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A CENTRAL LIMIT THEOREM FOR PIECEWISE CONVEX MAPPINGS OF THE UNIT INTERVAL

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Abstract. In this paper we prove a central limit theorem for a class of piecewise continuous and convex transformations from the unit interval into itself.

1. Introduction. Recently many authors have discussed the existence of an invariant measure for a transformation of a compact manifold into itself (Lasota and Yorke [2], [4], Lasota [5], [6], Krzyżewski and Szlenk [7], Kosjakin and Sandler [8], Bunimovič [9], Renyi [10], Boyarsky and Scarowsky [11]). Many of these results were proved by showing the existence of a fixed point for the Frobenius-Perron operator. In several situations one can strengthen these results. Namely, one can show that the iterates of the Frobenius-Perron operator are exponentially convergent to the fixed point. By using such result [3] we shall prove in this paper a central limit theorem for a class of piecewise continuous and convex transformations of the unit interval. We do not require that our mappings are C^1 and expanding. The central limit theorem for transformations of the unit interval into itself was given also by Boyarsky and Scarowsky [11], Walters [12], Ishitani [13], Wong [14[, Tran Vinh Hien [17]. These results were generalized by Keller [16]. But he required in his paper that transformations under consideration were piecewise C^2 and expanding.

Since our transformations are exact [2] and all iterates of our transformation have property $\inf|(T^n)'| > 1$ for n sufficiently large (Proposition 1) then Ratner's theorem in [18] implies the weak Bernoulli condition for their natural extensions. Therefore Ornstein and Friedman's theorem [19] implies that the natural extension of our tranformations are isomorphic to the Bernoulli shifts for which the classical central limit theorems hold. But we do not obtain automatically from this the same result for our transformations. In particular, it is not easy to verify that the assumptions of the classical central limit theorems for random variables given by Hölder continuous functions or functions of bounded variation over the unit interval are satisfied.

The present paper consists of three sections. In the first one we shall state our main result Theorem 1. The second section contains some preliminary notations and auxiliary results which may be of some interest in themselves, namely Theorem 2, Propositions 1, 2 and a lemma. The proof of Theorem 1 we shall give in the last section.

2. A central limit theorem. Let T be a given transformation of the unit interval into itself satisfying the following conditions:

(a) there is a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of the unit interval such that for each integer k ($k = 1, \cdots, r$) the restriction of T to the interval $[a_{k-1}, a_k)$ is continuous and convex,

(b) $T(a_{k-1}) = 0,\ T'(a_{k-1}) > 0,\ k = 1, \cdots, r,$

(c) $T'(0) > 1$.

In [2], it is shown that for such a transformation there exists a unique normalized absolutely continuous measure μ invariant under T, the density $g_\mu = d\mu/dm$ (m denotes the Lebesgue measure) is bounded and decreasing and moreover, the system $([0, 1], \mu, T)$ is exact.

Under the above assumptions on the transformation T we shall prove our main result:

THEOREM 1. *If either*

(i) *f is a function of bounded variation over* $[0, 1]$

or

(ii) *f is Hölder continuous*

then

$$\sigma^2 = E_\mu(f - E_\mu f)^2 + 2\sum_{j=1}^{\infty} E_\mu[(f - E_\mu f)(f \circ T^j - E_\mu f)] < \infty\ , \tag{1}$$

$$\lim_{n\to\infty}\left\{(1/\sqrt{n}\,)\sum_{j=0}^{n-1}(f \circ T^j - E_\mu f) < z\right\} = \phi_\sigma(z)\ , \tag{2}$$

$$\lim_{n\to\infty} m\left\{(1/\sqrt{n}\,)\sum_{j=0}^{n-1}(f \circ T^j - E_m(f \circ T^j)) < z\right\} = \phi_\sigma(z)\ , \tag{3}$$

where $E_\mu f = \int_0^1 f d\mu,\ E_m(f \circ T^j) = \int_0^1 f \circ T^j dm,\ \phi_\sigma(z) = (1/(2\pi\sigma))\int_{-\infty}^{z} \exp(-t^2/(2\sigma^2))dt$ *if* $\sigma > 0$ *and* $\phi_0(z) = \begin{cases} 1 & (z > 0) \\ 0 & (z \leqq 0) \end{cases}$.

