Tôhoku Math. Journ.
35 (1983), 225-247.

AN ALGEBRAIC APPROACH TO ISOPARAMETRIC HYPERSURFACES IN SPHERES II

JOSEF DORFMEISTER* AND ERHARD NEHER**

(Received November 26, 1981)

Introduction. In [1] it was shown that isoparametric hypersurfaces in spheres with 4 distinct principal curvatures can be equivalently described by isoparametric triple systems. These triple systems have a "Peirce decomposition"

$$V = V_{11} \oplus V_{10} \oplus V_{12}^{+} \oplus V_{12}^{-} \oplus V_{22} \oplus V_{20}$$

and every element in a Peirce space V_{ij} is a scalar multiple of a tripotent. Moreover, it was proved that this property essentially characterizes isoparametric triple systems.

In this paper we investigate the Peirce decomposition relative to a tripotent from a Peirce space V_{ij}. We also begin the study of the fine structure of V_{12} by introducing the subspaces Q and JV_{12}.

The results of this paper are used in [2] and [3] and lay the foundations for subsequent publications.

The paper is organized as follows. In §1 we compute all the triple products $\{uvw\}$ where each element u, v, w lies in some Peirce space V_{ij}. In §§2, 3 we compute the Peirce decompositions of V relative to tripotents from V_{10}, V_{20} and V_{12}. Finally, in §4 we introduce the space $Q \subset V_{12}$ and show how it is connected to elements of the dual triple satisfying Jordan composition rules. This space is also important for the investigation of isoparametric triple systems of FKM-type, [2], §8.

For definitions and notations we refer to [1].

The authors would like to thank the University of Virginia for its hospitality during the preparation of this paper.

1. Various triple products. In this section we consider an isoparametric triple V. We fix orthogonal tripotents (e_1, e_2) and denote by V_{ij} the Peirce spaces relative to (e_1, e_2). By [1, Remark 4.3. a] the elements $e = \lambda(e_1 + e_2)$ and $\hat{e} = \lambda(e_1 - e_2)$, $\lambda = (\sqrt{2})^{-1}$ are maximal tripotents of V.

1.1. In this subsection we compute the triple products where each

* Partially supported by NSF Grant MCS-8104848.
** Supported by Deutsche Forschungsgemeinschaft.

factor lies in some Peirce space relative to (e_1, e_2).

We recall that, besides the algebras $\circ$ and $*$ given by $x \circ y = \{xe_1y\}$ and $x*y = \{xe_2y\}$, we have the algebras $\square$ and $\hat{\square}$ defined by $x \square y = \{xey\}$ $x \mathbin{\hat{\square}} y = \{x\hat{e}y\}$.

We will prove the following two theorems simultaneously step by step.

THEOREM 1.1. *Let $V = \bigoplus V_{ij}$ be the Peirce decomposition of the isoparametric triple V relative to (e_1, e_2). Then for all $u_{ij}, v_{ij}, w_{ij} \in V_{ij}$ the following identities hold:*

$$\{uvw\} = 2(\langle u, v\rangle w + \langle u, w\rangle v + \langle v, w\rangle u) \tag{1.1}$$

for all $u, v, w \in V_{11} \oplus V_{10}$ and for all $u, v, w \in V_{22} \oplus V_{20}$.

$$\{u_{22} + u_{20}, v_{22} + v_{20}, w_{11}\} = 0\ . \tag{1.2}$$

$$\{u_{11} + u_{10}, v_{11} + v_{10}, w_{22}\} = 0\ . \tag{1.3}$$

$$\{u_{22}, v_{22}, w_{10}\} = 0\ . \tag{1.4}$$

$$\{u_{11}, v_{11}, w_{20}\} = 0\ . \tag{1.5}$$

$$\{u_{22}, v_{20}, w_{10}\} = u_{22}\circ(v_{20}\circ w_{10}) \in V_{12}\ . \tag{1.6}$$

$$\{u_{11}, v_{10}, w_{20}\} = u_{11}*(v_{10}*w_{20}) \in V_{21}\ . \tag{1.7}$$

$$\{u_{20}, v_{20}, w_{10}\} = u_{20}\circ(v_{20}\circ w_{10}) + v_{20}\circ(u_{20}\circ w_{10}) \in V_{12} \oplus V_{10}\ . \tag{1.8}$$

$$\{u_{10}, v_{10}, w_{20}\} = u_{10}*(v_{10}*w_{20}) + v_{10}*(u_{10}*w_{20}) \in V_{12} \oplus V_{20}\ . \tag{1.9}$$

$$\{u_{20}, v_{20}, w_{12}\} = u_{20}\circ(v_{20}\circ w_{12}) + v_{20}\circ(u_{20}\circ w_{12}) \in V_{12} \oplus V_{10}\ . \tag{1.10}$$

$$\{u_{20}, v_{20}, w_{12}\}_{12} = 2\langle u_{20}, v_{20}\rangle w_{12} - u_{20}*(v_{20}*w_{12}) - v_{20}*(u_{20}*w_{12})\ . \tag{1.10a}$$

$$\{u_{10}, v_{10}, w_{12}\} = u_{10}*(v_{10}*w_{12}) + v_{10}*(u_{10}*w_{12}) \in V_{12} \oplus V_{20}\ . \tag{1.11}$$

$$\{u_{10}, v_{10}, w_{12}\}_{12} = 2\langle u_{10}, v_{10}\rangle w_{12} - u_{10}\circ(v_{10}\circ w_{12}) - v_{10}\circ(u_{10}\circ w_{12})\ . \tag{1.11a}$$

$$\begin{aligned}\{u_{22}, v_{20}, w_{12}\} &= [v_{20}\circ(u_{22}\circ(u_{22}\circ w_{12}))]_{10} \in V_{10} \\ &= u_{22}\circ(v_{20}\circ w_{12}) + v_{20}\circ(u_{22}\circ w_{12})\ .\end{aligned} \tag{1.12}$$

$$\begin{aligned}\{u_{11}, v_{10}, w_{12}\} &= [v_{10}*(u_{11}*w_{12})]_{20} \in V_{20} \\ &= u_{11}*(v_{10}*w_{12}) + v_{10}*(u_{11}*w_{12})\ .\end{aligned} \tag{1.13}$$

$$\{u_{22}, v_{22}, w_{12}\} = 2\langle u_{22}, v_{22}\rangle w_{12} \in V_{12}\ . \tag{1.14}$$

$$\{u_{11}, v_{11}, w_{12}\} = 2\langle u_{11}, v_{11}\rangle w_{12} \in V_{12}\ . \tag{1.15}$$

$$\{u_{11}, v_{22}, w_{12}\} \in V_{12}\ . \tag{1.16}$$

$$(\{u_{11}^-, v_{22}^-, w_{12}^\varepsilon\})_{12}^\varepsilon = -\frac{\varepsilon}{2}(u_{11}^-*(v_{22}^-\circ w_{12}^\varepsilon) + v_{22}^-\circ(u_{11}^-*w_{12}^\varepsilon)) \quad \textit{for} \quad \varepsilon = \pm\ . \tag{1.16a}$$

(1.17) $$\{u_{22}, v_{10}, w_{12}\} = v_{10}\circ(u_{22}\circ w_{12}) + a_{12} \in V_{20} \oplus V_{12} .$$

(1.17a) $$(\{u_{22}, v_{10}, w_{12}\})^{\varepsilon}_{12} = -\frac{\varepsilon}{2}(u_{22}\circ(v_{10}*w^{\varepsilon}_{12}) + v_{10}*(u_{22}\circ w_{12}))^{\varepsilon}_{12} \quad \textit{for} \quad \varepsilon = \pm .$$

(1.18) $$\{u_{11}, v_{20}, w_{12}\} = v_{20}*(u_{11}*w_{12}) + b_{12} \in V_{10} \oplus V_{12} .$$

(1.18a) $$(\{u_{11}, v_{20}, w^{\varepsilon}_{12}\})^{\varepsilon}_{12} = -\frac{\varepsilon}{2}(u_{11}*(v_{20}\circ w^{\varepsilon}_{12}) + v_{20}\circ(u_{11}*w^{\varepsilon}_{12}))^{\varepsilon}_{12} \quad \textit{for} \quad \varepsilon = \pm .$$

(1.19) $$\{u_{22}, v_{12}, w_{12}\} = 2\langle v_{12}, w_{12}\rangle u_{22} + a_{11} + a_{10} \in V_{11} \oplus V_{22} \oplus V_{10} .$$

(1.20) $$\{u_{11}, v_{12}, w_{12}\} = 2\langle v_{12}, w_{12}\rangle u_{11} + a_{22} + a_{20} \in V_{11} \oplus V_{22} \oplus V_{20} .$$

(1.21) $$\{u_{20}, v^{\varepsilon}_{12}, w^{\varepsilon}_{12}\} = 2\langle v^{\varepsilon}_{12}, w^{\varepsilon}_{12}\rangle u_{20} - v^{\varepsilon}_{12}*(w^{\varepsilon}_{12}*u_{20}) - w^{\varepsilon}_{12}*(v^{\varepsilon}_{12}*u_{20}) + a^{-}_{11} + a_{10} \in V_{20} \oplus V^{-\varepsilon}_{12} \oplus V^{-}_{11} \oplus V_{10} .$$

(1.22) $$\{u_{10}, v^{\varepsilon}_{12}, w^{\varepsilon}_{12}\} = 2\langle v^{\varepsilon}_{12}, w^{\varepsilon}_{12}\rangle u_{10} - v^{\varepsilon}_{12}\circ(w^{\varepsilon}_{12}\circ u_{10}) - w^{\varepsilon}_{12}\circ(v^{\varepsilon}_{12}\circ u_{10}) + a^{-}_{22} + a_{20} \in V_{10} \oplus V^{-\varepsilon}_{12} \oplus V^{-}_{22} \oplus V_{20} .$$

(1.23) $$\{u_{20}, v^{+}_{12}, w^{-}_{12}\} \in V_{11} \oplus V_{10} \oplus V_{12} \oplus V_{20} .$$

(1.23a) $$\{u_{20}, v^{+}_{12}, w^{-}_{12}\}_{12} = -u_{20}*(v^{+}_{12}*w^{-}_{12}) - [v^{+}_{12}*(w^{-}_{12}*u_{20}) + w^{-}_{12}*(v^{+}_{12}*u_{20})]_{12} .$$

(1.23b) $$\{u_{20}, v^{+}_{12}, w^{-}_{12}\}_{20} = -[v^{+}_{12}*(w^{-}_{12}*u_{20}) + w^{-}_{12}*(v^{+}_{12}*u_{20})]_{20} = [v^{+}_{12}\circ(w^{-}_{12}\circ u_{20}) + w^{-}_{12}\circ(v^{+}_{12}\circ u_{20})]_{20} .$$

(1.24) $$\{u_{10}, v^{+}_{12}, w^{-}_{12}\} \in V_{22} \oplus V_{20} \oplus V_{12} \oplus V_{10} .$$

(1.24a) $$\{u_{10}, v^{+}_{12}, w^{-}_{12}\}_{12} = -u_{10}\circ(v^{+}_{12}\circ w^{-}_{12}) - [v^{+}_{12}\circ(w^{-}_{12}\circ u_{10}) + w^{-}_{12}\circ(v^{+}_{12}\circ u_{10})]_{12} .$$

(1.24b) $$\{u_{10}, v^{+}_{12}, w^{-}_{12}\}_{10} = -[v^{+}_{12}\circ(w^{-}_{12}\circ u_{10}) + w^{-}_{12}\circ(v^{+}_{12}\circ u_{10})]_{10} = [v^{+}_{12}*(w^{-}_{12}*u_{10}) + w^{-}_{12}*(v^{+}_{12}*u_{10})]_{10} .$$

(1.25) $$\{u^{\varepsilon}_{12}, v^{\varepsilon}_{12}, w^{\varepsilon}_{12}\} = \langle u^{\varepsilon}_{12}, v^{\varepsilon}_{12}\rangle w^{\varepsilon}_{12} + \langle u^{\varepsilon}_{12}, w^{\varepsilon}_{12}\rangle v^{\varepsilon}_{12} + \langle v^{\varepsilon}_{12}, w^{\varepsilon}_{12}\rangle u^{\varepsilon}_{12} .$$

(1.26) $$\{u^{+}_{12}, v^{+}_{12}, w^{-}_{12}\} = 3\langle u^{+}_{12}, v^{+}_{12}\rangle w^{-}_{12} - u^{+}_{12} \,\square\, (v^{+}_{12} \,\square\, w^{-}_{12}) - v^{+}_{12} \,\square\, (u^{+}_{12} \,\square\, w^{-}_{12}) \in V^{-}_{12} \oplus V_{10} \oplus V_{20} .$$

(1.26a) $$\{u^{+}_{12}, v^{+}_{12}, w^{-}_{12}\}_{20} = -u^{+}_{12}\circ(v^{+}_{12}*w^{-}_{12}) - v^{+}_{12}\circ(u^{+}_{12}*w^{-}_{12}) .$$

(1.26b) $$\{u^{+}_{12}, v^{+}_{12}, w^{-}_{12}\}_{10} = -u^{+}_{12}*(v^{+}_{12}\circ w^{-}_{12}) - v^{+}_{12}*(u^{+}_{12}\circ w^{-}_{12}) .$$

(1.26c) $$\{u^{+}_{12}, v^{+}_{12}, w^{-}_{12}\}_{12} = 3\langle u^{+}_{12}, v^{+}_{12}\rangle w^{-}_{12} - \frac{1}{2}[u^{+}_{12}*(v^{+}_{12}*w^{-}_{12}) - u^{+}_{12}\circ(v^{+}_{12}\circ w^{-}_{12})] \in V^{-}_{12} .$$

(1.27) $$\{u^{-}_{12}, v^{-}_{12}, w^{+}_{12}\} = 3 < u^{-}_{12}, v^{-}_{12}\rangle w^{+}_{12} - u^{-}_{12} \,\hat{\square}\, (v^{-}_{12} \,\hat{\square}\, w^{+}_{12}) - u^{-}_{12} \,\hat{\square}\, (u^{-}_{12} \,\hat{\square}\, w^{+}_{12}) \in V^{+}_{12} \oplus V_{10} \oplus V_{20} .$$

(1.27a) $$\{u^{-}_{12}, v^{-}_{12}, w^{+}_{12}\}_{20} = u^{-}_{12}\circ(v^{-}_{12}*w^{+}_{12}) + v^{-}_{12}\circ(u^{-}_{12}*w^{+}_{12}) .$$

(1.27b) $$\{u^{-}_{12}, v^{-}_{12}, w^{+}_{12}\}_{10} = u^{-}_{12}*(v^{-}_{12}\circ w^{+}_{12}) + v^{-}_{12}*(u^{-}_{12}\circ w^{+}_{12}) .$$

(1.27c) $\{u_{12}^-, v_{12}^-, w_{12}^+\}_{12} = 3\langle u_{12}^-, v_{12}^-\rangle w_{12}^+ - \frac{1}{2}[u_{12}^-*(v_{12}^-*w_{12}^+) + u_{12}^-\circ(v_{12}^-\circ w_{12}^+)] \in V_{12}^-$.

