Tôhoku Math. Journ.
35 (1983), 181-186.

REMARKS ON HOMOGENEOUS HYPERBOLIC COMPLEX MANIFOLDS

AKIO KODAMA*

(Received August 7, 1981)

Introduction. Let M be a connected complex manifold with a Hermitian metric dS_M^2. We denote by $\mathrm{Aut}(M)$ the group of all biholomorphic transformations of M and by $\mathrm{Iso}(M)$ the group of all isometries of M with respect to dS_M^2. We put $\mathrm{AI}(M) = \mathrm{Aut}(M) \cap \mathrm{Iso}(M)$. As is well-known, if M is hyperbolic in the sense of Kobayashi [6], then $\mathrm{Aut}(M)$ as well as $\mathrm{Iso}(M)$ is a Lie transformation group on M.

In this note, we prove two mutually independent theorems on homogeneous hyperbolic complex manifolds. We first show the following Theorem 1, which may be a supplement to Hano [3].

THEOREM 1. *Let $M = G/K$ be a hyperbolic complex manifold on which a connected Lie subgroup G of* $\mathrm{Aut}(M)$ *acts transitively, where K denotes the isotropy subgroup of G at a point p of M. Then M can be holomorphically and equivariantly immersed into an N-dimensional complex projective space $P_N(C)$ as an open subset of a complex homogeneous submanifold G_c/G_-. In particular, M is a Kaehler manifold with respect to the induced Kaehler metric.*

An analogue for a homogeneous Siegel domain $M = G/K$ is well-known (cf. [4], [5], [8]). After some preliminaries in Section 1, the above theorem will be proved in Section 2.

Let M be a connected complex manifold with a Hermitian metric dS_M^2. If M is hyperbolic, M admits no complex line, i.e., there is no holomorphic mapping from C into M other than the constant mappings. The converse to this assertion is not true in general by an example of Eisenman and Taylor [6, p. 130]. But, it was proved by Brody [1] that M is hyperbolic if and only if M admits no complex line, provided that M is compact. By a simple modification of Brody's proof, we obtain the following theorem in Section 3.

THEOREM 2. *Let M be a Hermitian complex manifold with compact*

* The author was partially supported by the Sakkokai Foundation. Work also supported in part by the Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture, Japan.

quotient $M/\mathrm{AI}(M)$. Then M is hyperbolic if and only if M admits no complex line.

As an immediate consequence of this fact, we have the following:

COROLLARY. *Let M be a Hermitian complex manifold on which the group $\mathrm{AI}(M)$ of all holomorphic isometries acts transitively. Then M is hyperbolic if and only if M admits no complex line.*

The author would like to express his thanks to Professor H. Shima for his valuable advice.

1. Preliminaries. For later purpose, we recall some notations in Hano [3].

Let $M = G/K$ be a complex manifold on which a connected Lie group G acts transitively and effectively as a group of holomorphic transformations, where K denotes the isotropy subgroup of G at a point p of M. We denote by $\mathfrak{g}$ the Lie algebra of left invariant vector fields on G and $\mathfrak{k}$ the subalgebra corresponding to K. To the invariant complex structure I on M, there corresponds a left invariant tensor field J on G satisfying the following conditions [7]:

$$J \cdot X = 0 \quad \text{for} \quad X \in \mathfrak{k}; \tag{1}$$

$$J^2 \cdot X + X \in \mathfrak{k} \quad \text{for} \quad X \in \mathfrak{g}; \tag{2}$$

$$J \cdot \mathrm{Ad}\, k \cdot X - \mathrm{Ad}\, k \cdot J \cdot X \in \mathfrak{k} \quad \text{for} \quad k \in K, \quad X \in \mathfrak{g}; \tag{3}$$

$$J \cdot [X, Y] - [J \cdot X, Y] - [X, J \cdot Y] - J \cdot [J \cdot X, J \cdot Y] \in \mathfrak{k} \quad \text{for} \quad X, Y \in \mathfrak{g}. \tag{4}$$

Denote by $\mathfrak{g}_c$ (resp. $\mathfrak{k}_c$) the complexification of $\mathfrak{g}$ (resp. $\mathfrak{k}$), we put

$$\mathfrak{g}_\pm = \{X \mp \sqrt{-1} J \cdot X \mid X \in \mathfrak{g}\}. \tag{5}$$