3. Definitions and auxiliary results. Let $([0, 1], \Sigma, m)$ be the measure space with Lebesgue measure m and let $L_1([0, 1], \Sigma, m) = L_1(m)$ be the space of all functions f defined on $[0, 1]$ for which $|f|$ is integrable. For a measurable nonsingular function $\tau\colon [0, 1] \to [0, 1]$ (i.e., if $A \in \Sigma$, $m(A) = 0$ implies $m(\tau^{-1}(A)) = 0$) we define the Frobenius-Perron operator

$P_\tau: L_1(m) \to L_1(m)$ by the formula

$$P_\tau f = \frac{d}{dx}\int_{\tau^{-1}([0,x])} f dm .$$

The operator P_τ is linear, continuous and satisfies the following conditions:

(4) P_τ is positive: $f \geqq 0 \Rightarrow P_\tau f \geqq 0$,

(5) P_τ preserves integrals: $\int_0^1 P_\tau f dm = \int_0^1 f dm$, $f \in L_1(m)$,

(6) P_τ^n is the Frobenius-Perron operator corresponding to τ^n,

(7) $P_\tau f = f$ iff the measure $d\mu = f dm$ is invariant under τ, i.e., $\mu(\tau^{-1}(A)) = \mu(A)$ for $A \in \Sigma$,

(8) $\int_0^1 (P_\tau f) g dm = \int_0^1 (g \circ \tau) f dm$ for any $f \in L_1(m)$ and $g \in L_\infty(m)$.

For convenience, we shall simply denote by P the Frobenius-Perron operator corresponding to T satisfying conditions (a)-(c).

The convergence of iterates P^n is discussed in [3] where is stated the following:

THEOREM 2. *There exist constants* $K > 0$, $c > 0$ *and* $s \in (0, 1)$ *such that for a nonnegative integrable function* f *with bounded variation over* [0, 1]

$$\| P^n f - g_\mu \| f \|_{L_1} \|_{L_1} \leqq s^n K\Big(\bigvee_0^1 f + c \| f \|_{L_1}\Big) , \tag{9}$$

where $\bigvee_0^1 f$ *denotes the variation of* f *over* [0, 1].

Now, we shall show two properties of the transformation T satisfying conditions (a)-(c) which will be needed for the proof of Theorem 1.

PROPOSITION 1. *There exists* $n_0 \in N$ *such that* $\inf_{x \in [0,1]} |(T^n)'(x)| > 1$ *for* $n > n_0$.

PROOF. Denote by $0 = a_0^n < a_1^n \cdots < a_{r_n}^n = 1$ the partition corresponding to T^n (i.e., T^n is convex on each interval $[a_{k-1}^n, a_k^n)$ and $T^n(a_{k-1}^n) = 0$) and by T_k^n the restriction of T^n to the interval $[a_{k-1}^n, a_k^n)$. The functions T_k^n are increasing, continuous and differentiable except on a set of at most countable number of points.

Consider, now, the set $S = \bigcup_{n=0}^\infty T^{-n}(\{a_0, a_1, \cdots, a_r\})$. Since $\mathrm{cl}\, S = [0, 1]$ (see the proof of Theorem 3 in [2] for details), there exists $m_0 \in N$ such that for any $n > m_0$

$$\max_i (a_i^n - a_{i-1}^n) < \eta = (a_1/2) \inf_{x \in [0,1]} |T'(x)| . \tag{10}$$

Let $n > m_0$. Define

$$A_n = T^{-n}([a_0, a_1)) = \bigcup_{i=1}^{r_n} (T_i^n)^{-1}([a_0, a_1))$$

and $B_n = [0, 1] \setminus A_n$. It is obvious that

$$T^n(x) < a_1 \quad \text{if} \quad x \in A_n \tag{11}$$

and

$$T^n(x) \geqq a_1 \quad \text{if} \quad x \in B_n . \tag{12}$$

Since the function T^n satisfies conditions analogous to (a)–(c) and in particular $(T_i^n)'$ $(i = 1, 2, \cdots, r_n)$ is increasing, by (10) and (12) for $x \in B_n$ we have $(T_i^n)'(x) \geqq a_1/\eta$. Hence, by (10)

$$(T \circ T_i^n)'(x) = T'(T_i^n(x))(T_i^n)'(x) \geqq (a_1/\eta) \inf_{x \in [0,1]} (T'(x)) = 2 \tag{13}$$

whenever $x \in B_n$. By the same arguments and (11) for $x \in A_n$ we have

$$(T \circ T_i^n)'(x) = T'(T_i^n(x))(T_i^n)'(x) \geqq T'(0)(T_i^n)'(a_{i-1}^n) \geqq T'(0) \inf_{x \in [0,1]} (T^n)'(x) . \tag{14}$$

Inequalities (13) and (14) give us $\inf (T^{n+1})' \geqq \min (2, T'(0) \inf (T^n)')$ and consequently, by induction, $\inf (T^n)' \geqq \min (2, [T'(0)]^{n-m_0} \inf (T^{m_0})')$. Hence, since $\inf (T^{m_0})' > 0$ and $T'(0) > 1$ we have the conclusion of Proposition 1.

