\xi\right| .$$

Then

$$(1) \qquad |\{x \in \boldsymbol{R};\ (S^*f)(x) > t\}| \leqq At^{-1}\|f\|_{H^1(\boldsymbol{R})}$$

for all $t > 0$, where the constant A depends only on α_0.

From this theorem, the following corollary is obtained by routine methods (cf. de Guzmán [11, §3.3]).

COROLLARY. *Under the same conditions as in Theorem 5, the following relation holds for all δ with $0 < \delta < 1$ and for all measurable set E of finite measure.*

$$\left[\int_E \{(S^*f)(x)\}^\delta dx\right]^{1/\delta} \leqq (1-\delta)^{-1/\delta}A|E|^{(1-\delta)/\delta}\|f\|_{H^1(\boldsymbol{R})} ,$$

where A is the same constant as in (1).

Both Theorem 5 and the corollary imply that, if $f \in H^1(\boldsymbol{R})$, then the lacunary partial means of the integral of $\hat{f}(\xi)e(x\xi)$ converge to $f(x)$ for almost all $x \in \boldsymbol{R}$.

To prove Theorem 5, some comments are needed on the lacunary partial sums of the Fourier series of power series type. For a power series $\Phi(z) = \sum_{m=0}^{\infty} c_m z^m \in H^1(\boldsymbol{D})$, let $(S_n\Phi)(\theta) = \sum_{m=0}^{n} c_m e(m\theta)$. It is stated in [28, p. 231, Th. (4.4)] that, *if a sequence $\{n(k)\}$ satisfies*

$$(2) \qquad n(k+1)/n(k) \geqq \alpha > 1 \qquad (k = 1, 2, \cdots),$$

then $(S_{n(k)}\Phi)(\theta) \to \Phi(e(\theta))$ a.e. *as* $k \to \infty$. Using the fact that the singular integral operators for ℓ^2-valued functions are of weak type (1, 1), and following carefully the proof of Theorem (4.4) in Zygmund's book, we have

$$(3) \qquad |\{\theta \in \boldsymbol{Q};\ \sup_k |(S_{n(k)}\Phi)(\theta)| > t\}| \leqq A_\alpha t^{-1} \|\Phi\|_1$$

for any $t > 0$ and any sequence $\{n(k)\}$ satisfying (2), where the constant A_α does not depend on $\{n(k)\}$ but only on α.

PROOF OF THEOREM 5. Our aim is to deduce (1) from (3). Let $\alpha = (\alpha_0 + 1)/2$ and $\beta = \max\{\alpha, 2/(\alpha_0 - 1)\}$. For a given $\varepsilon > 0$, we write

$$K = K(\varepsilon) = \min\{k;\ \beta \leqq [\varepsilon^{-1}R(k)]\}, \quad n(k) = [\varepsilon^{-1}R(K + k - 2)]$$

$(k = 2, 3, \cdots)$, $n(1) = 1$ and $n(0) = 0$, where $[\cdot]$ denotes the integral part of the number in the bracket. Then $n(k+1)/n(k) \geqq \alpha > 1$ $(k = 1, 2, \cdots)$. Let F be a real valued function in $C^\infty(\boldsymbol{T})$ and let Φ be the Poisson integral of $F + i\tilde{F}$. Then $\Phi \in H^1(\boldsymbol{D})$ and, by (3), the following relation is obtained.

$$(4) \qquad |\{\theta \in \boldsymbol{Q};\ \sup_k |(S_{n(k)}F)(\theta)| > t\}| \leqq A_\alpha t^{-1}(\|F\|_{L^1(\boldsymbol{T})} + \|\tilde{F}\|_{L^1(\boldsymbol{T})})$$

for all $t > 0$, where $S_{n(k)}F$ denotes the $n(k)$-th partial sum of the Fourier series of F. Let χ be the characteristic function of the set $\{\xi \in \hat{\boldsymbol{R}};\ |\xi| \leqq 1\}$ and define λ by

$$\lambda(\xi, k) = \chi(\xi/R(k)) \qquad (\xi \in \hat{\boldsymbol{R}},\ k = 1, 2, \cdots).$$

Defining $T = T_\lambda$ as in § 1 and the corresponding operators $\tilde{T}_\varepsilon$, $\varepsilon > 0$, as in § 2, we see that $(\tilde{T}_\varepsilon F)(\theta, k)$ is equal to the $[\varepsilon^{-1}R(k)]$-th partial sum of the Fourier series of F and so

$$(\tilde{T}_\varepsilon F)(\theta, k) = (S_{n(k-K+2)}F)(\theta) \qquad (k = K, K+1, \cdots)$$

and

$$|(\tilde{T}_\varepsilon F)(\theta, k)| \leqq \max_{0 \leqq n \leqq \beta} |(S_n F)(\theta)| \qquad (k = 1, \cdots, K-1).$$

