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CHARACTERIZATIONS OF S-DECOMPOSABLE OPERATORS ON A COMPLEX BANACH SPACE

Dedicated to the memory of the late Professor Teishirô Saitô

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Introduction. Several characterizations of decomposable operators have been found by Lange [2], Radjabalipour [4] and Tanahashi [6]. On the other hand, S-decomposable operators have been studied by Bacalu [1], Nagy [3] and Vasilescu [7], and we know that there are many similarities between decomposable operators and S-decomposable operators. This paper is a continuation of Tanahashi [6] and we show that S-decomposable operators have characterizations similar to decomposable operators. For example, a bounded linear operator T on a complex Banach space X is S-decomposable if and only if $X_T(F) = X(T, F)$ and the operator T^F on $X/X_T(F)$ induced by T satisfies $\sigma(T^F) \subset (\boldsymbol{C}\backslash F^i) \cup S$ for all closed sets $F \supset S$ in the complex plane $\boldsymbol{C}$ where F^i is the interior of F. This is a generalization of Radjabalipour [4] which is the case $S = \varnothing$, that is, T is a decomposable operator.

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1. Preliminaries. An operator T means a bounded linear transformation on a complex Banach space X. Then there exists a unique maximal open set Ω_T in the complex plane $\boldsymbol{C}$ with the property that if $G \subset \Omega_T$ is an open set and if $f: G \to X$ is an analytic function such that $(z - T)f(z) \equiv 0$ on G, then $f(z) \equiv 0$ on G. Let $S_T = \Omega_T^c$ be the complement of Ω_T in $\boldsymbol{C}$.

For an operator T and a closed set F in $\boldsymbol{C}$, we denote $X_T(F) = \{x \in X \mid \text{there exists an analytic function } f: F^c \to X \text{ such that } (z - T)f(z) \equiv x \text{ on } F^c\}$, and for any set E in $\boldsymbol{C}$ we denote $X_T(E) = \bigcup\{X_T(F) \mid F \subset E \text{ and } F \text{ is closed}\}$ (cf. [4]). These definitions are different from those of [1], [3] and [7]. But it is easy to show that if E contains S_T, then these definitions are equivalent.

$\mathrm{Lat}(T)$ is the lattice of all invariant subspaces of T and $T|Y$ denotes the restriction of T to $Y \in \mathrm{Lat}(T)$. For a closed set F in $\boldsymbol{C}$, we denote by $X(T, F)$ an invariant subspace of T such that (1) $\sigma(T|X(T, F)) \subset F$ and that (2) if $Y \in \mathrm{Lat}(T)$ satisfies $\sigma(T|Y) \subset F$, then $Y \subset X(T, F)$. Of course $X(T, F)$ may or may not exist, but if such an invariant subspace exists, then it is obviously unique. $Y \in \mathrm{Lat}(T)$ is called a spectral maximal space of T if $\sigma(T|Z) \subset \sigma(T|Y)$ implies $Z \subset Y$ for all $Z \in \mathrm{Lat}(T)$. We denote by $\mathrm{SM}(T)$ the family of all spectral maximal spaces of T. Then $X(T, F) \in \mathrm{SM}(T)$, and conversely $Y \in \mathrm{SM}(T)$ can be written $Y = X(T, F)$ with $F = \sigma(T|Y)$. (cf. [3]).

Let T be an operator and $S \subset \sigma(T)$ be a closed set. A family of open sets $\{G_1, \cdots, G_n; G_0\}$ is called an S-covering of $\sigma(T)$ if $G_1 \cup \cdots \cup G_n \cup G_0 \supset \sigma(T)$ and $\bar{G}_i \cap S = \varnothing$ for $i = 1, \cdots, n$. T is called S-decomposable if for every S-covering $\{G_1, \cdots, G_n; G_0\}$ of $\sigma(T)$ there exists a system $\{X_1, \cdots, X_n; X_0\}$ of spectral maximal spaces of T such that (1) $X = X_1 + \cdots + X_n + X_0$ and (2) $\sigma(T|X_i) \subset G_i$ for $i = 1, \cdots, n, 0$.

