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GENERALIZED INVERSE METHOD FOR SUBSPACE MAPS

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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1. Introduction. Let H be a Hilbert space and let $C(H)$ be the set of all closed linear subspaces in H. For a bounded linear operator A on H, define a map ϕ_A on $C(H)$, called the subspace map of A, by

$$\phi_A(M) = (AM)^- \qquad (M \in C(H)) ,$$

where "$-$" denotes the uniform closure. Identifying every closed subspace M with the corresponding (orthogonal) projection P_M or $\operatorname{proj} M$, we see that $C(H)$ is a subset of $B(H)$, the Banach space of all bounded linear operators on H and hence has the uniform, strong and weak (operator) topologies. It was shown in [8] (cf. [2]) that the subspace map ϕ_A is uniformly (and strongly) continuous on $C(H)$ if and only if the operator A is left-invertible, and moreover, in this case ϕ_A behaves well. For instance, $\phi_A(\mathscr{F})$ is uniformly (resp. strongly, weakly) closed if $\mathscr{F}$ is a uniformly (resp. strongly, weakly) closed subset of $C(H)$.

In this paper we shall show similar results on the subspace map ϕ_A under the weaker condition that the operator A has closed range, or equivalently, has the (Moore-Penrose) generalized inverse [1] [9]; using operator theory of generalized inverses, we shall discuss the local continuity and some other topological properties of ϕ_A of A with closed range, which will extend some results in [2] and [8].

Throughout this note we shall write $A \in (\mathrm{CR})$ when the operator A has closed range. The generalized inverse $A^\dagger$ of $A \in (\mathrm{CR})$ satisfies (and is determined by) the following four Penrose identities [1]

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger \ \text{ and } \ (A^\dagger A)^* = A^\dagger A .$$

If we denote by AH and $\ker A$ the range and the kernel of $A(\in (\mathrm{CR}))$ respectively, then the products $AA^\dagger$ and $A^\dagger A$ represent the projections onto AH and the orthogonal complement $(\ker A)^\perp$ of $\ker A$ respectively [1]. For two projections P and Q, write $P^\perp$ and $P \vee Q$ for the projection onto $(PH)^\perp$ and for that onto the closed linear span of PH and QH, respectively. Now, for our later discussion we state three lemmas on operators with closed range.

LEMMA 1.1 (e.g. [1, Section 8]). *Let* $A(\neq 0)\in B(H)$. *Then* $A\in(\mathrm{CR})$ *if and only if the lower bound* $\gamma(A)$ *of* A, *defined by*

$$\inf\{\|Ax\|;\ x\in(\ker A)^{\perp},\ \|x\|=1\}$$

is positive. In this case $A^*\in(\mathrm{CR})$, $|A|:=(A^*A)^{1/2}\in(\mathrm{CR})$ *and*

$$\|A^{\dagger}\|=\|(A^*)^{\dagger}\|=\|\,|A|^{\dagger}\|=\gamma(A)^{-1}\,. \tag{1.1}$$

LEMMA 1.2 ([4, Proposition 2.2 and Corollary 3.8]). *Let* $A, B\in(\mathrm{CR})$. *Then* $AB\in(\mathrm{CR})$ *if and only if* $A^{\dagger}ABB^{\dagger}\in(\mathrm{CR})$. *In this case*

$$\|(AB)^{\dagger}\|\leqq\|A^{\dagger}\|\,\|B^{\dagger}\|\,\|(A^{\dagger}ABB^{\dagger})^{\dagger}\|\,. \tag{1.2}$$

LEMMA 1.3 ([4, Section 2]). *Let* P *and* Q *be projections. Then the following conditions are equivalent.*

(1) $PQ\in(\mathrm{CR})$.
(2) $\|P^{\perp}Q(P\vee Q^{\perp})\|(=\|PQ^{\perp}(P^{\perp}\vee Q)\|)<1$.
(3) $P^{\perp}+Q\in(\mathrm{CR})$.
(4) $P^{\perp}H+QH$ *is closed.*

If $PQ\in(\mathrm{CR})$, *i.e., if one of* (1)-(4) *holds, then*

$$\|(PQ)^{\dagger}\|\leqq\|(P^{\perp}+Q)^{\dagger}\|\leqq(1-\|P^{\perp}Q(P\vee Q^{\perp})\|)^{-2}\,.$$