Then, both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are complex subalgebras of $\mathfrak{g}_c$ and

$$\mathfrak{g}_c = \mathfrak{g}_+ + \mathfrak{g}_-, \qquad \mathfrak{k}_c = \mathfrak{g}_+ \cap \mathfrak{g}_-, \qquad \mathrm{Ad}\, k \cdot \mathfrak{g}_\pm = \mathfrak{g}_\pm \tag{6}$$

for all $k \in K$. Finally, putting

$$\mathfrak{n}(\mathfrak{g}_-) = \{X \in \mathfrak{g}_c \mid [X, \mathfrak{g}_-] \subset \mathfrak{g}_-\} \tag{7}$$

and

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid J \cdot [X, Y] - [X, J \cdot Y] \in \mathfrak{k} \text{ for all } Y \in \mathfrak{g}\}, \tag{8}$$

we obtain the following:

LEMMA 1 ([3, Lemma 3]). *The subspace $\mathfrak{h}$ is a J-stable subalgebra of $\mathfrak{g}$ and $\mathfrak{h} = \mathfrak{n}(\mathfrak{g}_-) \cap \mathfrak{g}$.*

2. **Proof of Theorem 1.** We start with the following lemma, which is an essential part of the proof.

LEMMA 2 (cf. [3, Lemma 5]). *Let $M = G/K$ be a homogeneous hyperbolic complex manifold as in Theorem* 1. *Then we have* $\mathfrak{n}(\mathfrak{g}_-) = \mathfrak{g}_-$ *and* $\mathfrak{h} = \mathfrak{k}$. *Moreover, the group K is an open subgroup of the subgroup K_1 in G consisting of all elements g such that* $\mathrm{Ad}\, g \cdot \mathfrak{g}_- = \mathfrak{g}_-$.

PROOF. Once it is shown that $\mathfrak{h} = \mathfrak{k}$, the rest can be proved by exactly the same arguments as in [3, Lemma 5]. Now, supposing that $\mathfrak{h} \supsetneqq \mathfrak{k}$, we denote by H the analytic subgroup of G corresponding to $\mathfrak{h}$ and K_0 the identity component of K. Since $\mathfrak{k}$ is an ideal of $\mathfrak{h}$, K_0 is an invariant subgroup of H so that H/K_0 is a Lie group of positive dimension by our assumption. Moreover, as is remarked in [3, p. 130], H/K_0 is a complex Lie group by Lemma 1 and G/H admits an invariant complex structure such that the principal fibre bundle G/K_0 over G/H with structure group H/K_0 is holomorphic. Notice now that G/K_0 is hyperbolic, since it is a holomorphic covering space of the hyperbolic complex manifold G/K. Then, being a complex submanifold of G/K_0, the complex Lie group H/K_0 is also hyperbolic. But, this is a contradiction, since $\dim_C H/K_0 > 0$ and since the Kobayashi pseudo-distance of any complex Lie group vanishes identically in general. q.e.d.

Now, we put $\dim_C \mathfrak{g}_- = m$ and we denote by $\mathrm{Gr}(\mathfrak{g}_c; m)$ the Grassman manifold of all complex subspaces of complex dimension m in $\mathfrak{g}_c$. The group G acts on $\mathrm{Gr}(\mathfrak{g}_c; m)$ via its adjoint representation Ad. Since the subgroup K of G leaves $\mathfrak{g}_-$ invariant by (6), we can define a mapping φ of $M = G/K$ into $\mathrm{Gr}(\mathfrak{g}_c; m)$ by

$$\varphi(gK) = \mathrm{Ad}\, g \cdot \mathfrak{g}_- \quad \text{for all} \quad g \in G\,. \tag{9}$$

It is obvious that $\mathrm{Ad}\, g_1 \cdot \varphi(g_2 K) = \varphi(g_1 g_2 K)$. Moreover, denoting by K_1 the isotropy subgroup of G at the point $\mathfrak{g}_-$, we see by Lemma 2 that K is an open subgroup of K_1. Therefore, the mapping $\varphi\colon M = G/K \to \varphi(M) = G/K_1 \subset \mathrm{Gr}(\mathfrak{g}_c; m)$ is a G-equivariant immersion.