(1.28) $$(\{u_{10}, v_{20}, w_{12}^\varepsilon\})_{12}^\varepsilon = -\frac{\varepsilon}{2}[u_{10}*(v_{20}\circ w_{12}^\varepsilon) + v_{20}\circ(u_{10}*w_{12}^\varepsilon)]_{12}^\varepsilon .$$

For $x_{12} \in V_{12}$ we put $T(e_1, e_2)x_{12} =: \bar{x}_{12}$. We note $\bar{\bar{x}}_{12} = x_{12}$.

THEOREM 1.2. *Let $V = \bigoplus V_{ij}$ be the Peirce decomposition of an isoparametric triple relative to (e_1, e_2). Then for all $u_{ij}, v_{ij}, w_{ij}, x_{ij} \in V_{ij}$ the following identities hold:*

(1.29) $$u_{22}\circ(v_{20}\circ w_{12}) = -[v_{20}\circ(u_{22}\circ w_{12})]_{12} .$$

(1.30) $$v_{20}*w_{12} = \overline{v_{20}\circ w_{12}} + v_{20}\circ\overline{w_{12}} .$$

(1.31) $$u_{11}*(v_{10}*w_{12}) = -[v_{10}*(u_{11}*w_{12})]_{12} .$$

(1.32) $$v_{10}\circ w_{12} = \overline{v_{10}*w_{12}} + v_{10}*\overline{w_{12}} .$$

(1.33) $$T(u_{11}, v_{22})^2 w_{12} = \langle u_{11}, u_{11}\rangle\langle v_{22}, v_{22}\rangle w_{12} .$$

(1.33a) $$[T(w_{11}, v_{22})T(u_{11}, v_{22}) + T(u_{11}, v_{22})T(w_{11}, v_{22})]w_{12} = 2\langle u_{11}, w_{11}\rangle\langle v_{22}, w_{22}\rangle w_{12} .$$

(1.33b) $$[T(w_{11}, x_{22})T(u_{11}, v_{22}) + T(w_{11}, v_{22})T(u_{11}, x_{22}) + T(u_{11}, x_{22})T(w_{11}, v_{22}) + T(u_{11}, v_{22})T(w_{11}, x_{22})]w_{12} = 4\langle u_{11}, w_{11}\rangle\langle v_{22}, x_{22}\rangle w_{12} .$$

(1.34) $$v_{12}^\varepsilon\circ(w_{12}^\varepsilon\circ u_{22}^-) + w_{12}^\varepsilon\circ(v_{12}^\varepsilon\circ u_{22}^-) = 2\langle v_{12}^\varepsilon, w_{12}^\varepsilon\rangle u_{22}^- .$$

(1.35) $$v_{12}^\varepsilon*(w_{12}^\varepsilon*u_{11}^-) + w_{12}^\varepsilon*(v_{12}^\varepsilon*u_{11}^-) = 2\langle v_{12}^\varepsilon, w_{12}^\varepsilon\rangle u_{11}^- .$$

(1.36) $$v_{12}^\varepsilon\circ(w_{12}^\varepsilon\circ u_{20}) + w_{12}^\varepsilon\circ(v_{12}^\varepsilon\circ u_{20}) + [v_{12}^\varepsilon*(w_{12}^\varepsilon*u_{20}) + w_{12}^\varepsilon*(v_{12}^\varepsilon*u_{20})]_{20} = 2\langle v_{12}^\varepsilon, w_{12}^\varepsilon\rangle u_{20} .$$

(1.37) $$v_{12}^\varepsilon*(w_{12}^\varepsilon*u_{10}) + w_{12}^\varepsilon*(v_{12}^\varepsilon*u_{10}) + [v_{12}^\varepsilon\circ(w_{12}^\varepsilon\circ u_{10}) + w_{12}^\varepsilon\circ(v_{12}^\varepsilon\circ u_{10})]_{10} = 2\langle v_{12}^\varepsilon, w_{12}^\varepsilon\rangle u_{10} .$$

(1.38) $$[u_{12}^+*(v_{12}^+*w_{12}^-) + v_{12}^+*(u_{12}^+*w_{12}^-)]_{20} = u_{12}^+\circ(v_{12}^+*w_{12}^-) + v_{12}^+\circ(u_{12}^+*w_{12}^-) .$$

(1.39) $$[u_{12}^+\circ(v_{12}^+\circ w_{12}^-) + v_{12}^+\circ(u_{12}^+\circ w_{12}^-)]_{10} = u_{12}^+*(v_{12}^+\circ w_{12}^-) + v_{12}^+*(u_{12}^+\circ w_{12}^-) .$$

(1.40) $$[u_{12}^-*(v_{12}^-*w_{12}^+) + v_{12}^-*(u_{12}^-*w_{12}^+)]_{20} = -u_{12}^-\circ(v_{12}^-*w_{12}^+) - v_{12}^-\circ(u_{12}^-*w_{12}^+) .$$

(1.41) $$[u_{12}^-\circ(v_{12}^-\circ w_{12}^+) + v_{12}^-\circ(u_{12}^-\circ w_{12}^+)]_{10} = -u_{12}^-*(v_{12}^-\circ w_{12}^+) - v_{12}^-*(u_{12}^-*w_{12}^+) .$$

PROOF. (1.1): Follows from [1, (2.6) and Corollary 5.2] applied to $c = e_2$ and $c = e_1$.

(1.2) to (1.11) except (1.10a): From [1, (2.7)] we know $\{u_0, v_0, w_2\} = \{u_0, c, \{v_0, c, w_2\}\} + \{v_0, c, \{u_0, c, w_2\}\}$ for each minimal tripotent c and all $u_\kappa, v_\kappa, w_\kappa \in V_\kappa(c)$. We put $c = e_1$ and $c = e_2$, apply the multiplication rules of [1, Theorem 5.7] and easily get (1.2) to (1.11) except (1.10a). (1.10a) and (1.11a) will be proved later.

(1.12), (1.29), (1.30): From [1, (2.7)] we get $\{u_{22}, v_{20}, w_{12}\} = u_{22} \circ (v_{20} \circ w_{12}) + v_{20} \circ (u_{22} \circ w_{12})$. We may assume $\langle u_{22}, u_{22} \rangle = 1$. Then u_{22} is a minimal tripotent and $v_{20} \in V_{20}(e_1, u_{22})$, $w_{12} \in V_{12}(e_1, u_{22})$ by [1, Theorem 5.11]. Now [1] (5.7) shows $\{u_{22}, v_{20}, w_{12}\} \in V_{10}(e_1, u_{22})$. Using once more [1, Theorem 5.11] we get $\{u_{22}, v_{20}, w_{12}\} \in V_{10}$. The multiplication rules for the algebra "$\circ$" show $u_{22} \circ (v_{20} \circ w_{12}) \in V_{12}$ and $v_{20} \circ (u_{22} \circ w_{12}) \in V_{12} \oplus V_{10}$. This implies (1.12) and (1.29). Finally, (1.30) is the consequence of (1.12) for $u_{22} = e_2$.

(1.12), (1.31), (1.32): Interchange "1" and "2" in (1.12), (1.29), and (1.30).

(1.14), (1.15): Follow by linearization of [1, (5.1)].

(1.16), (1.16a), (1.33) to (1.33b): Here (1.16) follows from [1, (5.3)]. (1.33) is just [1, (5.2)] and (1.33a), (1.33b) are linearizations of (1.33). To verify (1.16a) we put $u_{11} = u_{11}^-$, $v_{22} = v_{22}^-$, $w_{11} = e_1$, $x_{22} = e_2$ in (1.33b) and get $0 = \overline{\{u_{11}^-, v_{22}^-, w_{12}\}} + v_{22}^- \circ (u_{11}^- * w_{12}) + u_{11}^- * (v_{22}^- \circ w_{12}) + \{u_{11}^-, v_{22}^-, w_{12}^-\}$. Therefore $0 = \varepsilon(\overline{\{u_{11}^-, v_{22}^-, w_{12}^\varepsilon\}} + \varepsilon\overline{\{u_{11}^-, v_{22}^-, w_{12}^\varepsilon\}} + u_{11}^- * (v_{22}^- \circ w_{12}^\varepsilon) + v_{22}^- \circ (u_{11}^- * w_{12}^\varepsilon)$; this is equivalent to (1.16a).

(1.17): Since $v_{10}, w_{12} \in V_2(e_1)$, $u_{22} \in V_0(e_1)$ we may apply [1, (2.8)] with $c = e_1$ and get $\{u_{22}, v_{10}, w_{12}\} = \langle v_{10} \circ u_{22}, w_{12} \rangle e_1 + [v_{10} \circ (w_{12} \circ u_{22}) + w_{12} \circ (v_{10} \circ u_{22})]_0 + a_2$. By [1, (5.8)] we have $v_{10} \circ u_{22} = 0$ and by [1, (5.12), (5.7)] we know $v_{10} \circ (w_{12} \circ u_{22}) \in V_{20} \subset V_0(e_1)$. Therefore $\{u_{22}, v_{10}, w_{12}\} = v_{10} \circ (u_{22} \circ w_{12}) + a_2$ with some $a_2 \in V_2(e_1) = V_{11}^- \oplus V_{10} \oplus V_{12}$. But $\langle \{u_{22}, v_{10}, w_{12}\}, V_{11}^- \oplus V_{10} \rangle = \langle w_{12}, \{u_{22}, v_{10}, V_{11}^- \oplus V_{10}\} \rangle = 0$ by (1.3) whence (1.17), (1.17a) and (1.18a) are proved later.

(1.18): Follows from (1.17) by interchanging "1" and "2".

(1.19), (1.34): We may assume $\langle u_{22}, u_{22} \rangle = 1$. Then u_{22} is a minimal tripotent of V with $V_{ij} = V_{ij}(e_1, e_2) = V_{ij}(e_1, u_{22})$ by [1, Theorem 5.11]. Therefore [1, (5.10), (5.11)] imply $\{v_{12} u_{22} w_{12}\} = 2\langle v_{12}, w_{12} \rangle u_{22} + a_0$, $a_0 \in V_0(u_{22})$. Since $V_0(u_{22}) = V_0(e_2)$, (1.19) follows. To verify (2.34) we note that by [1, (2.8)] the $V_0(e_1)$-component of $\{u_{22}^-, v_{12}^\varepsilon, w_{12}^\varepsilon\}$ is $[v_{12}^\varepsilon \circ (w_{12}^\varepsilon \circ u_{22}^-) + w_{12}^\varepsilon \circ (v_{12}^\varepsilon u_{22}^-)]_0$. Here we may drop the subscript "0" since $V_{12}^\varepsilon \circ (V_{12}^\varepsilon \circ u_{22}^-) \subset V_0(e_1)$. On the other hand, (1.19) means that the $V_0(e_1)$-component of $\{u_{22}^-, v_{12}, w_{12}\}$ is $2\langle v_{12}, w_{12} \rangle u_{22}^-$. This implies (1.34).

(1.20), (1.35): follow by symmetry from (1.19), (1.34).

(1.21), (1.36): We compute $\{u_{20}, v_{12}^\varepsilon, w_{12}^\varepsilon\}$ according to [1, (2.9)] for $c = e_2$ and get $\{u_{20}, v_{12}^\varepsilon, w_{12}^\varepsilon\} = 2\langle v_{12}^\varepsilon, w_{12}^\varepsilon \rangle u_{20} - u_{20} * (v_{12}^\varepsilon * w_{12}^\varepsilon)_0 - v_{12}^\varepsilon * (w_{12}^\varepsilon * u_{20})_0 - w_{12}^\varepsilon * (v_{12}^\varepsilon * u_{20})_0 + a_0$, $a_0 \in V_0(e)$. Because $(v_{12} * w_{12})_0 \in \boldsymbol{R} e_1$ we have $u_{20} * (v_{12}^\varepsilon * w_{12}^\varepsilon)_0 = 0$ by [1, (5.3)]. Further, $w_{12}^\varepsilon * u_{20} \in V_{10}$ whence $v_{12}^\varepsilon * (w_{12}^\varepsilon * u_{20})_0 = v_{12}^\varepsilon * (w_{12}^\varepsilon * u_{20}) \in V_{12}^{-\varepsilon} \oplus V_{20}$. Finally, $\langle \{u_{20}, v_{12}^\varepsilon, w_{12}^\varepsilon\}, e_1 \rangle = \langle u_{20}, v_{12}^\varepsilon \circ w_{12}^\varepsilon \rangle = 0$ implies $a_0 = a_{11}^- + a_{10}$.

This proves (1.21). It follows that the $V_0(e_1)$-component of $\{u_{20}, v^{\varepsilon}_{12}, w^{\varepsilon}_{12}\}$ equals $2\langle v^{\varepsilon}_{12}, w^{\varepsilon}_{12}\rangle u_{20} - [v^{\varepsilon}_{12}*(w^{\varepsilon}_{12}*u_{20}) + w^{\varepsilon}_{12}*(v^{\varepsilon}_{12}*u_{20})]_{20}$; on the other hand this component is $[v^{\varepsilon}_{12}\circ(w^{\varepsilon}_{12}\circ u_{20}) + w^{\varepsilon}_{12}\circ(v^{\varepsilon}_{12}\circ u_{20})]_0$ by [1, (2.8)]. Since $v^{\varepsilon}_{12}\circ(w^{\varepsilon}_{12}\circ u_{20}) \in V^-_{22}\oplus V_{20}$ we can drop the subscript "0" and get (1.36).

(1.22), (1.37): follow by interchanging "1" and "2".

(1.23), (1.23a), (1.23b): By [1, (2.8)] the $V_0(e_1)$-component of $\{u_{20}, v^+_{12}, w^-_{12}\}$ is $[v^+_{12}\circ(w^-_{12}\circ u_{20}) + w^-_{12}\circ(v^+_{12}\circ u_{20})]_0$. The multiplication rules for the algebra "$\circ$" show $v^+_{12}\circ(w^-_{12}\circ u_{20}) + w^-_{12}\circ(v^+_{12}\circ u_{20}) \in \boldsymbol{R}e_1 \oplus \boldsymbol{R}e_2 \oplus V_{20}$; but the e_2-component is zero here because $\langle v^+_{12}\circ(w^-_{12}\circ u_{20}), e_2\rangle + \langle w^-_{12}\circ(v^+_{12}\circ u_{20}), e_2\rangle = \langle v^+_{12}, w^-_{12}\circ u_{20}\rangle - \langle w^-_{12}, v^+_{12}\circ u_{20}\rangle = 0$. By [1, (2.9)] the $V_2(e_2)$-component of $\{u_{20}, v^+_{12}, w^-_{12}\}$ is $-u_{20}*(v^+_{12}*w^-_{12}) - v^+_{12}*(w^-_{12}*u_{20}) - w^-_{12}*(v^+_{12}*u_{20})$. The first summand lies in V_{12}, the last two lie in $V_{12}\oplus V_{20}$. This proves the assertions.