2. Convergence of generalized inverses. We begin by discussing perturbations of generalized inverses. First we remark that if $A, B\in(\mathrm{CR})$ then

$$B^{\dagger}-A^{\dagger}=B^{\dagger}(BB^{\dagger}-AA^{\dagger})+(B^{\dagger}B-A^{\dagger}A)A^{\dagger}-B^{\dagger}(B-A)A^{\dagger}\,. \tag{2.1}$$

Concerning the uniform perturbation, we know [10, Theorem 3.3] that

$$\|B^{\dagger}-A^{\dagger}\|\leqq 3\max\{\|B^{\dagger}\|^2,\|A^{\dagger}\|^2\}\|B-A\|\quad\text{for}\quad A, B\in(\mathrm{CR})\,. \tag{2.2}$$

However, for our discussions on the strong convergence we need:

LEMMA 2.1. *Let* $A, B\in(\mathrm{CR})$ *and let* $x\in H$. *Then*

$$\|(BB^{\dagger}-AA^{\dagger})x\|^2\leqq\|B^{\dagger}\|^2\|(B^*-A^*)(1-AA^{\dagger})x\|^2+\|(B-A)A^{\dagger}x\|^2\,. \tag{2.3}$$

PROOF. Put $P_A=AA^{\dagger}$ and $P_B=BB^{\dagger}(=B^{\dagger *}B^*)$. Then, we see

$$\|P_B(1-P_A)x\|\leqq\|B^{\dagger}\|\,\|B^*(1-P_A)x\|=\|B^{\dagger}\|\,\|(B^*-A^*)(1-P_A)x\|$$

and

$$\|(1-P_B)P_Ax\|^2\leqq\|(1-P_B)P_Ax\|^2+\|B(B^{\dagger}-A^{\dagger})P_Ax\|^2=\|(1-BA^{\dagger})P_Ax\|^2$$
$$=\|(B-A)A^{\dagger}x\|^2\,.$$

Hence, using the identity $P_B-P_A=P_B(1-P_A)-(1-P_B)P_A$, we have

$$\begin{aligned}\|(P_B - P_A)x\|^2 &= \|P_B(1 - P_A)x\|^2 + \|(1 - P_B)P_Ax\|^2 \\ &\leqq \|B^\dagger\|^2 \|(B^* - A^*)(1 - P_A)x\|^2 + \|(B - A)A^\dagger x\|^2 . \quad \text{q.e.d.}\end{aligned}$$

COROLLARY 2.2 ([6, Theorem 1]). *Let* $A, B \in (\mathrm{CR})$. *Then*

$$\|BB^\dagger - AA^\dagger\| \leqq \max\{\|B^\dagger\|, \|A^\dagger\|\}\|B - A\| .$$

PROOF. For $x \in H$ with $\|x\| = 1$, we have

$$\|(B^* - A^*)(1 - P_A)x\| \leqq \|B - A\| \|(1 - P_A)x\|$$

and

$$\|(B - A)A^\dagger x\| = \|(B - A)A^\dagger P_Ax\| \leqq \|B - A\| \|A^\dagger\| \|P_Ax\| .$$

Hence, by (2.3) and the identity $\|P_Ax\|^2 + \|(1 - P_A)x\|^2 = 1$, we can easily get the desired inequality. q.e.d.

Let A_n $(n = 1, 2, \cdots)$ and A be operators in $B(H)$. If the sequence $\{A_n\}$ converges to A uniformly (resp. strongly), then we write $A_n \to A$ (un) (resp. $A_n \to A$ (st)). On the uniform convergence of generalized inverses, we see the following by (2.2):

LEMMA 2.3 ([5, Proposition 2.3]). *Let* $\{A_n\}$ *be a sequence with* $A_n \in (\mathrm{CR})$ *for* $n \geqq 1$, *and let* $A_n \to A \in (\mathrm{CR})$ (un). *Then* $A_n^\dagger \to A^\dagger$ (un) *if and only if* $\sup_n \|A_n^\dagger\| < \infty$.

A similar fact holds for the strong convergence of generalized inverses:

LEMMA 2.4. *Let* $\{A_n\}$ *be a sequence with* $A_n \in (\mathrm{CR})$ *for* $n \geqq 1$, *and let* $A_n \to A \in (\mathrm{CR})$ (*st), *i.e.*, $A_n \to A$ (st) *and* $A_n^* \to A^*$ (st). *Then* $A_n^\dagger \to A^\dagger$ (*st) *if and only if* $\sup_n \|A_n^\dagger\| < \infty$.