For $Y \in \mathrm{Lat}(T)$, let T^Y be the operator on X/Y induced by T. If $Y = X_T(F)$ for a closed set F in $\boldsymbol{C}$, then we write T^F instead of T^Y. We denote by $\hat{x}$ the image of $x \in X$ under the canonical mapping of X onto X/Y.

2. Main results.

THEOREM. *Let T be an operator and $S \subset \sigma(T)$ be a closed set. Then the following assertions are equivalent.*

(1) *T is S-decomposable.*

(2) *$X_T(F) = X(T, F)$ for all closed sets $F \supset S$ and $X_T(G_1 \cup G_0) = X_T(G_1) + X_T(G_0)$, where G_1 and G_0 are arbitrary open sets with $\bar{G}_1 \cap S = \varnothing$ and $G_0 \supset S$.*

(3) *$X_T(F) = X(T, F)$ and $\sigma(T^F) \subset (\boldsymbol{C} \setminus F^i) \cup S$ for all closed sets $F \supset S$.*

(4) *If $G \supset S$ is open, then there exists $Y \in \mathrm{Lat}(T)$ such that $\sigma(T|Y) \subset \bar{G}$ and $\sigma(T^Y) \subset G^c \cup S$.*

We need some lemmas for the proof of Theorem.

LEMMA 1 ([3, Lemma 2]). *If a closed set F contains S_T and $X_T(F)$ is closed in X, then $X_T(F) = X(T, F)$.*

The proof of Lemma 2 is similar to [4, Theorem 2.10], hence we omit it here.

LEMMA 2. *Let $Y \in \mathrm{Lat}(T)$. If $\hat{x} \in X_{T^Y}(F)$ for a closed set F in $\boldsymbol{C}$, then $x \in X_T(F \cup \sigma(T|Y) \cup S_T)$.*

The following was inspired by [7, Theorem 4.1].

LEMMA 3. *Let T be an operator and $S \subset \sigma(T)$ be a closed set. If*

$X_T(H) = X(T, H)$ for all closed sets $H \supset S$, then $X_T(F) = X(T, F)$ for all closed sets F with $F \cap S = \varnothing$.

PROOF. Let F be a closed set with $F \cap S = \varnothing$. Let $Y = X_T(F \cup S)$. Then $\sigma(T|Y) \subset F \cup S$. Since F and S are disjoint closed sets, by the Riesz decomposition theorem (see [5, Theorem 2.10]), there exist Y_1 and Y_0 in Lat $(T|Y)$ such that $Y = Y_1 \oplus Y_0$ and $\sigma((T|Y)|Y_1) \subset F$, $\sigma((T|Y)|Y_0) \subset S$ where $\oplus$ denotes the topological direct sum. We show first $X_T(F) = Y_1$. Let $x \in Y_1$. Since $(T|Y)|Y_1 = T|Y_1$, we have $\sigma(T|Y_1) \subset F$, hence $(z - T)(z - T|Y_1)^{-1}x \equiv x$ on F^c. Thus $x \in X_T(F)$. Conversely let $x \in X_T(F)$. Then there exists an analytic function $f\colon F^c \to X$ such that $(z - T)f(z) \equiv x$ on F^c. By [7, Proposition 2.2], $f(z) \in X_T(F)$ for all $z \in F^c$, hence x and $f(z)$ belong to $X_T(F \cup S) = Y$, and so $(z - T)f(z) = (z - T|Y)f(z) = x$ on F^c. We can write $x = x_1 + x_0$ where $x_i \in Y_i$ for $i = 1, 0$. Let D be a Cauchy domain with boundary Γ such that $S \subset D$ and $\bar{D} \subset F^c$. Then,

$$x_0 = \frac{1}{2\pi i}\int_\Gamma (z - T|Y)^{-1}x dz = \frac{1}{2\pi i}\int_\Gamma f(z) dz = 0 .$$

Hence $x = x_1 \in Y_1$. Thus $X_T(F) = Y_1$ and $X_T(F)$ is closed. Since $\sigma(T|X_T(F)) = \sigma(T|Y_1) \subset F$, it is easy to show $X_T(F) = X(T, F)$.

PROOF OF THEOREM. We show the implications $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (2) \Rightarrow (1)$.

