Tôhoku Math. Journ.
35 (1983), 249-259.

UNIQUENESS IN THE CAUCHY PROBLEM FOR A CLASS OF DIFFERENTIAL OPERATORS WITH DOUBLE CHARACTERISTICS

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(Received February 8, 1982, revised July 30, 1982)

0. Introduction. Hörmander's theorem on the unique propagation of zeros of solutions of partial differential homogeneous equations through strongly pseudo-convex surfaces [6, Thm. 8.9.1] cannot be used if the principal part of the operator has a double zero. The presence of real double characteristics also precludes the use of Calderón's uniqueness results ([4] and [5]). On the other hand, there has been considerable development in very recent times on uniqueness for the non-characteristic Cauchy problem for operators with higher characteristics, for instance [1], [2], [3], [7], [9], [10] to mention a few (the last reference contains an extensive list of results known to date).

The purpose of this paper is to prove uniqueness in the non-characteristic Cauchy problem for a class of differential operators of order two with double real characteristics modelled upon the heat equation. Let Ω be an open subset of $\boldsymbol{R}^{n+1}$ and denote by $z=(x, y)$, $x \in \boldsymbol{R}^n$, $y \in \boldsymbol{R}$ the variable point in Ω. We consider an operator

$$P(z, D_z) = \sum_{|\alpha| \leq 2} a_\alpha(z) D_x^\alpha + c(z)\partial_y \tag{0.1}$$

with principal symbol $p(x, y; \xi)$. We shall assume that

$$p(x, y; \xi) \text{ and } c(x, y) \text{ are real and } c(x, y) \neq 0\,. \tag{0.2}$$

Let Σ be an oriented non-characteristic surface in Ω and consider a point z_0 in Σ. We prove in Theorem 1.1 that if Σ is "partially pseudo-convex" with respect to P in the direction of x at z_0 there is unique propagation of the zeros of the solutions of $Pu=0$ through Σ in a neighborhood of z_0 (see Definition 1.1 for the precise meaning of partial pseudo-convexity). It is interesting to note that a class of operators analogous to those studied here but modelled on the Schrödinger operator (i.e., with the coefficient $c(z)$ of ∂_y purely imaginary rather than real) was studied in [7] and sufficient and close to necessary "pseudo-convexity" conditions for uniqueness in the Cauchy problem were given. These conditions, however, bear on the sub-principal symbol whereas the notion of partial pseudo-convexity, naturally associated to (0.1) when $c(z)$ is real, depends on the principal symbol alone.

The proof of Theorem 1.1 is based on a Carleman estimate

$$\gamma\| |\phi|^{-\gamma-1/2}\nabla_x u\|^2 + \gamma^{-1}\| |\phi|^{-\gamma}u_y\|^2 + \gamma^3\| |\phi|^{-\gamma-2}u\|^2 \leqq C\| |\phi|^{-\gamma}Pu\|^2 \tag{0.3}$$

valid for u smooth and compactly supported in the negative side of the initial surface $\phi = \phi(z_o)$ and γ large. It is obtained by patching-up three microlocal energy estimates.

Concerning the necessity of our hypothesis, it is a consequence of a result of Alinhac that if partial pseudo-convexity is violated in a strict sense at a non-radial fiber point ξ over z_o where $dp(z_o; \xi) \neq 0$, there exists a zero order perturbation of P for which there is no uniqueness through Σ.

We are indebted to Professor Zuily who kindly posed this problem in the course of his lectures [10] at Recife. We also thank the anonymous referee who pointed out a flaw in the original proof of Lemma 2.1 and suggested the use of a partition of unity in the x-space to correct it.

1. Partial pseudo-convexity. Let Ω be an open subset of $\boldsymbol{R}^n \times \boldsymbol{R}^m$. We denote by $z = (x, y)$, $x \in \boldsymbol{R}^n$, $y \in \boldsymbol{R}^m$, the variable point in Ω. Given a point $z_o = (x_o, y_o)$ in Ω, consider the submanifold $\{(x_o, y) \in \Omega\}$. Its tangent space at z_o is the subspace of $T_{z_o}(\Omega)$ generated by $\partial/\partial y^i$, $i = 1, \cdots, m$. We shall denote by $N_{z_o} \subseteqq T^*_{z_o}(\Omega)$ the orthogonal to this subspace. In local coordinates,

$$N_{z_o} = \{(x_o, y_o; \xi, 0) \in T^*(\Omega)\} .$$

Given a differential operator $P(x, y, D_x, D_y)$ with real principal symbol $p(x, y; \xi, \eta)$ and smooth coefficients, we shall define the notion of partial pseudo-convexity of a surface Σ with respect to P in the direction of x. Let ϕ be a real valued smooth function defined in a neighborhood of z_o and assume that $\nabla\phi(z_o) \neq 0$. Then the equation

$$\phi(z) = \phi(z_o) \tag{1.1}$$

defines a non-singular oriented level surface in a neighborhood of z_o; we say a point z of the neighborhood of z_o is in the positive (resp. negative) side of the surface when $\phi(z) > \phi(z_o)$ (resp. $\phi(z) < \phi(z_o)$). In the following definition $\{\ ,\ \}$ denotes the Poisson bracket.