The corresponding equations (1.24), (1.24a), (1.24b) are shown by interchanging "1" and "2".

(1.25): This follows from [1. Lemma 5.4 and (2.13)].

(1.26) to (1.26c), (1.38), (1.39): The first assertion follows from [1, (2.14)]. Next we consider $\langle\{u^+_{12}, v^+_{12}, w^-_{12}\}, u_{20}\rangle = \langle\{u_{20}, v^+_{12}, w^-_{12}\}, u^+_{12}\rangle = -\langle u_{20}*(v^+_{12}*w^-_{12}), u^+_{12}\rangle - \langle v^+_{12}*(w^-_{12}*u_{20}), u^+_{12}\rangle - \langle w^-_{12}*(v^+_{12}*u_{20}), u^+_{12}\rangle = \langle -u^+_{12}*(v^+_{12}*w^-_{12}) - v^+_{12}*(u^+_{12}*w^-_{12}), u_{20}\rangle$ where we have used (1.23a) and $\langle u^+_{12}*v^+_{12}, w^-_{12}*u_{20}\rangle = 0$. Because $V^+_{12}*V^-_{12} \subset V^-_{11}\oplus V_{10}$ and $V^-_{11}*V^+_{12} \subset V^-_{12}$ we only have to consider $[v^+_{12}*(v^+_{12}*v^-_{12})_{10}]_{20}$. But from (1.32) follows $(v^+_{12}*v_{10})_{20} = v^+_{12}\circ v_{10}$. Moreover $v^-_{11}\circ v_{12} = 0$, hence $[u^+_{12}*(v^+_{12}*w^-_{12}) + v^+_{12}*(u^+_{12}*w^-_{12})]_{20} = u^+_{12}\circ(v^+_{12}*w^-_{12}) + v^+_{12}\circ(u^+_{12}*w^-_{12})$. This proves (1.26a), (1.38) and, by interchanging "1" and "2", also (1.26b) and (1.39). Finally, (1.26c) follows by expanding the right hand side of (1.26) and comparing with (1.26a) and (1.26b).

The analogous equations (1.27) to (1.27c), (1.40), (1.41) are shown by interchanging "1" and "2".

(1.28): In [1, (1.10)] we put $w = w^{\varepsilon}_{12}, u = u_{10}, v = v_{20}, x = e_1$, apply to e_2 and form the scalar product with x^{ε}_{12}. We get $0 = \langle w^{\varepsilon}_{12}*v_{20}, u_{10}\circ x^{\varepsilon}_{12}\rangle + \langle w^{\varepsilon}_{12}*u_{10}, v_{20}\circ x^{\varepsilon}_{12}\rangle + \langle \varepsilon w^{\varepsilon}_{12}, \{u_{10}, v_{20}, x^{\varepsilon}_{12}\}\rangle + \langle u_{10}*v_{20}, w^{\varepsilon}_{12}\circ x^{\varepsilon}_{12}\rangle + \langle u_{10}\circ w^{\varepsilon}_{12}, v_{20}*x^{\varepsilon}_{12}\rangle + \langle u_{10}\circ v_{20}, w^{\varepsilon}_{12}*x^{\varepsilon}_{12}\rangle + \langle\{u_{10}, v_{20}, w^{\varepsilon}_{12}\}, \varepsilon x^{\varepsilon}_{12}\rangle + \langle v_{20}\circ w^{\varepsilon}_{12}, u_{10}*x^{\varepsilon}_{12}\rangle - 6\langle w^{\varepsilon}_{12}, x^{\varepsilon}_{12}\rangle\langle u_{10}\circ v_{20}, e_2\rangle$. Here the last term vanishes because $\{e_1 e_2 v_{20}\} = 0$, and the fourth and sixth term vanish because $V_{10}\circ V_{20} \subset V_{12}$, $V_{10}*V_{20} \subset V_{12}$ and $V^{\varepsilon}_{12}\circ V^{\varepsilon}_{12} \subset \boldsymbol{R}e_1 + \boldsymbol{R}e_2$, $V^{\varepsilon}_{12}*V^{\varepsilon}_{12} \subset \boldsymbol{R}e_1 + \boldsymbol{R}e_2$. Finally, the first and the fifth summand vanish since $V_{20}*V_{12} \subset V_{10}$, $V_{10}\circ V_{12} \subset V_{20}$. The remaining summands give (1.28).

(1.10a): It is easy to see, using (1.10) and [1, Theorem 5.7], that $\{u_{20}, v_{20}, w^{\varepsilon}_{12}\}$ is orthogonal to $V^{-\varepsilon}_{12}$. We therefore have $\langle\{u_{20}, v_{20}, w_{12}\}, x_{12}\rangle = \langle\{w^{\varepsilon}_{12}, x^{\varepsilon}_{12}, u_{20}\}, v_{20}\rangle = 2\langle w^{\varepsilon}_{12}, x^{\varepsilon}_{12}\rangle\langle v_{20}, u_{20}\rangle - \langle x^{\varepsilon}_{12}*(w^{\varepsilon}_{12}*u_{20}) + w^{\varepsilon}_{12}*(x^{\varepsilon}_{12}*u_{20}), v_{20}\rangle$ by (1.21). From this the assertion easily follows.

(1.11a): Follows from (1.10a) by interchanging "1" and "2".

(1.17a): With the aid of [1, (2.14), Lemma 5.4 and Theorem 5.7] we compute $\langle\{u_{22}, v_{10}w_{12}^{+}\}, v_{12}^{+}\rangle = \langle\{w_{12}^{+}v_{12}^{+}v_{10}\}, u_{22}\rangle = -\langle w_{12}^{+}\Box(v_{12}^{+}\Box v_{10}) + v_{12}^{+}\Box(w_{12}^{+}\Box v_{10}), u_{22}\rangle = -\lambda\langle v_{12}^{+}\Box v_{10}, w_{12}^{+}\circ u_{22}\rangle - \lambda\langle w_{12}^{+}\Box v_{10}, v_{12}^{+}\circ u_{22}\rangle = -1/2\langle v_{12}^{+}*v_{10}, w_{12}^{+}\circ u_{22}\rangle - 1/2\langle w_{12}^{+}*v_{10}, v_{12}^{+}\circ u_{22}\rangle$. Now the assertion follows for $\varepsilon = +$. The case $\varepsilon = -$ is treated analogously.

(1.18a): Follows from (1.17a) by interchanging "1" and "2".

This finishes the proof of the theorems.

1.2. In this subsection we derive more identities which will be useful later.

LEMMA 1.3. *For all* $v_{ij} \in V_{ij}$ *we have*

(a) $2\langle v_{10}, v_{10}\rangle v_{10}\circ v_{12} = v_{10}\circ T(v_{10})v_{12} + T(v_{10})(v_{10}\circ v_{12})$,

(b) $2\langle v_{20}, v_{20}\rangle v_{20}*v_{12} = v_{20}*T(v_{20})v_{12} + T(v_{20})(v_{20}*v_{12})$.

PROOF. (a) By (1.1) we know $\{v_{10}v_{10}v_{10}\} = 6\langle v_{10}, v_{10}\rangle v_{10}$ and from [1, Theorem 5.7] we get $v_{10}\circ v_{10} = 2\langle v_{10}, v_{10}\rangle e_1$. In [1, (1.8)] we put $x = v_{10}$, $u = e_1$ and apply to v_{12}. We then derive $v_{10}\circ T(v_{10})v_{12} + T(v_{10})(v_{10}\circ v_{12}) + 2\langle v_{10}, v_{10}\rangle v_{10}\circ v_{12} + 2\langle v_{10}, v_{10}\rangle v_{10}\circ v_{12} - 6\langle v_{10}, v_{10}\rangle v_{10}\circ v_{12} = 0$. Hence the assertion.

(b) follows by interchanging "1" and "2" in (a).

LEMMA 1.4. *For all* $v_{ij} \in V_{ij}$ *we have*

(a) $2\langle v_{10}, v_{10}\rangle \bar{v}_{12} = \{v_{10}, v_{10}, \bar{v}_{12}\} + 2v_{10}\circ(v_{10}*v_{12})$,

(b) $2\langle v_{10}, v_{10}\rangle \bar{v}_{12} = \overline{\{v_{10}, v_{10}, v_{12}\}} + 2v_{10}*(v_{10}\circ v_{12})$,

(c) $2\langle v_{20}, v_{20}\rangle \bar{v}_{12} = \{v_{20}, v_{20}, \bar{v}_{12}\} + 2v_{20}*(v_{20}\circ v_{12})$,

(d) $2\langle v_{20}, v_{20}\rangle \bar{v}_{12} = \overline{\{v_{20}, v_{20}, v_{12}\}} + 2v_{20}\circ(v_{20}*v_{12})$,

(e) $2\langle v_{10}, v_{10}\rangle v_{12} = \{v_{10}v_{10}v_{12}\}_{12} + 2v_{10}\circ(v_{10}\circ v_{12})$,

(f) $2\langle v_{20}, v_{20}\rangle v_{12} = \{v_{20}v_{20}v_{12}\}_{12} + 2v_{20}*(v_{20}*v_{12})$.

PROOF. (a) and (b): In [1, (1.9)] we put $x = v_{10}$, $u = e_1$, $v + e_2$ and apply to v_{12}. Then we get $2v_{10}\circ(v_{10}*v_{12}) + 2v_{10}*(v_{10}\circ v_{12}) + \overline{\{v_{10}v_{10}v_{12}\}} + \{v_{10}v_{10}\bar{v}_{12}\} + 2\langle v_{10}, v_{10}\rangle\bar{v}_{12} - 6\langle v_{10}, v_{10}\rangle\bar{v}_{12} = 0$ where we have used $v_{10}\circ v_{10} = 2\langle v_{10}, v_{10}\rangle e_1$ from [1, Theorem 5.7], $\bar{v}_{10} = 0$ from [1, Lemma 5.1], and $v_{10}*v_{10} = 0$ from [1, Theorem 5.7]. This gives

$$(*)\quad 4\langle v_{10}, v_{10}\rangle\bar{v}_{12} = 2v_{10}\circ(v_{10}*v_{12}) + 2v_{20}*(v_{10}\circ v_{12}) + \overline{\{v_{10}v_{10}v_{12}\}} + \{v_{10}v_{10}\bar{v}_{10}\}\,.$$

From (1.11a) we get $\overline{\{v_{10}v_{10}v_{12}\}} = 2\langle v_{10}v_{10}\rangle\bar{v}_{12} - \overline{2v_{10}\circ(v_{10}\circ v_{12})}$. But $v_{10}\circ v_{12} \in V_{20}$ by [1, Theorem 5.7] whence $\overline{v_{10}\circ(v_{10}\circ v_{12})} = v_{10}*(v_{10}\circ v_{12})$ by [1, (5.16)]. This implies (b) and (b) together with $(*)$ gives (a). The assertions (c) and (d) follow by interchanging 1 and 2 in (a) and (b), and (e) and (f) are immediate consequences of (b) and (d).

1.3. In [1, 5.2] we introduced the notion of a triple of JC-type. Yet in each isoparametric triple we consider the subspace of V_{ij} consisting of elements which satisfy the Jordan composition rules. To be more precise, we define

$$JV_{10} := \{x_{10} \in V_{10};\ x_{10} * V_{12} \subset V_{20}\}\ ,\quad JV_{20} := \{x_{20} \in V_{20};\ x_{20} \circ V_{12} \subset V_{10}\}$$
$$JV_{12} := \{x_{12} \in V_{12};\ x_{12} \circ V_{20} \subset V_{10},\ x_{12} * V_{10} \subset V_{20}\}\ .$$

First, we note that some of the composition rules derived in Theorem 1.1 can obviously be improved if one or two of the factors has Jordan composition. Such rules are (1.8)-(1.13).

LEMMA 1.5. (a) $JV_{12} = JV_{12}^{+} \oplus JV_{12}^{-}$ *where* $JV_{12}^{\varepsilon} = JV_{12} \cap V_{12}^{\varepsilon} = \{x \in V_{12}^{\varepsilon};\ x \circ V_{12}^{-\varepsilon} \subset V_{22}^{-},\ x * V_{12}^{-\varepsilon} \subset V_{11}^{-}\}$.

(b) $(V')_2^0(e) \subset JV_{12}^{-}$, $(V')_2^0(\hat{e}) \subset JV_{12}^{+}$.

(c) *If* $V_{11}^{-} = 0 = V_{22}^{-}$, *then* $(V')_2^0(e) = JV_{12}^{-}$, $(V')_2^0(\hat{e}) = JV_{12}^{-}$.

PROOF. (a) From [1, (5.13)] we know $x_{12}^{\varepsilon} \circ V_{20} \subset V_{12}^{-\varepsilon} \oplus V_{10}$ and $V_{12}^{\varepsilon} * V_{10} \subset V_{12}^{-\varepsilon} \oplus V_{20}$. This implies $JV_{12} = (JV_{12} \cap V_{12}^{+}) \oplus (JV_{12} \cap V_{12}^{-})$. Further, $x_{12}^{\varepsilon} \in JV_{12} \cap V_{12}^{\varepsilon}$ iff $0 = \langle x_{12}^{\varepsilon} \circ V_{20}, V_{12}^{-\varepsilon}\rangle = \langle V_{20}, x_{12}^{\varepsilon} \circ V_{12}^{-\varepsilon}\rangle$ and $0 = \langle x_{12}^{\varepsilon} * V_{10}, V_{12}^{-\varepsilon}\rangle = \langle V_{10}, x_{12}^{\varepsilon} * V_{12}^{-}\rangle$ which by [1, (5.11)], is equivalent to $x_{12}^{\varepsilon} \circ V_{12}^{-} \subset V_{22}^{-}$ and $x_{12}^{\varepsilon} * V_{12}^{-\varepsilon} \subset V_{11}^{-}$.

(b) follows immediately from [1, Lemma 5.5 and Lemma 5.6], which, together with (a), also imply (c).

REMARK. (a) In general, $(V')_2^0(e) \oplus (V')_2^0(\hat{e}) \subsetneqq JV_{12}$ as one can see by looking at the isoparametric triple $V = \mathrm{Mat}(2, r; \boldsymbol{C})$, see [1, 1.15] for details.