$(1) \Rightarrow (3)$. Let T be S-decomposable. Then $S_T \subset S$ by [3, Lemma 4], and so $X_T(F) = X(T, F)$ for all closed sets $F \supset S$ by [3, Lemma 5]. We show $\sigma(T^F) \subset (\boldsymbol{C} \setminus F^i) \cup S$ for all closed sets $F \supset S$. Let $z_1 \in F^i \cap S^c$. Then there exists an open disc G_1 with center z_1 such that $\bar{G}_1 \subset F$ and $\bar{G}_1 \cap S = \varnothing$. We can choose another open set G_0 in $\boldsymbol{C}$ such that $z_1 \notin G_0$ and $\{G_1; G_0\}$ is an S-covering of $\sigma(T)$. Then there exist X_1 and X_0 in SM (T) such that $X = X_1 + X_0$ and $\sigma(T|X_i) \subset G_i$ for $i = 1, 0$. We have only to show that $z_1 - T^F$ is bijective. Let $y \in X$ be given. Then we can write $y = y_1 + y_0$ where $y_i \in X_i$ for $i = 1, 0$. Since $z_1 \in \rho(T|X_0)$, there exists $x = (z_1 - T|X_0)^{-1}y_0$, and so $(z_1 - T)x = y_0$. Hence $(z_1 - T^F)\hat{x} = \hat{y}_0 = \hat{y}_1 + \hat{y}_0 = \hat{y}$ because $X_1 \subset X_T(G_1) \subset X_T(F)$. Thus $z_1 - T^F$ is surjective. Let $(z_1 - T^F)\hat{x} = \hat{0}$. Then $(z_1 - T)x \in X_T(F)$. We can write $x = x_1 + x_0$ where $x_i \in X_i$ for $i = 1, 0$. Since $(z_1 - T)x_1 \in X_1 \subset X_T(G_1) \subset X_T(F)$, we have $(z_1 - T)x_0 = (z_1 - T)x - (z_1 - T)x_1 \in X_T(F)$. Hence $(z_1 - T)x_0 \in X_T(F) \cap X_T(\bar{G}_0) = X_T(F \cap \bar{G})$ because $S_T \subset S \subset F$ and $S \subset G_0$. Then there exists an analytic function $f\colon (F \cap \bar{G}_0)^c \to X$ such that $(z - T)f(z) \equiv (z_1 - T)x_0$. By [7, Proposition 2.2], we have $f(z) \in X_T(F \cap \bar{G}_0) \subset X_T(\bar{G}_0)$, hence $(z - T)(z_1 - T|X_T(\bar{G}_0))^{-1}f(z) = (z - T|X_T(\bar{G}_0))(z_1 - T|X_T(\bar{G}_0))^{-1}f(z) = (z_1 - T|X_T(\bar{G}_0))^{-1}(z - T|X_T(\bar{G}_0))f(z) = (z_1 - T|X_T(\bar{G}_0))^{-1}(z_1 - T)x_0 = x_0$ on F^c because $x_0 \in X_T(\bar{G}_0)$. This implies $x_0 \in X_T(F)$. Then $x = x_1 + x_0 \in X_T(F)$, hence $\hat{x} = \hat{0}$. Thus $z_1 - T^F$ is injective.

(3) $\Rightarrow$ (4). Let $Y = X_T(\bar{G})$ for all open sets $G \supset S$.

(4) $\Rightarrow$ (2). We show first $S_T \subset S$. Let D be an open set in C with $D \cap S = \emptyset$, and $f: D \to X$ be an analytic function such that $(z - T)f(z) \equiv 0$ on D. We have only to show $f(z) \equiv 0$ on D. We may assume D to be an open disc. Then there exists an open disc G with $\bar{G} \subset D$. Since $\bar{G}^c$ is an open set containing S, there exists $Y \in \mathrm{Lat}\,(T)$ such that $\sigma(T|Y) \subset G^c$ and $\sigma(T^Y) \subset \bar{G} \cup S$. Then $(z - T^Y)\widehat{f(z)} \equiv \hat{0}$ on D. Since $D \cap \rho(T^Y) \neq \emptyset$, we have $\widehat{f(z)} \equiv 0$ on $D \cap \rho(T^Y)$, hence $\widehat{f(z)} \equiv \hat{0}$ on D because $\widehat{f(z)}$ is analytic on D. Hence $f(z) \in Y$ on D, and so $(z - T)f(z) \equiv (z - T|Y)f(z) \equiv 0$ on D. If $f(z) \not\equiv 0$ on D, then it is easy to show $D \subset \sigma_p(T|Y) \subset \sigma(T|Y) \subset G^c$ where $\sigma_p(T|Y)$ is the point spectrum of $T|Y$. This is a contradiction. Thus $f(z) \equiv 0$ on D.