Let G_c be the complex analytic subgroup of $\mathrm{GL}(\mathfrak{g}_c; C)$ corresponding to the subalgebra $\mathrm{ad}\,\mathfrak{g}_c$, where ad denotes the adjoint representation of $\mathfrak{g}_c$. Consider the G_c-orbit through $\mathfrak{g}_-$ in $\mathrm{Gr}(\mathfrak{g}_c; m)$. Let G_- be the isotropy subgroup of G_c at the point $\mathfrak{g}_-$. Then the Lie algebra of G_- coincides with $\mathrm{ad}\,\mathfrak{g}_-$ by Lemma 2. Now, since $M = G/K$ is homogeneous hyperbolic, we know by [6] that G has no nondiscrete center, i.e., the center of $\mathfrak{g}$ reduces to $\{0\}$. By using this fact, we can show by exactly the same arguments as in [3, the proof of Proposition 1] that $\varphi(M) =$

G/K_1 is an open complex submanifold of the complex homogeneous space G_c/G_- and $\varphi: M = G/K \to G_c/G_-$ is a G-equivariant holomorphic immersion. Finally, by composing this φ and the standard Plücker imbedding of $\mathrm{Gr}(\mathfrak{g}_c; m)$ into a complex projective space $P_N(C)$, we obtain a desired projective immersion. q.e.d.

3. **Proof of Theorem 2.** We first fix some notations. Let $\Delta(r) = \{z \in C \mid |z| < r\}$ be the open disk of radius r with the normalized Poincaré-Bergman metric $\omega_r = r^4 dz \cdot d\bar{z}/(r^2 - |z|^2)^2$. Let ρ be the distance function on the unit disk $\Delta = \Delta(1)$ determined by ω_1. Given two complex manifolds X and Y, we denote by $\mathrm{Hol}(X, Y)$ the family of all holomorphic mappings $f: X \to Y$. For a given $f \in \mathrm{Hol}(\Delta(r), M)$, we put

$$|f'(z_0)| = |f_*(\partial/\partial z)_{z=z_0}|, \tag{10}$$

where $|\cdot|$ denotes a Hermitian metric in the complexified tangent bundle of M. With these notations we have the following:

LEMMA 3 ([1, Lemma 1.1]). *Let M be a complex manifold with compact quotient $M/\mathrm{AI}(M)$. Then M is hyperbolic if and only if $\sup_f |f'(0)| < \infty$, $f \in \mathrm{Hol}(\Delta, M)$.*

PROOF. First we remark that there is a compact subset K of M such that $\mathrm{AI}(M)\cdot K = M$. Indeed, since the natural projection $\pi: M \to M/\mathrm{AI}(M)$ is a continuous open mapping and $M/\mathrm{AI}(M)$ is compact by our assumption, we can see that there exists a compact subset K of M such that $\pi(K) = M/\mathrm{AI}(M)$. Clearly this implies $\mathrm{AI}(M)\cdot K = M$. Now, the proof of the "if part" is identical to that of [1, Lemma 1.1].

Conversely, supposing that $\sup_f |f'(0)| = \infty$, we have a sequence $\{f_n\}$ of holomorphic mappings $f_n: \Delta \to M$ with $|f_n'(0)| \uparrow \infty$. Since $M = \mathrm{AI}(M)\cdot K$ with compact subset K as above, we can choose $g_n \in \mathrm{AI}(M)$ in such a way that $(g_n \circ f_n)(0) \in K$ for all $n = 1, 2, \cdots$. We put $F_n = g_n \circ f_n$ for $n = 1, 2, \cdots$. Then, replacing the sequence $\{f_n\}$ in [1, Lemma 1.1] by our $\{F_n\}$, we can show that M is not hyperbolic. q.e.d.

LEMMA 4 ([1, Lemma 2.1]). *Let M be a complex manifold with a Hermitian metric $|\cdot|$. Given an $f \in \mathrm{Hol}(\Delta(r), M)$ with $|f'(0)| \geqq c \geqq 0$, there exists $\tilde{f} \in \mathrm{Hol}(\Delta(r), M)$ with*

$$\sup_{z \in \Delta(r)} |\tilde{f}'(z)|\cdot(r^2 - |z|^2)/r^2 = |\tilde{f}'(0)| = c\,. \tag{14}$$

LEMMA 5. *Let M be a complex manifold with compact quotient $M/\mathrm{AI}(M)$. Then M is a complete Hermitian manifold.*

PROOF. We can prove this fact in the same way as [2, Lemma 2.1]. q.e.d.