(b) We point out that, by definition, for elements of JV_{ij} the multiplication rules [1, (5.11), (5.13)] are sharpened: $x_{12}^{+} \circ y_{22}^{-} \in V_{22}^{-}$, $x_{12}^{+} * y_{12}^{-} \in V_{11}^{-}$, $x_{12}^{\varepsilon} \circ y_{20} \in V_{10}$, $x_{12}^{\varepsilon} * y_{10} \in V_{20}$, if one of the elements lies in JV_{ij}.

Before proceeding we recall that a subspace U of the isoparametric triple V is a subsystem if $\{u_1 u_2 u_3\} \in U$ for all $u_i \in U$. Further, a formal FKM-triple is a triple system whose triple product is as in [1, 1.5b], but does not necessarily satisfy (ISO 4).

THEOREM 1.6. *If* $V_{22}^{-} \circ JV_{12} \subset JV_{12}$ *and* $V_{11}^{-} * JV_{12} \subset JV_{12}$, *then* JV_{12} *is a subsystem of* V *and the restriction of the triple product of* V *to* JV_{12} *is the dual of a formal* FKM-*triple, given by*

$$P_j = T(e_1, x_2^{(j)}) \,|\, JV_{12}$$

where $x_2^{(0)}, \cdots, x_2^{(m)}$, $m + \dim V_{22}$, *is an orthonormal basis of* V_{22}.

PROOF. First we prove that JV_{12} is a subsystem. From (1.25) follows $\{JV_{12}^{\varepsilon}, JV_{12}^{\varepsilon}, JV_{12}^{\varepsilon}\} \subset JV_{12}^{\varepsilon}$ and it is therefore enough to show: $x \in JV_{12}^{\varepsilon}$, $y \in JV_{12}^{-\varepsilon}$ imply $\{xxy\} \in JV_{12}^{-\varepsilon}$. But this follows directly from (1.26), (1.27) and the assumption.

Let $v \in JV_{12}$. We compute $\{vvv\}$ according to [1, (2.9)], $c = e_1$, and note $\{vvv\}_0 = 0$. We get $\{vvv\} = 6\langle v, v\rangle v - 3v\circ(v\circ v)_0$. But $(v\circ v)_0 \in V_{22}$ and thus $(v\circ v)_0 = \sum_r \langle v\circ v, x_2^{(r)}\rangle x_2^{(r)}$ implies $(vvv) = 9\langle v, v\rangle v - 3[\langle v, v\rangle v + \sum_{r=0}^m \langle P_r v, v\rangle P_r v]$ which shows the second assertion.

2. Peirce decomposition relative to e_{20} and e_{10}. In this section, we investigate the Peirce decomposition relative to e_{20}. Of course, all results are also valid (with the obvious changes) for e_{10}.

2.1. Recall, by (1.1) each element of V_{20} with length 1 is a minimal tripotent. Throughout this section we fix $e_{20} \in V_{20}$ with $\langle e_{20}, e_{20}\rangle = 1$ and investigate the Peirce spaces of e_{20}. Note that (e_1, e_{20}) is a pair of orthogonal tripotents. We point out that throughout this section we may everywhere replace e_1 by $u_{11} \in V_{11} = \boldsymbol{R}e_1 \oplus V_2^0(e_1)$ and e_2 by $u_{22} \in V_{22} = \boldsymbol{R}e_2 \oplus V_2^0(e_2)$.

LEMMA 2.1. *The Peirce spaces of e_{20} have the following decomposition*

$$V_2(e_{20}) = ((V_{12} \oplus V_{10}) \cap V_2(e_{20})) \oplus (V_{20} \ominus \boldsymbol{R}e_{20}) \oplus V_{22}\,, \tag{2.1}$$

$$V_0(e_{20}) = ((V_{12} \oplus V_{10}) \cap V_0(e_{20})) \oplus V_{11}\,. \tag{2.2}$$

Further,

$$V_2^0(e_{20}) \subset V_{22} \oplus V_{20}\,. \tag{2.3}$$

PROOF. By (1.8) and (1.10) we know $T(e_{20})(V_{10} \oplus V_{12}) \subset V_{10} \oplus V_{12}$, hence $V_{12} \oplus V_{10} = ((V_{12} \oplus V_{10}) \cap V_0(e_{20})) \oplus ((V_{12} \oplus V_{10}) \cap V_2(e_{20}))$. By (1.1) we have $(V_{20} \ominus \boldsymbol{R}e_{20}) \oplus V_{22} \subset V_2(e_{20})$ and by (1.2) we get $V_{11} \subset V_0(e_{20})$. Altogether, this proves (2.1) and (2.2). Finally, we apply [1, Lemma 2.7] for $c = e_{20}$ and $y = e_1$ and get $V_2^0(e_{20}) \subset V_0(e_1) = V_{22} \oplus V_{20}$. This proves (2.3).

Obviously, in Lemma 2.1 the unpleasant parts of the Peirce space of e_{20} are $(V_{12} \oplus V_{10}) \cap V_2(e_{20})$ and $(V_{12} \oplus V_{10}) \cap V_0(e_{20})$. We will have a closer look at these spaces.

LEMMA 2.2. *Assume $z_{12} \in V_{12}$. Then*

(a) $z_{12} \in V_2(e_{20}) \Leftrightarrow e_{20}*z_{12} = 0 \Leftrightarrow e_{20}\circ\bar{z}_{12} \in V_{12}$

(b) $z_{12} \in V_0(e_{20}) \Leftrightarrow e_{20}\circ z_{12} = 0 \Leftrightarrow e_{20}*(e_{20}*z_{12}) = z_{12}$

(c) $z_{12} \in V_0(e_{20}) \Rightarrow \bar{z}_{12} \in V_2(e_{20})$.

PROOF. (a) We compute $T(e_{20})z_{12}$ according to [1, (2.9)] for $c = e_2$ and get

$$T(e_{20})z_{12} = a_0 + 2z_{12} - 2e_{20}*(e_{20}*z_{12}) \tag{$*$}$$

since $e_{20}*z_{12} = (e_{20}*z_{12})_0$ and $(e_{20}*e_{20})_0 = (e_2)_0 = 0$. Here $e_{20}*(e_{20}*z_{12}) \in V_{12}$ and $a_0 \in V_{10}$ because $T(e_{20})z_{12} \in V_{12} + V_{10}$ by (1.10). Therefore the condition $z_{12} \in V_2(e_{20})$ is equivalent to $[T(e_{20})z_{12}]_{10} = 0$ and $e_{20}*(e_{20}*z_{12}) = 0$. But since

$T(e_{20}, e_2)$ is symmetric, $T(e_{20}, e_2)^2 z_{12} = 0$ if and only if $T(e_{20}, e_2)z_{12} = 0$. Hence $z_{12} \in V_2(e_{20}) \Leftrightarrow e_{20} * z_{12} = 0$ and $[T(e_{20})z_{12}]_{10} = 0$. For the first equivalence it therefore remains to show that $e_{20} * z_{12} = 0$ implies $[T(e_{20})z_{12}]_{10} = a_0 = 0$. To do this we apply Lemma 1.3.(b) and get $0 = e_{20} * T(e_{20})z_{12}$. Using the expression $(*)$ for $T(e_{20})z_{12}$ we see $0 = e_{20} * a_0$. By [1, (5.16)] this implies $0 = e_{20} \circ a_0$, and from (1.8) we derive $T(e_{20})a_0 = 0$. But now we get $\langle a_0, a_0 \rangle = \langle a_0 + 2z_{12}, a_0 \rangle = \langle T(e_{20})z_{12}, a_0 \rangle = \langle z_{12}, T(e_{20})a_0 \rangle = 0$, hence $a_0 = 0$. The second equivalence follows from (1.30).

(b) By (1.10) the condition $z_{12} \in V_0(e_{20})$ is equivalent to $T(e_{20}, e_1)^2 z_{12} = 0$; hence equivalent to $T(e_{20}, e_1)z_{12} = 0$. The last assertion follows from $(*)$.

(c) Since $e_1 \in V_0(e_{20})$, $e_2 \in V_2(e_{20})$ we have $\bar{z}_{12} = \{e_1 e_2 z_{12}\} \in V_2(e_{20})$ by [1, (2.7)].

LEMMA 2.3. *Assume* $z_{12} \in V_{12}$. *Then*

(a) $z_{12} \in V_2(e_{20}) \Leftrightarrow e_{20} * z_{12} = 0 \Leftrightarrow \{e_{20} V_{22} z_{12}\} = 0$.

(b) $z_{12} \in V_0(e_{20}) \Leftrightarrow e_{20} \circ z_{12} = 0 \Leftrightarrow \{e_{20} V_{11} z_{12}\} = 0$.

(c) $z_{12} \in V_0(e_{20}) \Rightarrow \{V_{11}, V_{22}, z_{12}\} \subset V_2(e_{20})$.

PROOF. By Lemma 2.2.(a) we know $z_{12} \in V_2(e_{20}) \Leftrightarrow \{e_{20} e_2 z_{12}\} = 0$. Since $V_{12} = V_{12}(e_1, x_{22})$ for every $x_{22} \in V_{22}$ with $|x_{22}| = 1$ and since the condition $z_{12} \in V_2(e_{20})$ does not depend on e_2, the first equivalence implies $z_{12} \in V_2(e_{20}) \Leftrightarrow \{e_{20} V_{22} z_{12}\} = 0$. The remaining assertions are proven analogously.

COROLLARY 2.4. $e_{20} * V_{10} = \{e_{20} V_{22} V_{10}\}$ *(as vector spaces)*.

PROOF. Of course, $e_{20} * V_{10} \subset \{e_{20} V_{22} V_{10}\} \subset V_{12}$. Now assume $z_{12} \in V_{12}$ such that $\langle z_{12}, e_{20} * V_{10} \rangle = 0$. Then $0 = \langle e_{20} * z_{12}, V_{10} \rangle$ implies $e_{20} * z_{12} = 0$. Hence $\{e_{20} V_{22} z_{12}\} = 0$ by Lemma 2.3 which again implies $\langle z_{12}, \{e_{20} V_{22} V_{10}\} \rangle = 0$ and so proves the corollary.

LEMMA 2.5. (a) $e_{20} \circ (V_{12} \oplus V_{10}) \subset (V_{12} \oplus V_{10}) \cap V_2(e_{20})$.

(b) $0 \neq e_{20} \circ V_{10} \subset V_{12} \cap V_2(e_{20})$.

(c) *For* $x_{20} \in V_{20}$ *and* $w_{10}, y_{10} \in V_{10}$ *we have* $0 = x_{20} * (x_{20} \circ y_{10})$ *and* $\langle x_{20} \circ y_{10}, x_{20} * w_{10} \rangle = 0$.

PROOF. (a) Set $X = V_{12} \oplus V_{10}$. By (1.8) and (1.10) we know $T(e_{20}, e_1)(X \cap V_0(e_{20})) = 0$. Since $X \ominus (X \cap V_0(e_{20})) = X \cap V_2(e_{20})$, this implies $T(e_{20}, e_1)(X \cap V_2(e_{20})) \subset X \cap V_2(e_{20})$, hence (a).

By [1, Theorem 5.9 and (5.3)] we already know $0 \neq e_{20} \circ V_{10} \subset V_{12}$. Now (b) follows from (a).

(c) We may assume $x_{20} = e_{20}$. Then $e_{20} \circ y_{10} \in V_{12} \cap V_2(e_{20})$ by (b) and $0 = e_{20} * (e_{20} \circ y_{10})$ by Lemma 2.2.(a). The remaining assertion is now obvious.

LEMMA 2.6. *Assume* $z_{10} \in V_{10}$. *Then*

(a) $z_{10} \in V_0(e_{20}) \Leftrightarrow e_{20} \circ z_{10} = 0 \Leftrightarrow e_{20} * z_{10} = 0$.

(b) $z_{10} \in V_2(e_{20}) \Leftrightarrow e_{20} \circ (e_{20} \circ z_{10}) = z_{10} \Rightarrow e_{20} * (e_{20} * z_{10}) = z_{10}$.

PROOF. (a) $z_{10} \in V_0(e_{20}) \Leftrightarrow 0 = e_{20} \circ (e_{20} \circ z_{10}) \Leftrightarrow e_{20} \circ z_{10} = 0$ by (1.8), since $T(e_{20}, e_1)$ is symmetric. By [1, (5.16)] the last condition is equivalent to $e_{20} * z_{10} = 0$.

(b) The first equivalence follows from (1.8). By (1.30) and (1, (5.16)] we have $e_{20} \circ (e_{20} \circ z_{10}) = -\overline{e_{20} \circ (e_{20} * z_{10})} + e_{20} * (e_{20} * z_{10})$. Therefore $e_{20} \circ (e_{20} \circ z_{10}) = z_{10}$ implies $e_{20} * (e_{20} * z_{10}) = z_{10}$.

LEMMA 2.7. *If $y \in V_{12} \cap V_0(e_{20})$ with $\langle y, y \rangle = 1$, then y is a minimal tripotent such that $V_{11} \oplus V_{22} \subset V_2(y)$.*

PROOF. By [1, (2.6)] we know that y is a minimal tripotent. Let $x \in V_{ii}$, $i = 1, 2$, with $\langle x, x \rangle = 1$. Then x is a minimal tripotent with $y \in V_2(x)$ by (1.14), (1.15). Thus [1, Lemma 4.5.(a)] implies $x \in V_2(y)$, hence the lemma.

COROLLARY 2.8. *If $V_{12} \cap V_0(e_{20}) \neq 0$, then $V_2^0(e_{20}) \subset V_{20}$.*

PROOF. By assumption there is a $y \in V_{12} \cap V_0(e_{20})$ with $\langle y, y \rangle = 1$. Now [1, Lemma 2.7.(b)] implies $V_2^0(e_{20}) \subset V_0(y) \cap V_0(e_1)$. But by Lemma 2.7 we have $V_0(y) \cap V_0(e_1) \subset V_{20}$.

2.2. In §2.1 we investigated those parts of the Peirce spaces of e_{20} which lie in a Peirce space V_{ij}. We now look at those parts which do not split.

LEMMA 2.9. *For $x \in V$ we have*

$$e_{20} * x = 0 \Leftrightarrow x \in V_{11} \oplus V_{22}^- \oplus (V_{12} \cap V_2(e_{20})) \oplus V_{10} \cap V_0(e_{20}) \oplus (V_{20} \ominus \boldsymbol{R} e_{20}) .$$

PROOF. The multiplication rules for the algebra "*" show $e_{20} * (V_{11} \oplus V_{22}^- \oplus (V_{20} \ominus \boldsymbol{R} e_{20})) = 0$. Also, $e_{20} * (V_{12} \cap V_2(e_{20})) = 0 = e_{20} * (V_{10} \cap V_0(e_{20}))$ by Lemma 2.2.(a) and Lemma 2.6.(a).