PROPOSITION 2. *There are $C > 0$ and $q \in (0, 1)$ such that*

$$a_i^n - a_{i-1}^n \leqq Cq^n$$

where $0 = a_0^n < a_1^n < \cdots < a_{r_n}^n = 1$ is the partition corresponding to the transformation T^n.

PROOF. By Proposition 1, there exists n_0 such that $\inf (T^{n_0})' > 1$. Put $T^{n_0} = \psi$. It is obvious that $b_i^n - b_{i-1}^n \leqq (\max (b_i - b_{i-1}))/(\inf (\psi^n)')$, where $0 = b_0^n < b_1^n < \cdots < b_{p_n}^n = 1$ and $0 = b_0 < b_1 < \cdots < b_p = 1$ are partitions corresponding to ψ^n and ψ, respectively. By this, since $\{b_0^n, b_1^n, \cdots, b_{p_n}^n\} = T^{-nn_0}(\{a_0, a_1, \cdots, a_r\}) \cup \{1\}$ and $T^{-n}(\{a_0, a_1, \cdots, a_r\}) \subset T^{-m}(\{a_0, a_1, \cdots, a_r\})$ whenever $m \geqq n$, we obtain $\max (a_i^n - a_{i-1}^n) \leqq Cq^n$ where $C = \rho^{-1} \max (a_i - a_{i-1})$, $\rho = (\inf \psi')^{-1}$ and $q = \rho^{1/n_0}$. q.e.d.

Now, we give a lemma corresponding to the strong mixing coefficient for some stationary process. Before that, we establish some notations.

Let ξ_n be a process on the probability space $([0, 1], \Sigma, \mu)$ (Σ is the σ-field of the Borel sets) given by the formula $\xi_n = \chi(T^n)$ where $\chi = \sum_{i=1}^r \alpha_i \chi_{[a_{i-1}, a_i)}$ with $\alpha_i \neq \alpha_j$ for $i \neq j$ and χ_A denotes the characteristic function of the set A. Denote by $\mathfrak{M}_k^l$ the σ-field generated by the sets of the form $\{x \in [0, 1]: (\xi_k(x), \cdots, \xi_l(x)) \in A\}$ where $A \subset R^{l-k+1}$ is an $(l-k+1)$-dimensional cube. It is easy to see that $\mathfrak{M}_0^k$ is generated by the set of intervals $A_{j_0,\dots,j_k} = T_{j_k}^{-1} \cdots T_{j_0}^{-1}([0, 1))$ where T_i is the restriction of T to the interval $[a_{i-1}, a_i)$ and $j_i = 1, \cdots, r$, $i = 0, \cdots, k$. It is also obvious that $T^{k+1}(A_{j_0,\dots,j_k}) = T_{j_0} \circ \cdots \circ T_{j_k}(A_{j_0,\dots,j_k})$ is an interval with the left endpoint equal to zero.

We have the following;

LEMMA. *ξ_n is a stationary process with the strong mixing coefficients*

$$\alpha(n) = \sup_{k \geqq 0} \sup_{A \in \mathfrak{M}_0^k} \sup_{B \in \mathfrak{M}_{n+k}^\infty} |\mu(A \cap B) - \mu(A)\mu(B)|$$

satisfying

$$\alpha(n) \leqq s^n M \tag{15}$$

with s from Theorem 2 and with some $M > 0$.

PROOF. Since the measure μ is invariant under T the process ξ_n is stationary. It remains, therefore, only to prove (15).

By simple computations the Frobenius-Perron operator for the transformation T satisfying (a)-(c) has the form $Pf = \sum_{i=1}^r f(\varphi_i)|\varphi_i'|$ where

$$\varphi_i(x) = \begin{cases} T_i^{-1}(x) & \text{for } x \in T([a_{i-1}, a_i)) \\ a_i & \text{for } x \in [0, 1] \setminus T([a_{i-1}, a_i]) . \end{cases}$$

The functions φ_i are increasing, continuous and differentiable except on a set of at most a countable number of points. At these points φ_i' can be defined as the right hand side derivative. The functions φ_i' are decreasing and bounded (since $T'(a_i) > 0$). By its very definition the operator P is a map from L_1 into L_1, but the last formula enables us to consider P as a map from the space of functions defined on [0, 1] into itself. It is obvious that Pf is decreasing for any decreasing f.