DEFINITION 1.1. The oriented non-characteristic surface Σ defined by (1.1) will be called partially pseudo-convex with respect to P in the direction of x at z_o if

$$\{p, \{p, \phi\}\}(z_o, \zeta) > 0 \quad \text{for all} \quad (z_o, \zeta) \in N_{z_o} \setminus \{0\} \tag{1.2}$$

such that

$$p(z_0, \zeta) = 0 \quad \text{and} \quad \{p, \phi\}(z_0, \zeta) = 0\,. \tag{1.3}$$

If ψ defines the same surface Σ with the same orientation, then $\nabla\phi = \lambda\nabla\psi$ with $\lambda > 0$ at z_0 and $\{p, \{p, \phi\}\}(z_0, \zeta) = \lambda\{p, \{p, \psi\}\}(z_0, \zeta)$ if (z_0, ζ) verifies (1.3) so the definition is independent of the function used to define the oriented surface.

REMARK 1.1. If $m = 0$, then $N_{z_0} = T^*_{z_0}(\Omega)$ and partial pseudo-convexity is just pseudo-convexity. On the other hand if $\Sigma \subseteq \boldsymbol{R}^n$ is pseudo-convex with respect to $P(x, D_x)$, then $\Sigma \times \boldsymbol{R}^m$ will be partially pseudo-convex in the direction of x with respect to $P(x, D_x)$ in $\boldsymbol{R}^n \times \boldsymbol{R}^m$.

REMARK 1.2. Assume that P is given by

$$P = D_t^2 + \sum_{|\alpha| \leqq 2} a_\alpha(x, t, y) D_x^\alpha + c(x, t, y)\partial_y \tag{1.4}$$

with $x \in \boldsymbol{R}^{n-1}$, $t \in \boldsymbol{R}$, $y \in \boldsymbol{R}$ and Σ is given by $\phi(x, t, y) = -t = 0$ where $D_t = -\sqrt{-1}\,\partial/\partial t$, $D_x = -\sqrt{-1}(\partial/\partial x^1, \cdots, \partial/\partial x^{n-1})$ and $\partial_y = \partial/\partial y$ in a neighborhood of the origin. Then, Σ is partially pseudo-convex with respect to P in the direction of (x, t) at the origin if and only if the following condition holds:

$$(\partial a/\partial t)(0; \xi) > 0 \quad \text{for all } \xi \in \boldsymbol{R}^{n-1}\setminus\{0\} \text{ such that } a(0; \xi) = 0 \tag{1.5}$$

where we used the notation

$$a(x, t, y; \xi) = \sum_{|\alpha| = 2} a_\alpha(x, t, y)\xi^\alpha\,. \tag{1.6}$$

REMARK 1.3. If $\Sigma = \{\phi = \phi(z_0)\}$ is partially pseudo-convex with respect to P at z_0 and $\Sigma' = \{\psi = \psi(z_0)\}$ is another surface tangent to the former and with the same orientation, it follows that Σ' is also partially pseudo-convex with respect to P at z_0 if the second order derivatives of $\phi - \psi$ at z_0 are small enough. In particular, if F is a closed subset of Ω contained in the negative side of Σ, we may find Σ' tangent to Σ and partially pseudo-convex with respect to P so that F is contained in the negative side of $\Sigma' = \{\psi = \psi(z_0)\}$ and furthermore the sets

$$\{\psi \geqq \psi(z_0) - \varepsilon\} \cap F$$

are compact if $\varepsilon > 0$ is small enough. More generally we have:

PROPOSITION 1.1. *If the surface* (1.1) *is partially pseudo-convex with respect to P at $z_0 \in \Omega$ in the direction of x, then there exists a neighborhood ω of z_0 and a positive number δ such that every $\psi \in C^\infty(\omega)$ for which $|D^\alpha(\phi - \psi)| < \delta$ in ω, $|\alpha| \leqq 2$, has partially pseudo-convex level surfaces* *with respect to P in the direction of x at every point of ω.*

The proof is trivial.

We now state our main result.

THEOREM 1.1. *Let Ω be an open subset of $\boldsymbol{R}^{n+1}$, $z_o = (x_o, y_o) \in \Omega$ and consider the differential operator with C^∞ coefficients* (0.1) *satisfying* (0.2). *Let Σ be an oriented surface which is partially pseudo-convex with respect to P at (x_o, y_o) in the direction of x and non-characteristic with respect to P at (x_o, y_o).*

Then, there is a neighborhood ω of (x_o, y_o) such that if $u \in C^\infty(\Omega)$, $Pu = 0$ and u vanishes on the positive part of Σ, u must vanish identically in ω.