Assume now $e_{20} * x = 0$. Since $e_{20} * e_2 = 2e_{20}$, $e_{20} * e_{20} = 2e_2$, we know $\langle x, e_2 \rangle = 0 = \langle x, e_{20} \rangle$. By the conclusions above we may therefore assume $x \in V_{12} \oplus V_{10}$. Then we have $0 = e_{20} * x_{12} = e_{10} * x_{10}$, since $e_{20} * e_{12} \in V_{10}$, $e_{20} * x_{10} \in V_{12}$ and $0 = e_{20} * x$. Hence $x_{12} \in V_2(e_{20})$ and $x_{10} \in V_0(e_{20})$ by Lemma 2.2.(a) and Lemma 2.6.(a).

LEMMA 2.10. *Set*

$$A(e_{20}) = ((V_{12} \oplus V_{10}) \cap V_2(e_{20})) \ominus (V_{12} \cap V_2(e_{20})) ,$$
$$B(e_{20}) = ((V_{12} \oplus V_{10}) \cap V_0(e_{20})) \ominus (V_{10} \cap V_0(e_{20})) .$$

Then

(a) $e_{20}*A(e_{20}) = B(e_{20})$, $e_{20}*B(e_{20}) = A(e_{20})$,

(b) $e_{20}*(V_{10} \cap V_2(e_{20})) = V_{12} \cap V_0(e_{20})$, $e_{20}*(V_{12} \cap V_0(e_{20})) = V_{10} \cap V_2(e_{20})$,

(c) $T(e_{20}, e_2) | A(e_{20}) \oplus B(e_{20})$ *is a vector space automorphism,*

(d) $(V_{10} \cap V_2(e_{20})) \oplus (V_{12} \cap V_0(e_{20}))$ *is contained in the eigenspace of* $T(e_{20}, e_2)^2$ *for the eigenvalue* 1.

PROOF. Since $e_2 \in V_2(e_{20})$ we have $e_{20}*A(e_{20}) \subset V_0(e_{20})$. Further, $\langle e_{20}*a, x_{10}\rangle = \langle a, e_{20}*x_{10}\rangle = 0$ for $a \in A(e_{20})$, $x_{10} \in V_{10} \cap V_0(e_{20})$. Therefore $e_{20}*A(e_{20}) \subset B(e_{20})$. Similarly one proves $e_{20}*B(e_{20}) \subset A(e_{20})$. Hence $T(e_{20}, e_2)$ leaves invariant $A(e_{20}) \oplus B(e_{20})$. Moreover, the restriction of $T(e_{20}, e_2)$ to $A(e_{20}) \oplus B(e_{20})$ is injective by Lemma 2.9. This proves (a) and (c). To prove (b) we first note $e_{20}*(V_{10} \cap V_2(e_{20})) \subset V_{12} \cap V_0(e_{20})$ and $e_{20}*(V_{12} \cap V_0(e_{20})) \subset V_{10} \cap V_2(e_{20})$; this follows from $e_{20}*(V_2(e_{20}) \ominus \boldsymbol{R}e_2) \subset V_0(e_{20})$, $V_{10}*V_{20} \subset V_{12}$, $e_{20}*V_0(e_{20}) \subset V_2(e_{20})$ and $V_{20}*V_{12} \subset V_{10}$. The assertion is now a consequence of (c). Finally, (d) is just a restatement of parts of Lemma 2.2.(b) and Lemma 2.6.(b).

2.3. The Peirce spaces relative to e_{20} are much easier to handle if e_{20} lies in $JV_{20} = \{x_{20} \in V_{20};\ x_0 \in V_{12} \subset V_{10}\}$.

LEMMA 2.11. *Assume* $e_{20} \in JV_{20}$, *then* (a)

$$V_{12} = [V_{12} \cap V_0(e_{20})] \oplus [V_{12} \cap V_2(e_{20})]\,, \tag{2.4}$$

$$V_{10} = [V_{10} \cap V_0(e_{20})] \oplus [V_{10} \cap V_2(e_{20})]\,, \tag{2.5}$$

(b) $z_{12} \in V_0(e) \Leftrightarrow z_{12} = e_{20}*(e_{20}*z_{12}) \Leftrightarrow \bar{z}_{12} \in V_2(e_{20})$,

(c) $e_{20} \circ V_{10} = V_{12} \cap V_2(e_{20}) \neq 0$, $e_{20}*V_{10} = V_{12} \cap V_0(e_{20}) \neq 0$,

(d) $z_{10} \in V_2(e_{20}) \Leftrightarrow e_{20}*(e_{20}*z_{10}) = z_{10}$,

(e) $\dim V_{10} \cap V_2(e_{20}) = \dim V_{12} \cap V_0(e_{20}) = \dim V_{12} \cap V_2(e_{20})$,

(f) $V_{10} \cap V_2(e_{20}) \neq 0$,

(g) $V_2^0(e_{20}) \subset V_{20}$,

(h) $m_2 \geqq \dim V_{10} \geqq \dim V_{12} = m_1$, *in particular,* $\dim V > 4m_1$.

PROOF. (a) follows from (1.8) and (1.10).

(b) From Corollary 1.4.(f) we conclude $z_{12} \in V_0(e_{20}) \Leftrightarrow z_{12} = e_{20}*(e_{20}*z_{12})$. We already know $z_{12} \in V_0(e_{20}) \Rightarrow \bar{z}_{12} \in V_2(e_{20})$. Assume therefore $\bar{z}_{12} \in V_2(e_{20})$. Then $0 = e_{20}*\bar{z}_{12} = (e_{20} \circ \bar{z}_{12})^- + e_{20} \circ z_{12} = e_{20} \circ z_{12}$ by Lemma 2.2.(a), (1.30) and $e_{20} \circ V_{12} \subset V_{10}$ which holds by assumption. Hence $T(e_{20})z_{12} = 2e_{20} \circ (e_{20} \circ z_{12}) = 0$.

(c) By (2.5), Lemma 2.6 and Lemma 2.10 we have $e_{20}*V_{10} = e_{20}*((V_{10} \cap V_2(e_{20})) \oplus (V_{10} \cap V_0(e_{20}))) = e_{20}*(V_{10} \cap V_2(e_{20})) = V_{12} \cap V_0(e_{20})$. Using (b) and [1, (5.16)] we see $e_{20} \circ V_{10} = V_{12} \cap V_2(e_{20})$. By [1, Theorem 5.9] both spaces are nonzero.

(d) Let $z_{10} \in V_2(e_{20})$. Then $e_{20}*(e_{20}*z_{10}) = z_{10}$ by Lemma 2.6. Assume now $z_{10} = e_{20}*(e_{20}*z_{10})$. We know $V_{10} = (V_{10} \cap V_2(e_{20})) \oplus (V_{10} \cap V_0(e_{20}))$ by (2.5) and we have $A(e_{20}) = V_{10} \cap V_2(e_{20})$. We decompose $z_{10} = a_2 + a_0$, $a_2 \in A(e_{20})$, $a_0 \in V_{10} \cap V_0(e_{20})$ and get $z_{10} = e_{20}*(e_{20}*z_{10}) = e_{20}*(e_{20}*a_2) \in A(e_{20})$ by Lemma 2.6 and Lemma 2.10.(a). Hence the assertion.

(e) The first equality is a consequence of Lemma 2.10.(a), the second follows from (b).

(f) follows from (e) and (g) follows from (f) and Corollary 3.10.

(h) By (e) and [1, Corollary 5.5] we have $m_1 = \dim V_{12}^{\varepsilon} = (1/2) \dim V_{12} = \dim V_{12} \cap V_2(e_{20}) \leqq \dim V_{10} \leqq \dim (V_{10} + V_{11}^{-}) = m_2$. The remaining statement just means $2(m_1 + m_2 + 1) > 4m_1$.

COROLLARY 2.12. *If* $e_{20} \in JV_{20}$, *then*

$$V_{12} = e_{20} \circ V_{10} \oplus e_{20} * V_{10} \quad (\textit{orthogonal sum}) .$$

2.4. In this section we compute the Peirce spaces relative to the pair (e_1, e_{20}) of orthogonal tripotents. As above we denote by V_{ij} the Peirce spaces relative to (e_1, e_2).

LEMMA 2.13. *The Peirce spaces of* (e_1, e_{20}) *have the following description*:

(a) $V_{11}(e_1, e_{20}) = V_{11}$,

(b) $V_{22}(e_1, e_{20}) + V_{20}(e_1, e_{20}) = V_{22} + V_{20}$,

(c) $V_{10}(e_1, e_{20}) = (V_{12} \oplus V_{10}) \cap V_0(e_{20})$,

(d) $V_{12}(e_1, e_{20}) = (V_{12} \oplus V_{10}) \cap V_2(e_{20})$.

PROOF. (a) By definition $V_{11}(e_1, e_{20}) = \boldsymbol{R}e_1 \oplus V_2^0(e_1) = V_{11}$.

(b) By [1, Corollary 5.2] we have $V_{22}(e_1, e_{20}) + V_{20}(e_1, e_{20}) = V_0(e_1)$.

(c), (d) By [1, Corollary 5.2] we know $V_{10}(e_2, e_{20}) = [V_2(e_1) \cap V_0(e_{20})] \ominus V_{11}$. The assertions now follow from (2.1) and (2.2).

LEMMA 2.14. *Assume* $e_{20} \in JV_{20}$. *Then the Peirce space of* (e_1, e_{20}) *have the following decomposition*

(a) $V_{11}(e_1, e_{20}) = V_{11}$,

(b) $V_{22}(e_1, e_{20}) \subset V_{20}$, $V_{20}(e_1, e_{20}) = (V_{22} \oplus V_{20}) \ominus V_{22}(e_1, e_{20})$,

(c) $V_{10}(e_1, e_{20}) = [V_{10} \cap V_0(e_{20})] \oplus [V_{12} \cap V_0(e_{20})]$,

(d) $V_{12}(e_1, e_{20}) = [V_{10} \cap V_2(e_{20})] \oplus [V_{12} \cap V_2(e_{20})]$.

PROOF. (a) is Lemma 2.13.(a) and (b) follows from Lemma 2.11.(g). Finally (c) and (d) are easy consequences of (2.4) and (2.5).

3. Peirce decomposition relative to $e_{12} \in V_{12}$. In this section we consider Peirce decompositions relative to maximal and minimal tripotents contained in V_{12}.

3.1. In this subsection we consider the Peirce decomposition relative to a tripotent $e_{12}^{+} \in V_{12}^{+}$. Of course, the analogous results are valid for tripotents $e_{12}^- \in V_{12}^-$.

Recall, by (1.25) every element of V_{12}^+ with length 1 is a maximal tripotent. Throughout this subsection, we fix an element $e_{12}^+ \in V_{12}^+$ with $\langle e_{12}^+, e_{12}^+\rangle = 1$. We also recall that $e = \lambda(e_1 + e_2)$ and $\hat{e} = \lambda(e_1 - e_2)$, $\lambda = (\sqrt{2})^{-1}$, are maximal tripotents of V.

LEMMA 3.1. *The Peirce spaces of e_{12}^+ have the following decomposition*

$$V_1(e_{12}^+) = [V_{11}^- \oplus V_{22}^- \oplus V_{10} \oplus V_{20} \oplus (V')_{20}(e, \hat{e})] \cap V_1(e_{12}^+) \oplus (V_{12}^+ \ominus \boldsymbol{R}e_{12}^+) \oplus \boldsymbol{R}\hat{e} \,,$$
$$V_3(e_{12}^+) = [V_{11}^- \oplus V_{22}^- \oplus V_{10} \oplus V_{20} \oplus (V')_{20}(e, \hat{e})] \cap V_3(e_{12}^+) \oplus (V')_2^0(e) \oplus \boldsymbol{R}e \,.$$

PROOF. We have $(V_{12}^+ \ominus \boldsymbol{R}e_{12}^+) \oplus \boldsymbol{R}\hat{e} \subset V_1(e_{12}^+)$ by (1.25) and [1, Lemma 5.4]. Now, $T(e_{12}^+)u = 3u$ for $u \in \boldsymbol{R}e \oplus (V')_2^0(e)$ by [1, (2.10), Lemma 5.15] and (1.26). Hence $T(e_{12}^+)$ leaves invariant $V_{11}^- \oplus V_{22}^- \oplus V_{20} \oplus (V')_{20}(e, \hat{e})$. This proves the lemma.

In general, we cannot say more. However, as in §2, the situation is much nicer, if e_{12}^+ fulfills the Jordan composition rules. From 1.3 we recall $JV_{12} = JV_{12}^+ \oplus JV_{12}^-$ where $JV_{12}^\varepsilon = \{y_{12}^\varepsilon \in V_{12}^\varepsilon;\ y_{12}^\varepsilon \circ V_{20} \subset V_{10},\ y_{12}^\varepsilon * V_{10} \subset V_{20}\}$.

THEOREM 3.2. *Assume $e_{12}^+ \in JV_{12}^+$. Then* (a)

$$V_1(e_{12}^+) = [V_1(e_{12}^+) \cap (V_{11} \oplus V_{22})] \oplus [(V')_{20}(e, \hat{e}) \cap V_1(e_{12}^+)] \oplus [V_{21}^+ \ominus \boldsymbol{R}e_{12}^+] \oplus V_{10} \oplus V_{20} \tag{3.1}$$

$$V_3(e_{12}^+) = [V_3(e_{12}^+) \cap (V_{11} \oplus V_{22})] \oplus [(V')_{20}(e, \hat{e}) \cap V_3(e_{12}^+)] \oplus (V')_2^0(e) \,. \tag{3.2}$$

(b) *The map $V_1(e_{12}^+) \cap (V_{11} \oplus V_{22}) \to V_3(e_{12}^+) \cap (V_{11} \oplus V_{22})$, $x_{11} \oplus x_{22} \to x_{11} \oplus -x_{22}$ is a vector space isomorphism.*

(c) $\dim V_{11} = \dim V_{22} = \dim V_1(e_{12}^+) \cap (V_{11} \oplus V_{22}) = \dim V_3(e_{12}^+) \cap (V_{11} \oplus V_{22})$, $\dim V_{12}^- \cap V_1(e_{12}^+) = \dim V_{11}^-$, $\dim V_{12}^- \cap V_3(e_{12}^+) = m_1 - \dim V_{11}^-$.