Let B be a Borel set and $A = \bigcup_{j_1,\dots,j_k \in I} A_{j_1,\dots,j_k}$ where I is a subset of the k-th Cartesian product of the set $\{1, 2, \cdots, r\}$. We have

$$\begin{aligned} P^k(g_\mu \chi_A) &= \sum_{j_1,\dots,j_k=1}^{r} ((g_\mu \chi_A) \circ (\varphi_{j_1} \circ \cdots \circ \varphi_{j_k}))|(\varphi_{j_1} \circ \cdots \circ \varphi_{j_k})'| \\ &= \sum_{j_1,\dots,j_k \in I} ((g_\mu \chi_{A_{j_1,\dots,j_k}}) \circ (\varphi_{j_1} \circ \cdots \circ \varphi_{j_k}))|(\varphi_{j_1} \circ \cdots \circ \varphi_{j_k})'| \\ &= \sum_{j_1,\dots,j_k \in I} g_\mu(\varphi_{j_1} \circ \cdots \circ \varphi_{j_k}) \chi_{C_{j_1,\dots,j_k}} |(\varphi_{j_1} \circ \cdots \circ \varphi_{j_k})'| \\ &\leqq P^k g_\mu = g_\mu \end{aligned}$$

where $C_{j_1,\dots,j_k} = T^k(A_{j_1,\dots,j_k})$ is an interval with the left endpoint equal to zero. Hence, $P^k(g_\mu \chi_A)$ is decreasing, and moreover,

$$\bigvee_0^1 P^k(g_\mu \chi_A) \leqq \bigvee_0^1 g_\mu + \|g_\mu\|_{L_1} = \bigvee_0^1 g_\mu + 1 .$$

Now, by (7), (8) and (9) we have

$$\begin{aligned} |\mu(T^{-n-k}(B) \cap A) - \mu(A)\mu(B)| &= \left|\int_0^1 (P^n(P^k(g_\mu \chi_A)) - g_\mu \|g_\mu \chi_A\|_{L_1})\chi_B dm\right| \\ &\leqq \|P^n(P^k(g_\mu \chi_A)) - g_\mu \|g_\mu \chi_A\|_{L_1}\|_{L_1} \\ &\leqq s^n K\Big(\bigvee_0^1 P^k(g_\mu \chi_A) + c\|g_\mu \chi_A\|_{L_1}\Big) \leqq s^n K\Big(\bigvee_0^1 g_\mu + c + 1\Big) = s^n M \end{aligned}$$

where the constants K, c are from (9) and $M = K(\bigvee_0^1 g + c + 1)$. Therefore, since

$$\alpha(n) \leqq \sup_k \sup_{A \in \mathfrak{M}_0^k} \sup_{B \in \Sigma} |\mu(T^{-n-k}(B) \cap A) - \mu(A)\mu(B)|$$

we obtain $\alpha(n) \leqq s^n M s^{-1}$. q.e.d.

4. Proof of Theorem 1. The estimation of the strong mixing coefficients for a stationary process χ_n allows us to invoke [1, Theorem 18.6.2] directly, to obtain a central limit theorem. Thus, we must only to prove

$$E_\mu |f|^{2+\delta} < \infty \quad \text{for some} \quad \delta > 0 , \tag{16}$$

$$\sum_{k=1}^\infty [E_\mu |f - E_\mu\{f \mid \mathfrak{M}_0^k\}|^{(2+\delta)/(1+\delta)}]^{(1+\delta)/(2+\delta)} < \infty \tag{17}$$

where $E_\mu\{f \mid \mathfrak{M}_0^k\}$ is the conditional expectation of f with respect to a σ-field $\mathfrak{M}_0^k$, and

$$\sum_{n=1}^\infty (\alpha(n))^{\delta/(2+\delta)} < \infty \tag{18}$$

for any f satisfying the assumptions of Theorem 1.