Observe that (0.1) has double real characteristics (p vanishes to the second order at $(z_o; 0, \cdots, 1)$) so there are no pseudo-convex surfaces with respect to P. Furthermore, the equation $p(z, \zeta + \tau N) = 0$ always possesses the double root $\tau = 0$ when $\zeta = (0, \cdots, 1)$. If there exists a partially pseudo-convex surface with normal N at z_o, the double root $\tau = 0$ will split into a pair of simple roots (real or complex conjugate) at nearby points $(z; \zeta)$.

When $\sum_{|\alpha| \leq 2} a_\alpha(x, y) D_x^\alpha$ is elliptic in $\boldsymbol{R}^n$, as in the case of the heat operator in $\boldsymbol{R}^{n+1}$ (with y representing the time variable), all non-characteristic surfaces are partially pseudo-convex in the direction of x. In this case Theorem 1.1 implies a result of Mizohata [8]. A different example, is the "heat equation" based on the Tricomi operator: $P = D_t^2 + tD_x^2 + \partial_y$. In this case the oriented surface $\phi = 0$ is partially pseudo-convex in the (x, t)-direction if $\phi(x, t, y) = -t$ and is not if $\phi(x, t, y) = t$.

REMARK 1.4. Concerning the necessity of partial pseudo-convexity in Theorem 1.1, the following remarks are in order. Keeping the notation of the theorem, to say that Σ is not partially pseudo-convex with respect to P at z_o means that there exists $\xi_o \in \boldsymbol{R}^n \setminus \{0\}$ such that

$$p(z_o; \xi_o) = 0\,, \qquad \{p, \phi\}(z_o; \xi_o) = 0 \quad \text{and} \quad \{p, \{p, \phi\}\}(z_o; \xi_o) \leqq 0\,.$$

If we strengthen this to $\{p, \{p, \phi\}\}(z_o; \xi_o) < 0$ and assume furthermore that $dp(z_o; \xi_o) \neq 0$ and the characteristic set $\{(z; \zeta) \in T^*\boldsymbol{R}^{n+1} \setminus \{0\}; p(z; \zeta) = 0\}$ is transverse to the fibers, it follows from a recent result of Alinhac [1, Theorem 2] that there exists a smooth function b such that uniqueness in the Cauchy problem does not hold for $P + b$ through the oriented surface Σ.

COROLLARY 1.2. *Let Ω be an open subset of $\boldsymbol{R}^n$, $x_o \in \Omega$ and consider* *a second order operator*

$$Q = \sum_{|\alpha| \leqq 2} a_\alpha(x) D_x^\alpha$$

with real principal symbol and smooth coefficients. Let Σ be an oriented surface which is pseudo-convex with respect to Q at x_o. Then, there is a neighborhood ω of x_o such that if $u \in C^\infty(\Omega)$, $Qu = 0$ and u vanishes in the positive side of Σ, u must vanish identically in ω.

Indeed, this follows if we apply Theorem 1.1 to $Q + \partial/\partial y$. This result is a particular case of the aforementioned theorem of Hörmander since for second order operators with real principal symbol there is no difference between pseudo-convexity and strong pseudo-convexity.

2. Proof of Theorem 1.1. We may choose local coordinates (x, t, y), $x \in \boldsymbol{R}^{n-1}$, $t \in \boldsymbol{R}$, $y \in \boldsymbol{R}$, so that $z_o = (0, 0, 0)$, Σ is defined by $\phi(x, t, y) = -t = 0$ and (changing P by $-P$ if necessary) P is given by

$$P = \partial^2/\partial t^2 + \sum_{i,j=1}^{n-1} a_{ij}(\partial^2/\partial x^i \partial x^j) + c(\partial/\partial y) + \sum_{i=1}^{n-1} \alpha_i(\partial/\partial x^i) + \alpha(\partial/\partial t) + \beta \tag{2.1}$$

with

$$c(0) > 4 \tag{2.2}$$

$$a_{ij} = \overline{a_{ij}} = a_{ji}\,, \qquad i, j = 1, \cdots, n-1\,. \tag{2.3}$$

Observe that the definition of partial pseudo-convexity (Definition 1.1) does not make use of coordinates. Once a decomposition of $\boldsymbol{R}^{n+1}$ as the product of say, $\boldsymbol{R}^n \times \boldsymbol{R}$ is chosen, partial pseudo-convexity is invariant under coordinate changes of the form $(x, t, y) \mapsto (x'(x, t, y), t, y'(y))$. In these coordinates the partial pseudo-convexity is expressed by (1.5). Furthermore (see Remark 1.3) there is no loss of generality in assuming that the sets $\operatorname{supp} u \cap \{t \leqq \varepsilon\}$ are compact for small positive values of ε. Thus, Theorem 1.1 will follow in a standard way from:

LEMMA 2.1. *Assume that P is given by* (2.1) *and satisfies* (2.2). *Then there exists a neighborhood ω of the origin and a positive constant C, such that when γ is sufficiently large the estimate*

$$\begin{aligned}\gamma^3\| t^{-\gamma-2}u \|^2 + \gamma\| t^{-\gamma-1}(\partial u/\partial t)\|^2 \\ + \sum_{j=1}^{n-1} \| t^{-\gamma-1/2}(\partial u/\partial x^j)\|^2 + \gamma^{-1}\| t^{-\gamma}\partial u/\partial y\|^2 \leqq C\| t^{-\gamma}Pu\|^2\end{aligned} \tag{2.4}$$

holds for all $u \in C_c^\infty(\bar\omega \cap \{t \geqq 0\})$. Here $\|\ \ \|$ indicates the L^2-norm in all variables.

PROOF. It is enough to prove (2.4) with P replaced by $P_o = \partial_t^2 +$

$\sum_{i,j} \partial_i a_{ij} \partial_j + c\partial_y$ with $\partial/\partial x_j = \partial_j$. We may also assume that the support of u is a compact subset of $\{t > 0\}$ which will be useful later since in the course of the proof we use an operator depending smoothly on t for $t > 0$ but not defined for $t = 0$. The general case will follow easily considering the sequence $u(x, t - 1/n, y)$, $n \to \infty$.

We set $v = t^{-\gamma} u$. Thus, $P_0 u = t^{\gamma} \tilde{P} v$ with

$$\tilde{P} = \partial_t^2 + \sum_{i,j} \partial_i a_{ij} \partial_j + \gamma(\gamma - 1)t^{-2} + 2\gamma t^{-1}\partial_t + c\partial_y .$$

Consider the operator

$$P_1 = \tilde{P} + (1/2\gamma)\partial_y .$$

To prove (2.4) for large γ we need only to show the following estimate for P_1:

$$\gamma^3 \| t^{-2} v \|^2 + \gamma \| t^{-1} v_t \|^2 + \gamma \sum_i \| t^{-1/2} \partial_i v \|^2 + \gamma^{-1} \| v_y \|^2 \leqq C \| P_1 v \|^2 . \tag{2.5}$$

Indeed, $\| P_1 v \|^2 = \| \tilde{P} v + (1/2\gamma) v_y \|^2 \leqq 2 \| \tilde{P} v \|^2 + (1/2)\gamma^{-2} \| v_y \|^2$ and the second term on the right-hand side is absorbed by $\gamma^{-1} \| v_y \|^2$ as $\gamma \to \infty$. To obtain (2.5) write $P_1 = M + N$ where

$$M = \partial_t^2 + \sum_{i,j} \partial_i a_{ij} \partial_j + (\gamma^2 - \sigma\gamma)t^{-2} + (1/2\gamma)\partial_y ,$$
$$N = 2\gamma t^{-1}\partial_t + c\partial_y + (\sigma - 1)\gamma t^{-2} .$$

Note that we have split the term $\gamma(\gamma - 1)t^{-2}$ into $(\gamma^2 - \sigma\gamma)t^{-2}$ and $\gamma(\sigma - 1)t^{-2}$. Here σ is a real number that we will take later equal to 0, 1 and -1 to obtain three estimates that, combined, will yield (2.5). At any rate, $|\sigma| \leqq 1$. We may write

$$\| P_1 v \|^2 = \| Mv + Nv \|^2 \geqq 2 \operatorname{Re} (Mv, Nv) \tag{2.6}$$

where (,) indicates the inner product in L^2. To compute $2 \operatorname{Re} (Mv, Nv)$ it is convenient to use the formulas below. They are easily proved by integration by parts. We assume that v is compactly supported and α is a smooth real function:

$$2 \operatorname{Re} (v_t, \alpha v) = -(v, \alpha_t v) , \qquad 2 \operatorname{Re} (v_{tt}, \alpha v) = (v, \alpha_{tt} v) - 2(v_t, \alpha v_t)$$
$$2 \operatorname{Re} \sum_{i,j} (\partial_i a_{ij} \partial_j v, \alpha v) = \sum_{i,j} ([a_{ij}]_t \partial_j v, \alpha \partial_i v) + \sum_{i,j} (a_{ij} \partial_j v, \alpha_t \partial_i v)$$
$$- 2 \operatorname{Re} \sum_{i,j} (a_{ij} \partial_j v, (\partial\alpha/\partial x^j) v_t) .$$