(d) *The map $JV_{12}^+ \to \mathrm{End}\,(V_{10} \oplus V_{20})$: $x_{12}^+ \to T(e, x_{12}^+)$ induces a representation of the Clifford algebra over the Euclidean space $(JV_{12}^+, \langle\cdot\,,\cdot\rangle)$.*

PROOF. For $u \in V_1(e)$ we compute $T(e_{12}^+)u$ according to [1, (2.14)] and get

$$T(e_{12}^+)u = 3u - 2e_{12}^+ \square (e_{12}^+ \square u) \,. \tag{$*$}$$

We always have $e_{12}^+ \square (u_{11}^- + u_{22}^-) \subset V_{12}^-$. Also, the assumption about e_{12}^+ implies $e_{12}^+ \square V_{12}^- \subset V_{11}^- \oplus V_{22}^-$. Therefore $T(e_{12}^+)(V_{11}^- \oplus V_{22}^-) \subset V_{12}^- \oplus V_{22}^-$. Furthermore, $2e_{12}^+ \square (e_{12}^+ \square (u_{11}^- + u_{22}^-)) = e_{12}^+ *(e_{12}^+ * u_{11}^-) + e_{12}^+ \circ (e_{12}^+ * u_{11}^-) + e_{12}^+ *(e_{12}^+ \circ u_{22}^-) + e_{12}^+ \circ (e_{12}^+ \circ u_{22}^-) = u_{11}^- + u_{22}^- + e_{12}^+ \circ (e_{12}^+ * u_{11}^-) + e_{12}^+ *(e_{12}^+ \circ u_{22}^-)$, where we have used $u_{22}^- * V_{12} = 0$, $u_{11}^- \circ V_{12} = 0$, (1.34) and (1.35). We know $e_{12}^+ \circ (e_{22}^+ * u_{11}^-) \in V_{22}^-$, $e_{12}^+ *(e_{12}^+ \circ u_{22}^-) \in V_{11}^-$. Hence $(*)$ implies

$$u_{11}^- + u_{22}^- \in V_2(e_{12}^+) \Leftrightarrow e_{12}^+ *(e_{12}^+ \circ u_{22}^-) = -u_{11}^-,\ e_{12}^+ \circ (e_{12}^+ * u_{11}^-) = -u_{22}^-$$
$$u_{11}^- + u_{22}^- \in V_3(e_{12}^+) \Leftrightarrow e_{12}^+ *(e_{12}^+ \circ u_{22}^-) = u_{11}^-,\ e_{12}^+ \circ (e_{12}^+ * u_{11}^-) = u_{22}^- \,.$$

This shows (b) for $x \in V_{11}^{-} \oplus V_{22}^{-}$. Now $T(e_{12}^{+})e = 3e$ and $T(e_{12}^{+})\hat{e} = \hat{e}$ implies (b) in general.

By (*) we also have $T(e_{12}^{+})V_{12}^{-} \subset V_{12}^{-}$ and again (*) implies $(V')_2^0(e) \subset V_3(e_{12}^{+})$ since $e_{12}^{+} \square (V')_2^0(e) = 0$ by [1, Lemma 5.15].

For $w \in V_{10} \oplus V_{20}$ we have $e_{12}^{+} \circ w = e_{12}^{+} * w \in V_{10} \oplus V_{20}$ by (1.30), (1.32) and $e_{12}^{+} \in JV_{12}$. Hence $e_{12}^{+} \circ (e_{12}^{+} \circ w) = e_{12}^{+} \circ (e_{12}^{+} * w) = e_{12}^{+} * (e_{12}^{+} * w) = e_{12}^{+} * (e_{12}^{+} \circ w)$ and $T(e_{12}^{+})w = 3w - 2e_{12}^{+} \circ (e_{12}^{+} \circ w) - 2_{12}^{+} * (e_{12}^{+} * w) = w$ by (*), (1.36) and (1.37). Altogether, we have proven (a).

(c) The projection maps $V_1(e_{12}^{+}) \cap (V_{11} \oplus V_{22}) \to V_{ii}$, $i = 1, 2$, are injective by (b). Hence $\dim(V_1(e_{12}^{+}) \cap (V_{11} \oplus V_{22})) \leqq \dim V_{ii}$. But $\dim(V_{11} \oplus V_{22}) = 2\dim(V_1(e_{12}^{+}) \cap (V_{11} \oplus V_{22}))$ by (b) whence $\dim V_{11} = \dim V_1(e_{12}^{+}) \cap (V_{11} \oplus V_{22}) = \dim V_{22}$. Finally, $m_1 + 1 = \dim V_3(e_{12}^{+}) = \dim V_{11} + \dim(V_{12}^{-} \cap V_3(e_{12}^{+})) = \dim V_{11} + m_1 = \dim(V_{12}^{-} \cap V_1(e_{12}^{+}))$.

(d) Assume $y_{12}^{+} \in JV_{12}^{+}$ with $|y_{12}^{+}| = 1$. Then $V_{10} \oplus V_{20} \subset V_1(y_{12}^{+})$ by (a), $T(y_{12}^{+}, e)(V_{10} \oplus V_{20}) \subset V_{10} \oplus V_{20}$ as shown above, and (*) prove $y_{12}^{+} \square (y_{12}^{+} \square u) = u$ for $V_{10} \oplus V_{20}$.

3.2. In this subsection we consider the Peirce decomposition relative to a minimal tripotent $c \in V_{12}$. By applying [1, Lemma 4.5] we get

LEMMA 3.3. *$V_{11} \oplus V_{22} \subset V_2(c)$ for every minimal tripotent $c \in V_{12}$.*

We use this lemma to prove the following characterization of minimal tripotents in V_{12}.

LEMMA 3.4. *An element $c = c^{+} \oplus c^{-} \in V_{12}$ is a minimal tripotent if and only if $\langle c^{+}, c^{+} \rangle = 1/2 = \langle c^{-}, c^{-} \rangle$ and $c^{+} \circ c^{-} = 0$.*

PROOF. Assume $c \in V_{12}$ is a minimal tripotent. Then $c \circ c = T(c)e_1 = 2e_1$ by Lemma 3.3. We compute $c \circ c$ according to [1, (5.10)] and get $c \circ c = 2c^{+} \circ c^{-} + 2\langle c, c \rangle e_1 + \langle c, \bar{c} \rangle e_2$. Since $c^{+} \circ c^{-} \in V_{22}^{-} \oplus V_{22}$ by [1, (5.11)] we have $c^{+} \circ c^{-} = 0$ and $0 = \langle c, \bar{c} \rangle = \langle c^{+}, c^{+} \rangle - \langle c^{-}, c^{-} \rangle$. Since $1 = \langle c^{+}, c^{+} \rangle + \langle c^{-}, c^{-} \rangle$ this implies $\langle c^{\varepsilon}, c^{\varepsilon} \rangle = 1/2$.

Assume now $c^{+} \circ c^{-} = 0$ and $\langle c^{\varepsilon}, c^{\varepsilon} \rangle = 1/2$. Then also $c^{+} * c^{-} = 0$ by [1, (5.20)]. Hence $c^{+} \square c^{-} = 0 = c^{+} \hat{\square} c^{-}$. Now we compute $\{ccc\}$ according to (1.26), (1.27) and get $\{ccc\} = \{c^{+}c^{+}c^{+}\} + 3\{c^{+}c^{+}c^{-}\} + 3\{c^{-}c^{-}c^{+}\} + \{c^{-}c^{-}c^{-}\} = 3\langle c^{+}, c^{+} \rangle c^{+} + 9\langle c^{+}, c^{+} \rangle c^{-} + 9\langle c^{-}, c^{-} \rangle c^{+} + 3\langle c^{-}, c^{-} \rangle c^{-} = 6c$.

COROLLARY 3.5. *For $c \in V_{12}$ the following are equivalent:*

(a) *c is a minimal tripotent,*

(b) *$\bar{c}$ is a minimal tripotent,*

(c) *$T(x_1, x_2)c$ is a minimal tripotent for every pair $x_1 \in V_1$, $x_2 \in V_2$ with $|x_i| = 1$.*

PROOF. The equivalence of (a) and (b) follows from Lemma 3.4, and this equivalence implies the equivalence of (a) and (c) since $V_{12} = V_{12}(x_1, x_2)$ by [1, Theorem 5.11].

COROLLARY 3.6. *Let* $e_{12}^+ \in V_{12}$, $|e_{12}^+| = 1$ *and* $e_{12}^- \in V_3(e_{12}^+)$, $|e_{12}^-| = 1$. *Then* $c := \lambda e(_{12}^+ + e_{12}^-)$ *is a minimal tripotent.*

PROOF. By Lemma 3.4 we only have to check $e_{12}^+ \circ e_{12}^- = 0$. But $e_{12}^- \in V_3(e_{12}^+)$ implies $3e_{12}^- = \{e_{12}^+, e_{12}^+, e_{12}^-\} = 3e_{12}^- - 2e_{12}^+ \square (e_{12}^+ \square e_{12}^-)$ by (1.26). Therefore $e_{12}^+ \square (e_{12}^+ \square e_{12}^-) = 0$ and, consequently, also $e_{12}^+ \square e_{12}^- = 0$. From [1, (5.11)] it follows $e_{12}^+ \circ e_{12}^- = 0$ and the corollary is proven.

LEMMA 3.7. *Let* $c \in V_{12}$ *be a minimal tripotent. Then* $\bar{c} \in V_0(c)$ *and* $(c, \bar{c})$ *is a pair of orthogonal tripotents.*

PROOF. We have $e_1, e_2 \in V_2(c)$ by Lemma 3.3. But then $\bar{c} = \{e_1 e_2 c\} \in V_0(c)$ follows from [1, (2.5)].

THEOREM 3.8. *Let c be a minimal tripotent of JV_{12}. Then $\bar{c}$ is a tripotent orthogonal to c and we have*

(a) $V_2(c) = V_{11} \oplus V_{22} \oplus [V_{12} \cap V_2(c)] \oplus [(V_{10} \oplus V_{20}) \cap V_2(c)]$, $V_0(c) = [V_{12} \cap V_0(c)] \oplus [(V_{10} \oplus V_{20}) \cap V_0(c)]$.

(b) $\overline{V_{12} \cap V_\mu(c)} = V_{12} \cap V_\mu(\bar{c})$ *for* $\mu = 0, 2$.

(c)
$$\begin{aligned} V_{12} \cap V_{12}(c, \bar{c}) &= \{z \in V_{12};\ \langle z^+, c^+\rangle = 0 = \langle z^-, c^-\rangle,\ c^+ \circ z^- = 0 = c^- \circ z^+\} \\ &= [V_{12}^+ \cap V_{12}(c, \bar{c})] \oplus [V_{12}^- \cap V_{12}(c, \bar{c})] \end{aligned}$$

with $V_{12}^\varepsilon \cap V_{12}(c, \bar{c}) = V_{12}^{-\varepsilon}(c, \bar{c})$.

(d) $V_{12} = [V_{12} \cap V_{12}(c, \bar{c})] \oplus [V_{12} \cap V_0(\bar{c})] \oplus [V_{12} \cap V_0(c)]$.

(e) $V_{12} \cap V_0(\bar{c}) = [V_{12} \cap V_2(c)] \ominus [V_{12} \cap V_{12}(c, \bar{c})]$.

(f) *Let* $u \in V_{10} \oplus V_{20}$, *then*

$$u \in V_2(c) \Leftrightarrow u \in V_0(\bar{c}) \Leftrightarrow \{c^+ c^- u\} = (1/2)u\ .$$

PROOF. The first statement is clear by Lemma 3.7. Assume now $z \in V_{12}$ with $\langle z^+, c^+\rangle = 0 = \langle z^-, c^-\rangle$. We compute $T(c)z$ by using $c^+ \square c^- = 0 = c^+ \hat{\square} c^-$, the formulas (1.26) and (1.27): $T(c)z = T(c^+)z^+ + T(c^+)z^- + 2\{c^+c^-z^+\} + 2\{c^+c^-z^-\} + T(c^-)z^+ + T(c^-)z^- = (1/2)z^+ + (3/2)z^- - 2c^+ \square (c^+ \square z^-) - 2c^+ \square (z^+ \square c^-) - 2c^- \hat{\square} (z^- \hat{\square} c^+) + (3/2)z^+ - 2c^- \hat{\square} (c^- \hat{\square} z^+) + (1/2)z^-$. Because $c^+, c^- \in JV_{12}$ this formula implies $T(c)z \in V_{12}$. We already know $V_{11} \oplus V_{22} \subset V_2(c)$. Therefore we get (a).

(b) The expression above for $T(c)z$ shows $\overline{T(c)z} = T(\bar{c})\bar{z}$ which implies (b).

(c) For $z \in V_{12}$ with $\langle z^\varepsilon, c^\varepsilon\rangle = 0$ we have $T(c)z = 2z = T(\bar{c})z \Leftrightarrow c^+ \square (c^+ \square z^-) + c^- \hat{\square} (c^- \hat{\square} z^+) \pm [c^+ \square (z^+ \square c^-) + c^- \hat{\square} (z^- \hat{\square} c^+)] = 0 \Leftrightarrow c^+ \square$

$(c^+ \square z^-) + c^- \hat{\square} (c^- \hat{\square} z^+) = 0 = c^+ \square (z^+ \square c^-) + c^- \hat{\square} (z^- \hat{\square} c^+)$. Since $c^+ \square z^- \in V_{11}^- \oplus V_{22}^-$ we get $c^+ \square (c^+ \square z^-) \in V_{12}^-$, similarly $c^- \hat{\square} (c^- \hat{\square} z^+) \in V_{12}^+$. Therefore the last equation implies $c^+ \square (c^+ \square z^-) = 0 = c^- \hat{\square} (c^- \hat{\square} z^+)$, which is equivalent to $c^+ \square z^- = 0 = c^- \hat{\square} z^+$. Because $c^+ \circ z^- \in V_{22}^-$, $c^+ * z^- \in V_{11}^-$ we have $c^+ \circ z^- = 0$, and in the same way $c^- \circ z^+ = 0$. On the other hand, the conditions $c^+ \circ z^- = 0 = c^- \circ z^+$ imply $c^+ \square z^- = 0 = c^+ \hat{\square} z^- = c^- \square z^+ = c^- \hat{\square} z^+$, which shows $z \in V_{12}(c, \bar{c})$. Obviously, $V_{12} \cap V_{12}(c, \bar{c})$ is invariant under $T(e_1, e_2)$, which implies the second equation of (c). Finally, for $z^\varepsilon \in V_{12}^\varepsilon \cap V_{12}(c, \bar{c})$ we have $T(c, \bar{c})z = T(c^+, c^+)z^\varepsilon - T(c^-, c^-)z^\varepsilon$, which for $\varepsilon = +$ equals $(1/2)z^+ - (3/2)z^+ = -z^+$ and for $\varepsilon = -$ equals $(3/2)z^- - (1/2)z^- = z^-$.

(d) and (e) are obvious.