Let f be of bounded variation. The inequality (16) is obvious for each $\delta > 0$. The condition (18) follows immediately from (10). Therefore, it remains only to prove (17). Denote by Q_k the set of intervals $A_{j_0,\dots,j_k}$. By Proposition 2 the length of each interval from Q_k is not greater than Cq^n, where $q \in (0, 1)$. From this, for any f of bounded variation we have

$$\begin{aligned} E_\mu |f - E_\mu\{f \mid \mathfrak{M}_0^k\}|^2 &\leqq \sum_{A \in Q_k} \int_A \Big[f - (1/\mu(A))\int_A f d\mu\Big]^2 d\mu \\ &\leqq \sum_{A \in Q_k} \int_A (\bigvee_A f)^2 d\mu \leqq \bigvee_0^1 f \sum_{A \in Q_k} \int_A (\bigvee_A f) d\mu \\ &\leqq \Big(\bigvee_0^1 f\Big) \sum_{A \in Q_k} (\bigvee_A f) \sup_{A \in Q_k} \mu(A) \leqq \Big(\bigvee_0^1 f\Big)^2 C q^n . \end{aligned}$$

Hence, since $L^{\theta}(\mu) \subset L^2(\mu)$ for $\theta = (2+\delta)/(1+\delta) < 2$, we obtain (17).

Now, let f be Hölder continuous. We have

$$\begin{aligned}\sup |f - E_\mu\{f|\mathfrak{M}_0^k\}| &\leqq \sup_{A \in Q_k} \sup_{x \in A} \left| f(x) - (1/\mu(A)) \int_A f d\mu \right| \\ &\leqq \sup_{A \in Q_k} \sup_{x,y \in A} |f(x) - f(y)| \leqq \bar{K} m(A)^\beta \leqq \bar{K} C^\beta (q^\beta)^n\end{aligned}$$

for some $\bar{K} > 0$ and $\beta > 0$. This yields (17) and completes the proof of (1) and (2). Now, in the same method as in [13] we prove (3). Let

$$z_n^1 = (1/\sqrt{n}) \sum_{k=0}^{n-1} (f \circ T^k - E_\mu f) \quad \text{and} \quad z_n^2 = (1/\sqrt{n}) \sum_{k=0}^{n-1} (f \circ T^k - E_m(f \circ T^k)) .$$

We have

$$\begin{aligned}&|E_\mu(\exp(i\xi z_n^1)) - E_m(\exp(i\xi z_n^2))| \\ &\quad \leqq |E_\mu(\exp(i\xi z_n^1)) - E_m(\exp(i\xi z_n^1))| + |E_m(\exp(i\xi z_n^1)) - E_m(\exp(i\xi z_n^2))| \\ &\quad \leqq E_\mu \left| 1 - \exp\left\{(i\xi/\sqrt{n}) \sum_{k=0}^{w} (f \circ T^k - E_\mu f)\right\} \right| \\ &\qquad + E_m \left| 1 - \exp\left\{(i\xi/\sqrt{n}) \sum_{k=0}^{w} (f \circ T^k - E_\mu f)\right\} \right| \\ &\qquad + \left| (E_\mu - E_m) \exp\left\{(i\xi/\sqrt{n}) \sum_{k=w+1}^{n-1} (f \circ T^k - E_\mu f)\right\} \right| \\ &\qquad + E_m \left| 1 - \exp\left\{(i\xi/\sqrt{n}) \sum_{k=0}^{w} (f \circ T^k - E_\mu f)\right\} \right| \\ &\qquad + E_m \left| 1 - \exp\left\{(i\xi/\sqrt{n}) \sum_{k=0}^{w} (f \circ T^k - E_m(f \circ T^k))\right\} \right| \\ &\qquad + E_m \left| 1 - \exp\left\{(i\xi/\sqrt{n}) \sum_{k=w+1}^{n-1} (E_\mu f - E_m(f \circ T^k))\right\} \right| .\end{aligned}$$

Hence, setting $w = [\log n]$, by (9) we obtain

$$\lim_{n \to \infty} |E_\mu(\exp(i\xi z_n^1)) - E_m(\exp(i\xi z_n^2))| = 0$$

uniformly on any bounded interval. This gives us (3) and completes the proof of Theorem 1.

Under the same assumptions as in Theorem 1, without any difficulties we may prove the following:

THEOREM 3. *If $\sigma > 0$, then*

$$\lim_{n \to \infty} \nu\left\{(1/\sqrt{n}) \sum_{k=0}^{n-1} (f \circ T^k - E_\nu(f \circ T^k)) < z\right\} = \phi_\sigma(z)$$

whenever $\bigvee_0^1 d\nu/dm < \infty$.

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