We show next $X_T(F) = X(T, F)$ for all closed sets $F \supset S$. Since $F \supset S_T$, we have only to show that $X_T(F)$ is closed by Lemma 1. We may assume $F \subset \sigma(T)$. Let $G \supset F$ be any open set. Then there exists $Y \in \mathrm{Lat}\,(T)$ such that $\sigma(T|Y) \subset \bar{G}$ and $\sigma(T^Y) \subset G^c \cup S$. Since G^c and S are disjoint closed sets, by the Riesz decomposition theorem, there exist Z_1 and Z_0 in $\mathrm{Lat}\,(T^Y)$ such that $X/Y = Z_1 \oplus Z_0$, and we write $T^Y = U_1 \oplus U_0$ with $\sigma(U_1) \subset G^C$ and $\sigma(U_0) \subset S$ where $U_i = T^Y|Z_i$ for $i = 1, 0$. Let $x \in X_T(F)$ be given. Then there exists an analytic function $f: F^C \to X$ such that $(z - T)f(z) \equiv x$ on F^C. Hence we can write $(z - T^Y)\widehat{f(z)} = (z - U_1)g_1(z) \oplus (z - U_0)g_0(z) = \hat{x}_1 \oplus \hat{x}_0 = \hat{x}$ on F^C for $\widehat{f(z)} = g_1(z) \oplus g_0(z)$ with $g_i(z) \in Z_i$ for $i = 1, 0$. Since $\sigma(U_1) \subset G^C$, we can extend $g_1(z)$ analytically on $F^C \cup G = C$. And since $g_1(z) = (z - U_1)^{-1}\hat{x}_1 \to \hat{0}$ as $|z| \to \infty$, we have $g_1(z) \equiv \hat{0}$ by Liouville's theorem, and $\hat{x}_1 = \hat{0}$. Hence $\hat{x} = \hat{x}_0 \in Z_0$. Thus $X_T(F) \subset \Pi^{-1}(Z_0)$ where $\Pi: X \to Z_1 \oplus Z_0 = X/Y$ is the canonical mapping. We show next $\Pi^{-1}(Z_0) \subset X_T(\bar{G})$. Let $x \in \Pi^{-1}(Z_0)$ be given. Then $\hat{x} \in Z_0$, and so $(z - T^Y)(z - U_0)^{-1}\hat{x} = \hat{x}$ on $\rho(U_0) \supset S^C$. Hence $\hat{x} \in X_{T^Y}(S)$, and so $x \in X_T(S \cup \sigma(T|Y) \cup S_T) \subset X_T(\bar{G})$ by Lemma 2. Hence $X_T(F) \subset \Pi^{-1}(Z_0) \subset X_T(\bar{G})$. Since $G \supset F$ is any open set and $S_T \subset S \subset F$, we have $X_T(F) \subset \bigcap \{\Pi^{-1}(Z_0) | F \subset G\} \subset \bigcap \{X_T(\bar{G}) | F \subset G\} = X_T(F)$. Thus $X_T(F) = \bigcap \{\Pi^{-1}(Z_0) | F \subset G\}$ and $X_T(F)$ is closed.