In the sequel C will denote a large positive constant that need not be the same in each expression and v will be a smooth function with compact support contained in $\omega \cap \{t > 0\}$. We will need to shrink the neighborhood ω a number of times in the course of the proof. Typically, C is chosen to dominate certain coefficients of P_1 or its derivatives, thus, shrinking ω does not affect the size of C. Finally, γ is a big parameter that will eventually tend to infinity. We compute the terms appearing in $2\operatorname{Re}(Mv, Nv)$:

$$I_1 = 2\operatorname{Re}(\partial_t^2 v, 2\gamma t^{-1}\partial_t v) = 2\gamma\|t^{-1}\partial_t v\|^2 ; \tag{2.7}$$

$$\begin{aligned} I_2 &= 2\operatorname{Re}(\partial_t^2 v, c\partial_y v) = (\partial_t v, c_y\partial_t v) - 2\operatorname{Re}(\partial_y v, c_t\partial_t v) \\ |I_2| &\leqq C\gamma\|v_t\|^2 + \gamma^{-1}\|v_y\|^2 \leqq 2^{-1}\gamma\|t^{-1}v_t\|^2 + \gamma^{-1}\|v_y\|^2 , \end{aligned} \tag{2.8}$$

shrinking ω in the t-direction;

$$I_3 = 2\operatorname{Re}(\partial_t^2 v, (\sigma - 1)\gamma t^{-2}v) = (2\gamma\|t^{-1}\partial_t v\|^2 - 6\gamma\|t^{-2}v\|^2)(1-\sigma) ; \tag{2.9}$$

$$\begin{aligned} I_4 &= 2\operatorname{Re}(\textstyle\sum \partial_i a_{ij}\partial_j v, 2\gamma t^{-1}\partial_t v) \\ &= 2\gamma \textstyle\sum ((\partial_t a_{ij})\partial_j v, t^{-1}\partial_i v) - 2\gamma \sum (a_{ij}\partial_j v, t^{-2}\partial_i v) ; \end{aligned} \tag{2.10}$$

$$I_5 = 2\operatorname{Re}(\textstyle\sum \partial_i a_{ij}\partial_j v, c\partial_y v) , \qquad |I_5| \leqq C\gamma\|\nabla_x v\|^2 + \gamma^{-1}\|v_y\|^2 \tag{2.11}$$

where $\nabla_x v$ indicates the gradient of v in the x variables;

$$I_6 = 2\operatorname{Re}(\textstyle\sum \partial_i a_{ij}\partial_j v, (\sigma - 1)\gamma t^{-2}v) = 2\gamma(1-\sigma)\sum (a_{ij}\partial_j v, t^{-2}\partial_i v) ; \tag{2.12}$$

$$I_7 = 2\operatorname{Re}((\gamma^2 - \sigma\gamma)t^{-2}v, 2\gamma t^{-1}\partial_t v) = 6\gamma(\gamma^2 - \sigma\gamma)\|t^{-2}v\|^2; \tag{2.13}$$

$$\begin{aligned} I_8 &= 2\operatorname{Re}((\gamma^2 - \sigma\gamma)t^{-2}v, c\partial_y v) = -(\gamma^2 - \sigma\gamma)(v, c_y t^{-2}v) \\ |I_8| &\leqq C\gamma^2\|t^{-1}v\|^2 \leqq C\gamma^2\|t^{-2}v\|^2 \end{aligned} \tag{2.14}$$

assuming $|t| < 1$;

$$I_9 = 2\operatorname{Re}((\gamma^2 - \sigma\gamma)t^{-2}v, (\sigma - 1)\gamma t^{-2}v) = \gamma(\gamma^2 - \sigma\gamma)(\sigma - 1)\|t^{-2}v\|^2 ; \tag{2.15}$$

$$I_{10} = 2\operatorname{Re}((2\gamma)^{-1}\partial_y v, 2\gamma t^{-1}\partial_t v) , \qquad |I_{10}| \leqq \gamma^{-1}\|v_y\|^2 + \gamma\|t^{-1}v_t\|^2 ; \tag{2.16}$$

$$I_{11} = 2\operatorname{Re}((2\gamma)^{-1}\partial_y v, c\partial_y v) \geqq \gamma^{-1}\inf_{\omega} c(z)\|v_y\|^2 \geqq 4\gamma^{-1}\|v_y\|^2 , \tag{2.17}$$

taking advantage of (2.2) and shrinking ω conveniently; finally

$$I_{12} = 2\operatorname{Re}((2\gamma)^{-1}\partial_y v, (\sigma - 1)\gamma t^{-2}v) = 0 . \tag{2.18}$$