(f) Since $\sqrt{2}c^\varepsilon \in JV_{12}^\varepsilon$ is a maximal tripotent we have $V_{10} \oplus V_{20} \subset V_1(\sqrt{2}c^\varepsilon)$ by (3.1). Therefore $T(c^\varepsilon)u = (1/2)u$. Because $T(c)u = T(c^+)u + T(c^-)u + 2\{c^+c^-u\}$ this shows $T(c)u = 2u \Leftrightarrow 2\{c^+c^-u\} = u \Leftrightarrow T(\bar{c})u = 0$.

We finish this section by establishing the Peirce decomposition relative to $(c, \bar{c})$.

THEOREM 3.9. *Let c be a minimal tripotent of JV_{12}. Then the Peirce spaces relative to $(c, \bar{c})$ have the following description*:

(a) $V_{11}(c, \bar{c}) \subset V_{12} \cap V_0(\bar{c})$, $V_{22}(c, \bar{c}) \subset V_{12} \cap V_0(c)$.

(b) $V_{10}(c, \bar{c}) = [(V_{10} \oplus V_{20}) \cap V_0(\bar{c})] \oplus [V_{12} \cap V_0(\bar{c}) \ominus V_{11}(c, \bar{c})]$,
$V_{20}(c, \bar{c}) = [(V_{10} \oplus V_{20}) \cap V_0(c)] \oplus [V_{12} \cap V_0(c) \ominus V_{22}(c, \bar{c})]$.

(c) $V_{12}(c, \bar{c}) = V_{11} \oplus V_{22} \oplus [V_{12} \cap V_{12}(c, \bar{c})]$.

PROOF. We start with (c). By applying Theorem 3.8.(a) for c and $\bar{c}$ we get $V_{12}(c, \bar{c}) = V_2(c) \cap V_2(\bar{c}) = V_{11} \oplus V_{22} \oplus X$ where

$$X = [V_{12} \cap V_2(c) \oplus (V_{10} \oplus V_{20}) \cap V_2(c)] \cap [V_{12} \cap V_2(\bar{c}) \oplus (V_{10} \oplus V_{20}) \cap V_2(\bar{c})] .$$

Let $a = a_{12} + a_{10} + a_{20} \in X$. Then $a_{10} + a_{20} \in V_2(c)$, whence $a_{10} + a_{20} \in V_0(\bar{c})$ by Theorem 3.8.(f). Therefore $0 = \langle a, a_{10} + a_{20} \rangle = \langle a_{10} + a_{20}, a_{10} + a_{20} \rangle$, i.e., $a_{10} + a_{20} = 0$ and $X = V_{12} \cap V_2(c) \cap V_2(\bar{c}) = V_{12} \cap V_{12}(c, \bar{c})$.

(a) We know $e \in V_3(\sqrt{2}c^+) = V_{12}^+(c, \bar{c}) \oplus \boldsymbol{R}c^-$, whence $\{ec^+x\} \in V_{12}(c, \bar{c})$ for $x \in V_{11}(c, \bar{c})$ by [1, (5.5), (5.12)]. We also know $V_{11}(c, \bar{c}) \subset V_2(c) \ominus V_{12}(c, \bar{c})$. Therefore $x = x_{12} + x_{10} + x_{20}$ and $\{ec^+x\} = c^+ \square x_{12}^+ + c^+ \square x_{12}^- + c^+ \square (x_{10} + x_{20})$. By [1, (2.10)] we get $c^+ \square x_{12}^+ = 3\langle c^+, x_{12}^+ \rangle e$. Further, by Lemma 1.5, $c^+ \square x_{12}^- \in V_{11}^- \oplus V_{22}^-$ and $c^+ \square (x_{10} + x_{20}) \in V_{10} \oplus V_{20}$. Therefore $\{ec^+x\} \in V_{12}(c, \bar{c})$ implies $c^+ \square (x_{10} + x_{20}) = 0$ by (c), which in turn forces $x_{10} + x_{20}$ to vanish by Theorem 3.2.(d).

(b) follows from (a) and (c).

4. **The fine structure of V_{12}.** In this section we introduce a certain

subspace of $V'_{10} \oplus V'_{20}$. This subspace measures how far V_{12} differs from JV_{12}. For triples of FKM-type (which are considered in [2]) this subspace is an "obstruction" to the uniqueness of the Clifford sphere ([2, §8]).

4.1. One of the methods in [4] for showing that a given isoparametric triple system of FKM-type is not homogeneous is to show that the dimension of $V_2^0(c) = \bigcap\{\ker[T(c, y) \mid V_2(c)];\ y \in V_0(c)\}$ varies with the minimal tripotent c, i.e., is not constant. We also saw that $V_2^0(c)$ has nice properties in general. Hence it is reasonable to study the similarly defined space $\sum_{y \in V_0(c)} \ker[T(c, y) \mid V_2(c)]$.

Since we are dealing with an Euclidean space it is equivalent to consider

$$Q(c) := \Big(\sum_{y \in V_0(c)} \ker[T(c, y) \mid V_2(c)]\Big)^{\perp} \tag{4.1}$$

where c is a minimal tripotent and $\perp$ is the orthogonal complement in $V_2(c)$.

LEMMA 4.1. (a) $Q(c) = V_2(c) \cap \bigcap\{V_2(y);\ y \in V_0(c),\ |y| = 1\}$,
(b) $Q(c) = Q(d)$ *for* $d \in V_2^0(c)$ *with* $|d| = 1$.

PROOF. (a) We have $Q(c) = \bigcap\{(\ker[T(c, y) \mid V_2(c)])^{\perp};\ y \in V_0(c)\}$. Since $\ker T(c, y) = \ker T(c, sy)$ for any $s \in \boldsymbol{R}\text{-}\{0\}$ we may assume that the intersection is taken over $y \in V_0(c)$ with $|y| = 1$. Then (c, y) are orthogonal tripotents and [1, (5.2), (5.3)] for $(c, y) = (e_1, e_2)$ show $(\ker[T(c, y) \mid V_2(c)])^{\perp} = V_2(c) \cap V_2(y)$. Hence (a).

(b) We know $V_0(c) = V_0(d)$ and $V_2(c) \cap V_0(y) = V_2(d) \cap V_0(y)$ for $y \in V_0(c)$ by [1, Corollary 2.18.(b)].

We specialize c to e_1 and get the following description of $Q(e_1)$ relative to our standard Peirce decomposition:

THEOREM 4.2. *The following conditions are equivalent*:
(a) $x \in Q(e_1)$
(b) $x \in V_{12}$ *and* $x * V_{20} = 0$.

PROOF. Assume $x \in Q(e_1)$. Then $x \in V_{12} = V_2(e_1) \cap V_2(e_2)$ and $x * V_{20} = 0$ by Lemma 2.3. On the other hand, if $x \in V_{12}$ and $x * V_{20} = 0$ then $x \in V_2(e_1) \cap V_2(e_{20})$ for every $e_{20} \in V_{20}$ with $|e_{20}| = 1$ by Lemma 3.3. Also, $x \in V_2(y_{22})$ for $y_{22} \in V_{22}$, $y_{22} = 1$ by [1, (5.1)]. Assume now $y \in V_0(c)$ and $|y| = 1$. Then $y = sz_{22} + tz_{20}$ with $s^2 + t^2 = 1$ and $|z_{22}| = 1 = |z_{20}|$. Further, we have $T(y)x = s^2T(z_{22})x + t^2T(z_{20})x + 2stT(z_{22}, z_{20})x = 2x + 2st\{z_{22}z_{20}x\}$ since $x \in V_2(z_{22}) \cap V_2(z_{20})$. Because $\{z_{22}z_{20}x\} = 0$ by Lemma 2.3 it follows that $x \in V_2(y)$, whence $x \in Q(e_1)$.

We have the following equivalent description for $x \in Q(e_1)$:

LEMMA 4.3. *For* $x \in V_{12}$ *the following are equivalent:*
(1) $x * V_{20} = 0$, (2) $x*(V_{22} + V_{20}) = 0$, (3) $\{x V_{22} V_{20}\} = 0$, (4) $x * V_{10} \subset V_{12}$, (5) $\{x V_{22} V_{10}\} \subset V_{12}$, (6) $\langle x, V_{10} * V_{20}\rangle = 0$, (7) $\langle x, \{V_{10} V_{22} V_{20}\}\rangle = 0$, (8) $\langle \bar{x}, V_{10} \circ V_{20}\rangle = 0$, (9) $\langle \bar{x}, \{V_{10} V_{11} V_{20}\}\rangle = 0$, (10) $\bar{x} \circ V_{20} \subset V_{12}$, (11) $\{\bar{x} V_{11} V_{20}\} \subset V_{12}$, (12) $\bar{x} \circ V_{10} = 0$, (13) $x \circ (V_{11} + V_{20}) = 0$, (14) $\{\bar{x} V_{11} V_{10}\} = 0$, (15) $x \in Q(e_1)$.

PROOF. (1) $\Leftrightarrow$ (3): Lemma 2.3. (1) $\Leftrightarrow$ (2): [1, (5.5)]. (1) $\Leftrightarrow$ (4): [1, (5.13), (5.7)]. (3) $\Leftrightarrow$ (5): (1.12) and (1.17). (1) $\Leftrightarrow$ (6): [1, (5.7)]. (3) $\Leftrightarrow$ (7): (1.12). (6) $\Leftrightarrow$ (8): [1, (5.16)]. The equivalence of (8)–(14) follows from the equivalence of (1)–(7) by interchanging 1 and 2. Finally, (15) $\Leftrightarrow$ (2) by Theorem 4.2 and [1, (5.5)].

COROLLARY 4.4. (a) $\overline{Q(e_1)} = Q(e_2)$,
(b) $Q(x_1) = Q(e_1)$ *for all* $x_1 \in V_{11}$ *with* $|x_1| = 1$,
(c) $x_{11} * Q(e_1) = Q(e_2)$ *for all* $x_{11} \in V_{11}$, $x_{11} \neq 0$.

PROOF. (a) By Theorem 4.2 and Lemma 4.3 we know $Q(e_1) \subset V_{12}$, $Q(e_2) \subset V_{12}$ and, for $x \in V_{12}$, $x \in Q(e_1) \Leftrightarrow x * V_{20} = 0 \Leftrightarrow \bar{x} \circ V_{10} = 0 \Leftrightarrow \bar{x} \in Q(e_2)$.
(b) This is an immediate consequence of Lemma 4.3.
(c) We may assume $|x_{11}| = 1$ and have $x_{11} * Q(e_1) = x_{11} * Q(x_{11}) = Q(e_2)$ by (b) and (a).

4.2. We now define

$$Q := Q(e_1, e_2) := Q(e_1) \cap Q(e_2)\,. \tag{4.2}$$

REMARK. It is important to notice that for $x \in Q$ *all* properties of Lemma 4.3 hold. In particular, we know by Theorem 4.2 and Lemma 4.3 that

$$\begin{aligned} Q &= \{g \in V_{12}\colon g * V_{20} = 0 = g \circ V_{10}\} \\ &= \{g \in V_{12};\, 0 = \langle g, V_{10} * V_{20} + V_{10} \circ V_{20}\rangle\} = \{g \in V_{12};\, g \,\square\, (V_{10} + V_{20}) \subset V_{12}\}\,. \end{aligned} \tag{4.3}$$

Moreover, Corollary 4.4 implies

$$Q = Q^+ \oplus Q^- \quad \text{where} \quad Q^\varepsilon = Q \cap V_{12}^\varepsilon\,. \tag{4.4}$$

LEMMA 4.5. (a) $x_{11}^- * Q^\varepsilon = Q^{-\varepsilon}$ *for all* $x_{11}^- \in V_{11}^-$, $x_{11}^- \neq 0$.
(b) *If* $V_{11}^- \neq 0$ *or* $V_{22}^- \neq 0$ *then* $\dim Q^+ = \dim Q^-$.

PROOF. (a) From Corollary 4.4 we know $x_{11}^- * Q(e_1) = Q(e_2)$ and $x_{11}^- * Q(e_2) = Q(e_1)$, whence $x_{11}^- * Q = Q$. But $x_{11}^- * V_{12}^\varepsilon = V_{12}^{-\varepsilon}$, and the assertion follows.
(b) follows immediately from (a).

Note that we may define $Q' = Q'(e, \hat{e})$ for the dual triple V'. The following result interrelates Q' and JV_{ij} from § 1.3.

THEOREM 4.6. *$Q' = JV_{10} \oplus JV_{20}$ and $Q' \cap (V')^+_{12} = JV_{20}$, $Q' \cap (V')^-_{12} = JV_{10}$.*

PROOF. We use [1, Lemma 5.14 and Corollary 5.18]. Since $Q^+ = \{g \in V^+_{12};\ g*(V^-_{22} + V_{20}) = 0 = g \circ (V^-_{11} + V_{10})\}$ we have $[Q']^+ = Q' \cap (V')_{12} = \{g \in V^-_{22} \oplus V_{20};\ g \,\hat{\Box}\, V^+_{12} = 0 = g \,\Box\, V^-_{12}\}$. For $g \in Q' \cap (V')^+_{12}$ and all $x^-_{12} \in V^-_{12}$ this implies $0 = 2x^-_{12} \,\Box\, (x^-_{12} \,\Box\, g) = x^-_{12} \circ (x^-_{12} \circ g^-_{22}) + x^-_{12} * (x^-_{12} \circ g^-_{22}) + x^-_{12} * (x^-_{12} * g_{20}) + x^-_{12} * (x^-_{12} \circ g_{20}) + x^-_{12} \circ (x^-_{12} * g_{20}) + x^-_{12} \circ (x^-_{12} \circ g_{20})$. The multiplication rules for algebras "$\circ$" and "$*$" ([1, Theorem 5.7]) imply $x^-_{12} * (x^-_{12} \circ g^-_{22}) \in V^-_{11} + V_{10}$, $x^-_{12} * (x^-_{12} * g_{20}) \in V^+_{12} + V_{20}$, $x^-_{12} * (x^-_{12} \circ g_{20}) \in V^-_{11} + V_{10} + V_{20} + V^+_{12}$, $x^-_{12} \circ (x^-_{12} * g_{20}) \in V_{20}$. Also $x^-_{12} \circ (x^-_{12} \circ g^-_{22}) = \langle x^-_{12}, x^-_{12} \rangle g^-_{22}$ by (1.34) and $x^-_{12} \circ (x^-_{12} \circ g_{20}) \in V_{20}$ by (1.36). Hence $0 = [2x^-_{12} \,\Box\, (x^-_{12} \,\Box\, g)]^-_{22} = \langle x^-_{12}, x^-_{12} \rangle g^-_{22}$. Since this is valid for every $x^-_{12} \in V^-_{12}$, and since $V^-_{12} \neq 0$ by [1, Corollary 5.5], we have proved $Q' \cap (V')^+_{12} = \{g \in V_{20};\ g \,\hat{\Box}\, V^+_{12} = 0 = g \,\Box\, V^-_{12}\}$. From (1.30) it follows that $v_{20} \,\hat{\Box}\, x^+_{12} = \lambda(v_{20} \circ x^+_{12} - v_{20} * x^+_{12}) = -\lambda(\overline{v_{20} \circ x^+_{12}}) = \lambda(v_{20} \circ x^+_{12})_{12}$ and $v_{20} \,\Box\, x^-_{12} = \lambda(v_{20} \circ x^-_{12} + v_{20} * x^-_{12}) = \overline{v_{20} \circ x^-_{12}} = \lambda(v_{20} \circ x^-_{12})_{12}$ for every $v_{20} \in V_{20}$. Therefore $g_{20} \in Q' \cap (V')^+_{12} \Leftrightarrow g_{20} \circ V_{12} \subset V_{10} \Leftrightarrow g_{20} \in JV_{20}$. In the same way one proves $Q' \cap (V')^-_{12} = JV_{10}$.