We show $X_T(G_1 \cup G_0) = X_T(G_1) + X_T(G_0)$ for all open sets G_1 and G_0 with $\bar{G}_1 \cap S = \emptyset$ and $G_0 \supset S$. It is clear that $X_T(G_1 \cup G_0) \supset X_T(G_1)$, $X_T(G_0)$ and $X_T(G_1 \cup G_0) \supset X_T(G_1) + X_T(G_0)$. Conversely let $x \in X_T(G_1 \cup G_0)$ be given. Then there exists a closed set $F \subset G_1 \cup G_0$ and an analytic function $f: F^C \to X$ such that $(z - T)f(z) \equiv x$ on F^C. Then we can choose open sets D_1 and D_0 such that $\bar{D}_i \subset G_i$, for $i = 1, 0$, and $F \cup S \subset D_1 \cup D_0$. And we can choose an open set D_2 such that $S \subset D_2$, $\bar{D}_2 \subset D_0$ and $\bar{D}_2 \cap \bar{G}_1 = \varnothing$. Since $G = (D_1 \cap D_0) \cup D_2$ is an open set containing S, there exists $Y \in \mathrm{Lat}(T)$ such that $\sigma(T|Y) \subset \bar{G}$ and $\sigma(T^Y) \subset G^c \cup S$. Then $(z - T^Y)\hat{f}(z) \equiv \hat{x}$ on F^c. Since $\rho(T^Y) \supset G \cap S^c \supset D_1 \cap D_0$, we can extend $\widehat{f(z)}$ analytically on $F^c \cup (D_1 \cap D_0) = \{(F \setminus D_1) \cup (F \setminus D_0)\}^c$. Hence we can write $\hat{x} = \hat{x}_1 + \hat{x}_0$ where $\hat{x}_i \in X_{T^Y}(F \setminus D_j)$, for $i \neq j$, by [4, Theorem 2.3]. Hence $x = x_1 + x_0 + y$ for some $y \in Y$. Then by Lemma 2, $x_1 \in X_T((F \setminus D_0) \cup \bar{G})$, and so $x_1 \in X_T(\bar{D}_1) + X_T(\bar{D}_0) \subset X_T(G_1) + X_T(G_0)$ by [4, Theorem 2.3]. Similarly $x_0 \in X_T((F \setminus D_1) \cup \bar{G} \subset (X_T(G_0)$ and $y \in Y \subset X_T(\bar{G}) \subset X_T(G_0)$. Thus $x \in X_T(G_1) + X_T(G_0)$ and $X_T(G_1 \cup G_0) \subset X_T(G_1) + X_T(G_0)$.

(2) $\Rightarrow$ (1). Let $\{G_1, \cdots, G_n; G_0\}$ be any S-covering of $\sigma(T)$. Then we can choose an S-covering $\{D_1, \cdots, D_n; D_0\}$ of $\sigma(T)$ such that $\bar{D}_i \subset G_i$ for $i = 1, \cdots, n, 0$. Then $X \subset X_T(\sigma(T)) \subset X_T(D_1 \cup \cdots \cup D_n \cup D_0) = X_T(D_1) + X_T(D_2 \cup \cdots \cup D_n \cup D_0) = \cdots = X_T(D_1) + \cdots + X_T(D_n) + X_T(D_0) \subset X_T(\bar{D}_1) + \cdots + X_T(\bar{D}_n) + X_T(\bar{D}_0)$. Since $D_0 \supset S$, we have $X_T(\bar{D}_0) = X(T, \bar{D}_0)$ by the assumption. And since $\bar{D}_i \cap S = \varnothing$ for $i = 1, \cdots, n$, we have $X_T(\bar{D}_i) = X(T, \bar{D}_i)$ for $i = 1, \cdots, n$ by Lemma 3. Hence $X = X(T, \bar{D}_1) + \cdots + X(T, \bar{D}_n) + X(T, \bar{D}_0)$. This implies that T is S-decomposable.

REMARK. Lemmas 1 and 2 hold for all closed linear operators on a complex Banach space. Then can Theorem be extended to all closed linear operators?

REFERENCES

[1] I. BACALU, S-decomposable operators in a Banach space, Rev. Roum. Math. Pures et Appl. 20 (1975), 1101–1107.

[2] R. LANGE, Equivalent conditions for decomposable operators, Proc. Amer. Math. Soc. 82 (1981), 401–406.

[3] B. NAGY, Closed S-decomposable operators, Annales Univ. Sci. Budapest, Sectio Math. 22–23 (1979–1980), 143–149.

[4] M. RADJABALIPOUR, Decomposable operators, Bull. Iranian Math. Soc. 9 (1978), 1L–49L.

[5] M. RADJAVI AND P. ROSENTHAL, Invariant subspaces, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[6] K. TANAHASHI, A characterization of decomposable operators, Tôhoku Math. J. 34 (1982), 295–300.

[7] F.-H. VASILESCU, Residually decomposable operators in Banach spaces, Tôhoku Math. J. 21 (1969), 509–522.

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