Writing $2\operatorname{Re}(Mv, Nv) = \sum_{i=1}^{12} I_i$ and collecting the identities and estimates furnished by (2.7) to (2.18) we obtain

$$\begin{aligned} 2\operatorname{Re}(Mv, Nv) \geqq{}& \|t^{-2}v\|^2(\gamma^3(5+\sigma) + O(\gamma^2)) + \gamma\|t^{-1}v_t\|^2(5/2 - 2\sigma) \\ &+ \gamma[\textstyle\sum ((\partial_t a_{ij})\partial_j v, t^{-1}\partial_i v) - 2\sigma \sum (a_{ij}\partial_j v, t^{-2}\partial_i v)] \\ &+ \gamma^{-1}\|v_y\|^2 - C\gamma\|\nabla_x v\|^2 . \end{aligned} \tag{2.19}$$

Combining (2.6) and (2.19) we obtain for large γ and $|\sigma| \leqq 1$

$$\begin{aligned} \|P_1 v\|^2 \geqq{}& 3\gamma^3\|t^{-2}v\|^2 + 2^{-1}\gamma\|t^{-1}v_t\|^2 + \gamma^{-1}\|v_y\|^2 \\ &+ \gamma \textstyle\sum ((\partial_t a_{ij})\partial_j v, t^{-1}\partial_i v) - 2\sigma\gamma \sum (a_{ij}\partial_j v, t^{-2}\partial_i v) \\ &+ \gamma C(\Delta_x v, v) \end{aligned} \tag{2.20}$$

where Δ_x indicates the Laplacian in the x-variables so $\|\nabla_x v\|^2 = -(\Delta_x v, v)$. Now we let σ take the values $\sigma = -1$, $\sigma = 0$ and $\sigma = 1$ in (2.20) in order to get the three key estimates. To shorten the formulas we write

$$|||v|||_\gamma^2 = \gamma^3\|t^{-2}v\|^2 + \gamma\|t^{-1}v_t\|^2 + \gamma^{-1}\|v_y\|^2 .$$

Thus, for $\sigma = -1$ we get

$$C\|P_1 v\|^2 \geqq |||v|||_\gamma^2 + 2\gamma \sum (a_{ij}\partial_j v, t^{-2}\partial_i v) + \gamma C(t^{-1}\Delta_x v, v) . \tag{2.21}$$

Letting $\sigma = 0$, we obtain

$$C\|P_1 v\|^2 \geqq |||v|||_\gamma^2 + \gamma \sum ((\partial_t a_{ij})\partial_j v, t^{-1}\partial_i v) + \gamma C(\Delta_x v, v) . \tag{2.22}$$

As $\sigma = 1$ we see that

$$C\|P_1 v\|^2 \geqq |||v|||_\gamma^2 - 2\gamma \sum (a_{ij}\partial_j v, t^{-2}\partial_i v) + \gamma C(t^{-1}\Delta_x v, v) . \tag{2.23}$$

Should one of the quadratic forms $a(0;\xi)$, $a_t(0;\xi)$ or $-a(0;\xi)$ be positive, we could get (2.5) from (2.21), (2.22) or (2.23) respectively. Microlocally we are always in one of these cases so we need only patch together the microlocal forms of those estimates in the usual way. In view of (1.5) we can find a neighborhood ω of the origin and three open cones Γ_1, Γ_2, Γ_3 in $\boldsymbol{R}^{n-1}$ such that

$$a(x, t, y; \xi) > 0 , \qquad (x, t, y; \xi) \in \omega \times \Gamma_1 \setminus \{0\} \tag{2.24}$$

$$a_t(x, t, y; \xi) > 0 , \qquad (x, t, y; \xi) \in \omega \times \Gamma_2 \setminus \{0\} \tag{2.25}$$

$$a(x, t, y; \xi) < 0 , \qquad (x, t, y; \xi) \in \omega \times \Gamma_3 \setminus \{0\} \tag{2.26}$$

$$\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \boldsymbol{R}^{n-1} \setminus \{0\} \tag{2.27}$$

where we have used notation (1.6).

Let $\{\phi_i\}$, $i = 1, 2, 3$, be a smooth partition of unity of $S^{n-2} = \{\xi \in \boldsymbol{R}^{n-1}; |\xi| = 1\}$ subordinated to the covering $\{\Gamma_i \cap S^{n-2}\}$. Consider a smooth partition of unity in $\boldsymbol{R}^{n-1}$, $\{\chi_i\}$ so that $\chi_i(\xi) = \phi_i(\xi/|\xi|)$ for $|\xi| \geqq 1/2$ and set

$$v_i(x, t, y) = \psi_i(x, D_x)v(x, t, y) = \frac{\alpha(x)}{(2\pi)^{n-1}} \int e^{ix\cdot\xi}\chi_i(\xi)\hat{v}(\xi, t, y)d\xi$$