W now complete the proof of Theorem 2 along the same line as that for [1, Theorem 4.1]. Let K be a compact subset of M such that $\mathrm{AI}(M)\cdot K = M$ as before. It is well-known (cf. [6]) that if M is hyperbolic, M admits no complex line.

Suppose that M is not hyperbolic. By Lemma 3 there exists a sequence $\{f_n\}$ of holomorphic mappings $f_n: \Delta \to M$ with $|f_n'(0)| \uparrow \infty$, or equivalently, we have a sequence $\{f_n\}$ of holomorphic mappings $f_n: \Delta(r_n) \to M$ with $|f_n'(0)| = 1$ and $r_n \uparrow \infty$. Applying Lemma 4, we now obtain a sequence of mappings $\tilde{f}_n \in \mathrm{Hol}(\Delta(r_n), M)$ satisfying

$$\sup_{z\in\Delta(r_n)} |\tilde{f}_n'(z)|\cdot(r_n^2 - |z|^2)/r_n^2 = |\tilde{f}_n'(0)| = 1\,, \tag{15}$$

from which we have

$$|\tilde{f}_n'(z)| \leqq 4/3 \quad \text{on} \quad \Delta(r_n/2)\,. \tag{16}$$

Obviously, this implies that the sequence $\{\tilde{f}_k\}_{k\geqq n}$ is equicontinuous on $\Delta(r_n/2)$ for arbitrarily fixed n. Moreover, M is a complete Hermitian manifold by Lemma 5. We conclude therefore by Wu [9, Lemma 1.1] that $\{\tilde{f}_k\}_{k\geqq n}$ is a normal family on $\Delta(r_n/2)$. Changing $\tilde{f}_n$ for a suitable holomorphic mapping of the form $g_n\circ\tilde{f}_n$, $g_n\in\mathrm{AI}(M)$, if necessary, we may assume that $\tilde{f}_n(0)\in K$, that is, $\tilde{f}_n(\{o\})\cap K \neq \emptyset$ for all $n = 1, 2, \cdots$. Then, the normality guarantees that some subsequence $\{\tilde{f}_{k_i}\}$ of $\{\tilde{f}_k\}_{k\geqq n}$ converges on $\Delta(r_n/2)$ to a holomorphic mapping of $\Delta(r_n/2)$ into M. By the usual diagonal argument, we can now extract a subsequence $\{\tilde{f}_{n_i}\}$ of $\{\tilde{f}_n\}$ which converges on $\Delta(r_n)$ to a holomorphic mapping $F_n: \Delta(r_n) \to M$ for all $n = 1, 2, \cdots$. By means of this sequence $\{F_n\}$, we can define a holomorphic mapping $F: \boldsymbol{C} \to M$ by $F(z) = F_n(z)$ for all $z\in\Delta(r_n)$, $n = 1, 2, \cdots$. This mapping F cannot be constant, since $|F'(0)| = \lim_{i\to\infty} |\tilde{f}_{n_i}'(0)| = 1$. Therefore we have shown that if M admits no complex line, then M is hyperbolic. q.e.d.

References

[1] R. Brody, Compact manifolds and hyperbolicity, Trans. Amer. Math. Soc. 235 (1978), 213–219.

[2] J. E. Fornaess and N. Sibony, Increasing sequences of complex manifolds, Math. Ann. 255 (1981), 351–360.

[3] J. Hano, Equivariant projective immersion of a complex coset space with non-degenerate canonical hermitian form, Scripta Math. 29 (1971), 125–139.

[4] S. Kaneyuki, Homogeneous Bounded Domains and Siegel Domains, Lecture Notes in Math. 241, Springer-Verlag, Berlin, Heidelberg and New York, 1971.

[5] W. Kaup, Y. Matsushima and T. Ochiai, On the automorphisms and equivalences of generalized Siegel domains, Amer. J. Math. 92 (1970), 475–498.

[6] S. Kobayashi, Hyperbolic Manifolds and Holomorphic Mappings, Marcel Dekker, New

York, 1970.
[7] J. L. KOSZUL, Sur la forme hermitienne canonique des espaces homogenes complexes, Canad. J. Math. 7 (1955), 562-576.
[8] N. TANAKA, On infinitesimal automorphisms of Siegel domains, J. Math. Soc. Japan 22 (1970), 180-212.
[9] H. WU, Normal families of holomorphic mappings, Acta Math. 119 (1967), 193-233.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KANAZAWA UNIVERSITY
KANAZAWA, 920
JAPAN