COROLLARY 4.7.
$$Q^+ = JV'_{20} = \{g^+_{12} \in V'_{20};\ g^+_{12} \circ V_{20} \subset V'_{10},\ g^+_{12} * V_{10} \subset V'_{10}\}$$
$$Q^- = JV'_{10} = \{g^-_{12} \in V'_{10};\ g^-_{12} * V_{10} \subset V'_{10},\ g^+_{12} \circ V_{20} \subset V'_{20}\}.$$

PROOF. Obviously, $V'' = V$ and $Q'' = Q$. Hence $Q^+ = Q'' \cap (V'')^+_{12} = JV'_{20}$ and similarly, $Q^- = JV'_{10}$. By definition, $JV'_{20} = \{g^+_{12} \in V'_{20};\ g^+_{12} \,\Box\, V'_{12} \subset V'_{10}\}$. Hence, because $V'_{12} = V^-_{11} \oplus V^-_{22} \oplus V_{10} \oplus V_{20}$, the relation $g^+_{12} \,\Box\, (V^-_{11} \oplus V^-_{22}) \subset V'_{10}$ is trivial and with (1.30), (1.32) we get $g^+_{12} \,\Box\, v_{20} = \lambda(g^+_{12} \circ v_{20} + g^+_{12} * v_{20}) = \lambda(2g^+_{12} \circ v_{20}) + \lambda(\overline{g^+_{12} \circ v_{20}}) = \lambda(g^+_{12} \circ v_{20})^-_{12} + 2\lambda(g^+_{12} \circ v_{20})_{10}$ and $g^+_{12} \,\Box\, v_{10} = \lambda(g^+_{12} \circ v_{10} + g^+_{12} * v_{10}) = \lambda(2g^+_{12} * v_{10}) + \lambda(\overline{g^+_{12} * v_{10}}) = \lambda(g^+_{12} * v_{10})^-_{12} + 2\lambda(g^+_{12} * v_{10})_{20}$. Hence $JV'_{20} = \{g \in V^+_{12};\ g \circ V_{20} \subset V'_{10},\ g * V_{10} \subset V'_{10}\}$. The second assertion is shown similarly.

COROLLARY 4.8. *$Q = 0$ or $Q' = 0$.*

PROOF. If $Q' \neq 0$ we may assume $Q' \cap (V')^+_{12} = JV_{20} \neq 0$. Then Corollary 2.12 and (4.3) imply $Q = 0$.

COROLLARY 4.9. (a) *Q^+ and V'_{22} are orthogonal subspaces of V^+_{12},*
(b) *Q^- and V'_{11} are orthogonal subspaces of V^-_{12}.*

4.3. In addition to our considerations in §3.1 we consider here the Peirce decomposition of a maximal tripotent $g \in Q^+$.

THEOREM 4.10. *Assume $g \in Q^+$ with $\langle g, g \rangle = 1$. Then*

(a) $V_1(g) = \boldsymbol{R}\hat{e} \oplus [(V^-_{11} \oplus V_{10} \oplus V^-_{22} \oplus V_{20}) \cap V_1(g)] \oplus [V^-_{12} \cap V_1(g)] \oplus [V^+_{12} \ominus \boldsymbol{R}g]$,
$V_3(g) = \boldsymbol{R}e \oplus [(V^-_{11} \oplus V_{10} \oplus V^-_{22} \oplus V_{20}) \cap V_3(g)] \oplus [V^-_{12} \cap V_3(g)]$.

(b) *Denote by* $V_1 = \boldsymbol{R}e_1 \oplus V_{11}^- \oplus V_{10} = V_0(e_2)$ *and* $V_2 = \boldsymbol{R}e_2 \oplus V_{22}^- \oplus V_{20} = V_0(e_1)$. *The map*

$$(V_1 \oplus V_2) \cap V_3(g) \to (V_1 \oplus V_2) \cap V_1(g) \colon x_1 \oplus x_2 \to x_1 - x_2$$

is a vector space isomorphism.

(c) *The following inequality holds*: $m_2 \leqq m_1$.

PROOF. Because of (4.3) and Lemma 4.3 we have $g \circ V_{20} \subset V_{12}^-$, $g * V_{20} = 0$, $g * V_{10} \subset V_{12}^-$ and $g \circ V_{10} = 0$. Using this and [1, (5.5)] we compute for $x_1 \in V_{11}^- \oplus V_{10}$: $2g \square (g \square x_1) = g \circ (g * x_1) + g * (g * x_1) = g \circ (g * x_1) + x_1$; in the last step we have used (1.35) and (1.37). Note that the first summand lies in $V_{22}^- \oplus V_{20}$. Analogously, for $x_2 \in V_{22}^- \oplus V_{20}$ it follows $2g \square (g \square x_2) = g * (g \circ x_2) + g \circ (g \circ x_2) = g * (g \circ x_2) + x_2$. Here the first summand lies in $V_{11}^- \oplus V_{10}$. Since $T(g)(x_1 + x_2) = 3(x_1 + x_2) - 2g \square (g \square (x_1 + x_2))$ by [1, (2.14)]. We see that $V_{11}^- \oplus V_{10} \oplus V_{20}$ is $T(g)$-invariant; therefore V_{12}^- is also $T(g)$-invariant and (a) follows from Lemma 3.1.

To prove (b) we first remark that we need only consider $x_i \in V_{ii}^- \oplus V_{i0}$ since $e \in V_3(g)$ and $\hat{e} \in V_1(g)$. In this case the formulas above show $x_1 + x_2 \in V_3(g) \Leftrightarrow g \square (g \square (x_1 + x_2)) = 0 \Leftrightarrow x_2 = -g \circ (g * x_1)$ and $x_1 = -g * (g \circ x_2)$. And similarly $x_1 + x_2 \in V_1(g) \Leftrightarrow g \square (g \square (x_1 + x_2)) = x_1 + x_2 \Leftrightarrow x_2 = g \circ (g * x_1)$ and $x_1 = g * (g \circ x_2)$.

Now (b) easily follows. Finally, by [1, Corollary 5.5 and Theorem 2.2.(c)] we get $m_2 + 1 = \dim (V_{11}^- + V_{10}) + 1 = \dim V_3(g) \cap (V_1 \oplus V_2) \leqq \dim V_3(g) = m_1 + 1$.

We have the obvious but important

COROLLARY 4.11. $m_1 < m_2 \Rightarrow Q = 0$.

REMARK. There are examples of isoparametric triple systems with $m_1 = m_2$ and $Q \neq 0$ (see [3, Theorem 5.17]). But $Q' = 0$ by Corollary 4.8.

The last step in the proof of Theorem 4.10 shows

COROLLARY 4.12. *Let* $g \in Q^+$, $\langle g, g \rangle = 1$. *Then* $m_1 = m_2$ *iff* $V_3(g) \subset V_{11} \oplus V_{10} \oplus V_{22} \oplus V_{20}$.

4.4. In this subsection we consider the relations between Q^ε and JV_{12}^ε. For the following theorem we recall that we are still working with a fixed pair of orthogonal tripotents (e_1, e_2). Actually $Q = Q(e_1, e_2)$, $V_{ij} = V_{ij}(e_1, e_2)$. We also recall from [3] that V is called of *algebra type* relative to (e_1, e_2) if $V_{10} = 0 = V_{20}$.

THEOREM 4.13. *Assume* $Q^\varepsilon \neq 0$ *and* $JV_{12}^\varepsilon \neq 0$. *Then either* V *is of algebra type relative to* (e_1, e_2) *or* Q^ε *and* JV_{12}^ε *are orthogonal subspaces of* V_{12}^ε.

PROOF. Let $v \in JV_{12}^{+} = \{v \in V_{12}^{+};\ v \square V_{12}^{-} \subset V_{11}^{-} + V_{22}^{-}\}$ and $q \in Q^{+} = \{q \in V_{12}^{+},\ q \square (V_{10} + V_{20}) \subset V_{12}^{-}\}$. We may assume $|v| = |q| = 1$. We know $V_{12}^{+} \subset V_3(e) = \boldsymbol{R}\hat{e} \oplus V_{12}^{+}$ and $V_{10} \subset V_1(e)$. We put $A := T(q, e) \mid V_1(e)$ and $B := T(v, e) \mid V_1(e)$. Then $T(V_3(e), e) \mid V_1(e)$ is a cubic space, whence by [1, Lemma 3.7] we have $B^2A + AB^2 + BAB = 2\langle q, v\rangle B + A$. If V is not of algebra type then $V_{10} \neq 0$ by [1, Corollary 5.10], and we have $B^2Av_{10} = v \square (v \square (q \square v_{10})) \in V_{12}^{-}$, $AB^2v_{10} = q \square (v \square (v \square v_{10})) \in V_{12}^{-}$, $BAB = v \square (q \square (v \square v_{10})) \in V_{11}^{-} + V_{22}^{-}$, $Bv_{10} = v \square v_{10} \in V_{20}$ and $Av_{10} = q \square v_{10} \in V_{12}^{-}$ for all $v_{10} \in V_{10}$. Therefore $0 = 2\langle q, v\rangle Bv_{10}$. If $0 = Bv_{10} = v \square v_{10}$ then $\{vvv_{10}\} = 3v_{10} - 2v \square (v \square v_{10}) = 2v_{10}$; but $\{vvv_{10}\} = v_{10}$ by (3.1). Therefore $Bv_{10} \neq 0$ and $\langle q, v\rangle = 0$ follows.

COROLLARY 4.14. *$Q^{\varepsilon} \cap JV_{12}^{\varepsilon} \neq 0 \Leftrightarrow Q^{\varepsilon} = JV_{12}^{\varepsilon} \neq 0 \Leftrightarrow V$ is of algebra type.*

By the above results we know that JV_{12} and Q are either equal or orthogonal. In the second case we know nothing about $V_{12} \ominus (JV_{12} \oplus Q)$. What we may say if $V_{12} = JV_{12} \oplus Q$ is contained in the next theorem. It uses the notation of an FKM-triple. For a definition see [2].

THEOREM 4.15. *Assume $V_{12} = Q(V) \oplus JV_{12}$.*

(a) *$W := V_{10} \oplus V_{20}$ is a subsystem of V.*

(b) *If $y_1, \cdots, y_m$ is an orthonormal basis of JV_{12}^{+}, then for $w \in W$ we have (with $y_0 := \hat{e}$)*

$$\{www\} = 3\left[\langle w, w\rangle w + \sum_{j=0}^{m} \langle y_j \square w, w\rangle y_j \square w\right].$$

(c) *W is a formal FKM-triple with $m_1(W) = \dim JV_{12}^{+}$.*

PROOF. (a) Since $\{u_{i0}v_{i0}w_{i0}\} \in W$ for $i = 1, 2$ by (1.1) we only have to consider a triple product of type $\{u_{10}v_{10}w_{20}\}$ (the case $\{u_{20}v_{20}w_{10}\}$ follows by symmetry). The formula (1.9) says $\{u_{10}v_{10}w_{20}\} = u_{10}*(v_{10}*w_{20}) + v_{10}*(u_{10}*w_{20})$. By (4.3) we have $Q(V)^{\perp} = (V_{10}*V_{20} + V_{10}\circ V_{20})^{\perp} = JV_{12}$. Hence $v_{10}*w_{20}$ and $u_{10}*w_{20}$ lie in V_{12} and therefore $u_{10}*(v_{10}*w_{20}) + v_{10}*(u_{10}*w_{20})$ lies in V_{20}.

(b) Since $W \subset V_1(e)$ we may apply [1, (2.16)] and get $\{www\} = 3w \square (w \square w)$; here the element a_3 disappears since we already know $\{www\} \in V_1(e)$. By [1, (2.12)] we have $w \square w = \langle w, w\rangle e + (w \square w)_3$. We will show $(w \square w)_3 \in \boldsymbol{R}\hat{e} \oplus JV_{12}^{+}$. We have $(w \square w)_3 = 2(w_{10} \square w_{20})_3 + (w_{10} \square w_{10})_3 + (w_{20} \square w_{20})_3$. The first summand lies in V_{12}^{+} and is orthogonal to Q, hence it is in V_{12}^{+}. The second and third summands lie in $\boldsymbol{R}e_1 \oplus \boldsymbol{R}e_2$. Hence their 3-component relative to e is $(\langle w_{10} \square w_{10}, \hat{e}\rangle + \langle w_{20} \square w_{20}, \hat{e}\rangle)e = (1/2)(\langle w_{10}, w_{10}\rangle - \langle w_{20}, w_{20}\rangle)\hat{e}$, because $w_{10} \square \hat{e} = (1/2)w_{10}$ and $w_{20} \square \hat{e} = -(1/2)w_{20}$. This shows $(w \square w)_3 \in \boldsymbol{R}\hat{e} \oplus JV_{12}$ whence $w \square w = \langle w, w\rangle e + \sum_{j=0}^{m} \langle w \square w, y_j\rangle y_j$. Therefore $\{www\} = 3[\langle w, w\rangle w + \sum_{j=0}^{m} \langle y_j \square w, w\rangle y_j \square w]$.

(c) follows from (b) and Theorem 3.2.(d).

REFERENCES

[1] J. DORFMEISTER AND E. NEHER, An algebraic approach to isoparametric hypersurfaces in spheres I, this volume, 187–224.

[2] J. DORFMEISTER AND E. NEHER, Isoparametric triple systems of FKM-type I, II, III, to appear.

[3] J. DORMEISTER AND E. NEHER, Isoparametric triple systems of algebra type, to appear in Osaka Math. J.

[4] D. FERUS, H. KARCHER AND H. F. MÜNZNER, Cliffordalgebren und neue isoparametrische Hyperflächen, Math. Z. 177 (1981), 479–502.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GEORGIA
ATHENS, GA 30602
U. S. A.

AND

MATHEMATISCHES INSTITUT DER UNIVERSITÄT
EINSTEINSTR. 64/IV
4400 MÜNSTER
WEST-GERMANY