where $\hat{v}(\xi, t, y)$ is the partial Fourier transform of $v(x, t, y)$ in the variable x and $\alpha(x)$ is a smooth function that is identically 1 in a neighborhood of the origin and has small compact support so that inequalities (2.21), (2.22) and (2.23) are valid for v_i. We shall apply (2.21) to v_1, (2.22) to v_2 and (2.23) to v_3 and add the three estimates up. We get

$$\begin{aligned} C\sum_{i=1}^{3} \|P_1 v_i\|^2 \geqq& \sum_{i=1}^{3} |||v_i|||_\gamma^2 + \gamma[2(a(x, t, y; D_x)v_1, t^{-2}v_1) \\ &+ (a_t(x, t, y; D_x)v_2, t^{-1}v_2) - 2(a(x, t, y; D_x)v_3, t^{-2}v_3)] \\ &+ C\gamma[(\Delta_x v_1, t^{-1}v_1) + (\Delta_x v_2, v_2) + (\Delta_x v_3, t^{-1}v_3)] \\ =& \sum_{i=1}^{3} |||v_i|||_\gamma^2 + \gamma(\mathscr{A}(x, t, y; D_x)v, v) \end{aligned} \tag{2.28}$$

where we have written

$$\mathscr{A}(x, t, y; D_x) = 2t^{-2}(\psi_1^* a\psi_1)(D_x) + t^{-1}(\psi_2^* a_t\psi_2)(D_x) - 2t^{-2}(\psi_3^* a\psi_3)(D_x) + C[t^{-1}\{\psi_1^* \Delta_x\psi_1 + \psi_3^* \Delta_x\psi_3\} + \psi_2^* \Delta_x\psi_2](D_x) .$$

The operator $\mathscr{A}(x, t, y; D_x)$ is a classical pseudo-differential operator of order two in D_x depending smoothly on (t, y), $t > 0$. The principal symbol $s(x, t, y; \xi)$ of $\mathscr{A}$ is

$$s(x, t, y; \xi) = [2t^{-2}(\chi_1^2 - \chi_3^2)a(x, t, y; \xi) + t^{-1}\chi_2^2 a_t(x, t, y; \xi) - C|\xi|^2(t^{-1}\chi_1^2 + \chi_2^2 + t^{-1}\chi_3^2)]\alpha^2(x) .$$

It follows from (2.24), (2.25), (2.26) and the choice of $\chi_i(\xi)$ that there exists a positive constant k such that

$$s(x, t, y; \xi) > t^{-1}k|\xi|^2 , \quad (x, t, y) \in \omega , \quad \xi \in \boldsymbol{R}^{n-1} \tag{2.29}$$

if we decrease ω enough; in particular $\alpha(x) = 1$ for $(x, t, y) \in \omega$. Applying the Gårding-Fefferman-Phong inequality to $t^2\mathscr{A}(x, t, y; D_x) - tk|D_x|^2$ we obtain

$$\begin{aligned} &(\mathscr{A}(x, t, y; D_x)v, v) \geqq k\|t^{-1/2}\nabla_x v\|^2 - k'\|t^{-1}u\|^2 , \\ &\operatorname{supp} v \subseteqq \omega \cap \{t > 0\} . \end{aligned} \tag{2.30}$$

It is another immediate consequence of the calculus of pseudo-differential operators that

$$\begin{aligned} &\|[P_1, \psi_i]v\|^2 \leqq C\|\nabla_x v\|^2 + 2\|[c, \psi_i]v_y\|^2 , \\ &\sum_{i=1}^{3} \|v_i\|^2 \sim \|v\|^2 , \qquad \sum_{i=1}^{3} |||v_i|||_\gamma^2 \sim |||v|||_\gamma^2 . \end{aligned} \tag{2.31}$$

At this point we make an additional assumption that will be dropped later:

(2.32) the coefficient c of ∂_y is independent of x, in particular $[c, \psi_i] \equiv 0$, $i = 1, 2, 3$.

Using (2.28), (2.30), (2.31) and (2.32) we get

$$C\|P_1 v\|^2 \geqq |||v|||_\gamma^2 + \gamma k\|t^{-1/2}\nabla_x v\|^2 - C\|\nabla_x v\|^2 - k'\gamma\|t^{-1}u\|^2$$

which implies (2.5) for γ large.