These relations and (4) together imply that

$$|\{\theta \in \boldsymbol{Q};\ \sup_k |(\tilde{T}_\varepsilon F)(\theta, k)| > t\}| \leqq A'_\alpha t^{-1}(\|F\|_{L^1(\boldsymbol{T})} + \|\tilde{F}\|_{L^1(\boldsymbol{T})})$$

for all $t > 0$ and all $\varepsilon > 0$. Therefore,

$$|\{x \in \boldsymbol{R};\ \sup_k |(Tf)(x, k)| > t\}| \leqq A'_\alpha t^{-1}(\|f\|_{L^1(\boldsymbol{R})} + \|\tilde{f}\|_{L^1(\boldsymbol{R})})$$

for all $t > 0$ and $f \in \mathscr{S}_0(\boldsymbol{R})$ by (ii) of Theorem 3. The right hand side is equal to $A't^{-1}\|f\|_{H^1(\boldsymbol{R})}$ and

$$(Tf)(x, k) = \int_{|\xi|\leq R(k)} \hat{f}(\xi)e(x\xi)d\xi .$$

Thus, we get the theorem by the density of $\mathscr{S}_0(\boldsymbol{R})$ in $H^1(\boldsymbol{R})$.

REFERENCES

[1] A. BENEDEK AND R. PANZONE, The spaces L^P, with mixed norm, Duke Math. J. 28 (1961), 301-324.

[2] A. P. CALDERÓN, Ergodic theory and translation-invariant operators, Proc. Nat. Acad. Sci. U. S. A. 59 (1968), 349-353.

[3] A. P. CALDERÓN AND A. ZYGMUND, Singular integrals and periodic functions, Studia Math. 14 (1954), 249-271.

[4] R. R. COIFMAN ET Y. MEYER, Au delà des opérateurs pseudo-différentiels, Astérisque 57 (1978).

[5] R. R. COIFMAN AND G. WEISS, Operators associated with representations of amenable groups, singular integrals induced by ergodic flows, the rotation method and multipliers, Studia Math. 47 (1973), 285-303.

[6] A. CÓRDOBA AND B. LÓPEZ-MELERO, Spherical summation: A problem of E. M. Stein, Ann. Inst. Fourier (Grenoble) 31 (1981), 147-152.

[7] C. FEFFERMAN, Inequalities for strongly singular convolution operators, Acta Math. 124 (1970), 9-36.

[8] C. FEFFERMAN AND E. M. STEIN, H^p spaces of several variables, Acta Math. 129 (1972), 137-193.

[9] T. M. FLETT, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. London Math. Soc. 7 (1957), 113-141.

[10] D. GOLDBERG, A local version of real Hardy spaces, Duke Math. J. 46 (1979), 27-42.

[11] M. DE GUZMÁN, Real variable methods in Fourier analysis, Mathematics Studies 46, North-Holland, Amsterdam-New York-Oxford, 1981.

[12] R. A. HUNT, On $L(p, q)$ spaces, L'Enseign. Math. 12 (1966), 249-276.

[13] S. IGARI, Lectures on Fourier series of several variables, Lecture Notes at Univ. Wisconsin, 1968.

[14] S. IGARI, Functions of L^p-multipliers, Tôhoku Math. J. 21 (1969), 304-320.

[15] S. IGARI, On the almost everywhere convergence of lacunary Riesz-Bochner means of Fourier transform of two variables, preprint (1980).

[16] S. IGARI, Decomposition theorem and lacunary convergence of Riesz-Bochner means of Fourier transforms of two variables, Tôhoku Math. J. 33 (1981), 413-419.

[17] M. JODEIT, JR., Restrictions and extensions of Fourier multipliers, Studia Math. 34 (1970), 215-226.

[18] M. KANEKO, The absolute Cesàro summability and the Littlewood-Paley function, Tôhoku Math. J. 24 (1972), 223-232.

[19] C. E. KENIG AND P. A. TOMAS, Maximal operators defined by Fourier multipliers, Studia Math. 68 (1980), 79-83.

[20] K. DE LEEUW, On L_p multipliers, Ann. of Math. 81 (1965), 364-379.

[21] E. M. STEIN, On some functions of Littlewood-Paley and Zygmund, Bull. Amer. Math. Soc. 67 (1961), 99-101.

[22] E. M. STEIN, Singular integrals and differentiability properties of functions, Princeton Mathematical Series 30, Princeton Univ. Press, Princeton, New Jersey, 1970.

[23] E. M. STEIN AND G. WEISS, Introduction to Fourier analysis on Euclidean spaces, Princeton Mathematical Series 32, Princeton Univ. Press, Princeton, New Jersey, 1971.
[24] G. SUNOUCHI, Theorems on power series of the class H^p, Tôhoku Math. J. 8 (1956), 125–146.
[25] G. SUNOUCHI, On functions regular in a half-plane, Tôhoku Math. J. 9 (1957), 37–44.
[26] D. WATERMAN, On functions analytic in a half-plane, Trans, Amer. Math. Soc. 81 (1956), 167–194.
[27] A. ZYGMUND, On the Littlewood-Paley function $g^*(\theta)$, Proc. Nat. Acad. Sci. U. S. A. 42 (1956), 208–212.
[28] A. ZYGMUND, Trigonometric series, vol. II, 2nd. ed. Cambridge Univ. Press, London-New York, 1968.

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