3. End of the proof of Lemma 2.1. To deal with the general case, consider the operator

$$P_{x_0} = \partial_t^2 + \sum_{i,j=1}^{n-1} a_{ij}(x, t, y)\partial^2/\partial x^i\partial x^j + c(x_0, t, y)\partial_y \tag{3.1}$$

defined in $\omega_1 = U \times (-\varepsilon, \varepsilon) \times (-\rho, \rho)$ where U is a neighborhood of the origin in R^{n-1} and $x_o \in U$ is fixed. If U, ε and ρ are small enough, the first part of the proof applies to P_{x_0} (it verifies (2.32) in addition to the hypothesis of Lemma 2.1) to conclude that there exist M, γ_o such that

$$\text{(3.2)} \qquad M\|t^{-\gamma}P_{x_0}u\|^2 \geqq \gamma^3\|t^{-\gamma-2}u\|^2 + \gamma\{\|t^{-\gamma-1}u_t\|^2 + \|t^{-\gamma-1/2}\nabla_x u\|^2\} + \gamma^{-1}\|t^{-\gamma}u_y\|^2$$

for $\gamma > \gamma_o$, $x_o \in U$, $u \in C_c^\infty(\omega_1 \cap \{t \geqq 0\})$. Then we proceed to show (2.4) by partition of unity in the x-space. Let $\theta \in C^\infty(R^{n-1})$ be equal to 1 on $|x| \leqq 1$ and 0 on $|x| \geqq 2$. For $\delta > 0$, $k = (k_1, \cdots, k_{n-1}) \in Z^{n-1}$ and $\gamma > \delta^{-2}n$ set

$$\theta_k(x) = \theta(\delta^{-1}\gamma^{1/2}x - k)/(\sum_{m \in Z^{n-1}} \theta^2(\delta^{-1}\gamma^{1/2}x - m))^{1/2},$$
$$x_k = \delta\gamma^{-1/2}(k_1, \cdots, k_{n-1}).$$

Then, $\sum_k \theta_k^2 = 1$ on R^{n-1} and $\operatorname{supp}(\theta_k) \subseteqq \{|x - x_k| \leqq 2\gamma^{-1/2}\delta\}$. For $u \in C_c^\infty(\omega \cap \{t \geqq 0\})$ and $\bar\omega \subset \omega_1$ define

$$u_k(x, t, y) = \theta_k(x)u(x, t, y)$$

and observe that for δ fixed and $\gamma > \gamma_o(\delta)$, u_k has support in ω_1 and $x_k \in U$ whenever $u_k \neq 0$. Then

$$P_{x_k}u_k = Pu_k + (c(x_k, t, y) - c(x, t, y))\partial_y u_k$$
$$Pu_k = \theta_k Pu + 2\sum a_{ij}(\partial\theta_k/\partial x^i)(\partial u/\partial x^j) + \sum a_{ij}(\partial^2\theta_k/\partial x^i\partial x^j)u.$$

Hence,

$$|P_{x_k}u_k - \theta_k Pu|^2 \leqq C\{\delta^2\gamma^{-1}|\partial_y u_k|^2 + \delta^{-2}\gamma|\beta_k\nabla_x u|^2 + \delta^{-4}\gamma^2|\beta_k u|^2\}$$

where $\beta_k(x)$ denotes the characteristic function of the support of θ_k and C is constant. It follows that

$$\text{(3.3)} \qquad \sum_{k \in Z^{n-1}} \|t^{-\gamma}(P_{x_k}u_k - \theta_k Pu)\|^2 \leqq C_1\{\delta^2\gamma^{-1}\|t^{-\gamma}\partial_y u\|^2 + \delta^{-2}\gamma\|t^{-\gamma}\nabla_x u\|^2 + \delta^{-4}\gamma^2\|t^{-\gamma}u\|^2\}.$$

Thus, in view of (3.2) and (3.3)

$$\begin{aligned}\|t^{-\gamma}Pu\|^2 &= \sum_k \|t^{-\gamma}\theta_k Pu\|^2 \geqq \sum_k ((1/2)\|t^{-\gamma}P_{x_k}u_k\|^2 - 2\|t^{-\gamma}(P_{x_k}u_k - \theta_k Pu)\|^2)\\
&\geqq (1/2M)\sum_k (\gamma^3\|t^{-\gamma-2}u_k\|^2 + \gamma\{\|t^{-\gamma-1}\partial_t u_k\|^2 + \|t^{-\gamma-1/2}\nabla_x u_k\|^2\}\\
&\quad + \gamma^{-1}\|t^{-\gamma}\partial_y u_k\|^2)\\
&\quad - C_2\{\delta^2\gamma^{-1}\|t^{-\gamma}\partial_y u\|^2 + \delta^{-2}\gamma\|t^{-\gamma}\nabla_x u\|^2 + \delta^{-4}\gamma^2\|t^{-\gamma}u\|^2\}\\
&\geqq C_3[\gamma^3\|t^{-\gamma-2}u\|^2 + \gamma\{\|t^{-\gamma-1}u_t\|^2 + \|t^{-\gamma-1/2}\nabla_x u\|^2\} + \gamma^{-1}\|t^{-\gamma}u_y\|^2]\end{aligned}$$

for a positive constant C_3, if we choose δ and ε sufficiently small and fixed and $\gamma > \gamma'_0$. This proves the lemma.

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