PROOF. The "only if" part is obtained from the uniform boundedness theorem. To see the "if" part, put first $B = A_n$ in (2.1) and (2.3). Then we have (for $x \in H$)

$$\text{(2.5)} \quad \begin{aligned}\|(A_n^\dagger - A^\dagger)x\| \leqq \|A_n^\dagger\| \|(A_nA_n^\dagger - AA^\dagger)x\| + \|(A_n^\dagger A_n - A^\dagger A)A^\dagger x\| \\ + \|A_n^\dagger\| \|(A_n - A)A^\dagger x\|\end{aligned}$$

and

$$\text{(2.6)} \quad \|(A_nA_n^\dagger - AA^\dagger)x\|^2 \leqq \|A_n^\dagger\|^2 \|(A_n^* - A^*)(1 - AA^\dagger)x\|^2 + \|(A_n - A)A^\dagger x\|^2 .$$

Next, replacing, in (2.6), A_n and A by their adjoints A_n^* and A^* respectively (cf. $B^{*\dagger} = B^{\dagger *}$ for $B \in (\mathrm{CR})$), we have

$$\text{(2.7)} \quad \|(A_n^\dagger A_n - A^\dagger A)x\|^2 \leqq \|A_n^\dagger\|^2 \|(A_n - A)(1 - A^\dagger A)x\|^2 + \|(A_n^* - A^*)A^{*\dagger}x\|^2 .$$

Hence, since $\{\|A_n^\dagger\|\}$ is bounded, we conclude $A_n^\dagger x \to A^\dagger x$ from the above inequalities (2.5)-(2.7). Taking the adjoints of A_n and A, we can also obtain $A_n^{\dagger *}x \to A^{\dagger *}x$. q.e.d.

REMARK. In Lemma 2.3 we can replace the sequence $\{A_n\}$ by a net $\{A_\alpha\}$ (directed by a set). Similarly, in Lemma 2.4 we can replace $\{A_n\}$ by a net $\{A_\alpha\}$ with $\sup_\alpha \|A_\alpha\| < \infty$.

PROPOSITION 2.5. *Let $A \in (\mathrm{CR})$ and let $\{P_\alpha\}$ be a net of projections such that $P_\alpha \to P$* (un) (resp. (st)). *Suppose, furthermore, that $AP_\alpha \in (\mathrm{CR})$ for all α and $AP \in (\mathrm{CR})$. Then $(AP_\alpha)^\dagger \to (AP)^\dagger$* (un) (resp. (st)) *if and only if $\sup_\alpha \|(AP_\alpha)^\dagger\| < \infty$.*

PROOF. The equivalence on the uniform convergence is immediate from (2.2) (or the above remark). For the strong convergence, by the above remark, it suffices to note that $AP_\alpha \to AP$ (*st) and $\|AP_\alpha\| \leqq \|A\|$ when $P_\alpha \to P$ (st). q.e.d.

COROLLARY 2.6 ([8, Corollary 1 to Proposition 1]). *Let $A \in B(H)$, and let $\{M_\alpha\}$ be a net in $C(H)$ converging to $M \in C(H)$ uniformly* (resp. *strongly*). *If A is bounded below on $M_0 \in C(H)$ (i.e., there exists $\varepsilon > 0$ such that $\|Ax\| \geqq \varepsilon \|x\|$ for every $x \in M_0$), and if $M_\alpha \subset M_0$ for all α, then AM_α, $AM \in (CH)$ and $\{AM_\alpha\}$ converges to AM uniformly* (resp. *strongly*).

PROOF. Write $P_\alpha = \operatorname{proj} M_\alpha$, $P_0 = \operatorname{proj} M_0$ and $P = P_M$ $(=\operatorname{proj} M)$. Then, by our assumption we have $P_\alpha \to P$ (un) (resp. (st)), $P_\alpha \leqq P_0$ and $\|AP_0x\| \geqq \varepsilon \|P_0x\|$ for $x \in H$. From the last inequality we see that $B := AP_0 \in (\mathrm{CR})$ and $B^\dagger B = P_0$. Since $AP_\alpha = AP_0P_\alpha = BP_\alpha$ and $B^\dagger BP_\alpha P_\alpha^\dagger = P_\alpha \in (\mathrm{CR})$ (cf. $P_\alpha^\dagger = P_\alpha$), we see, by Lemma 1.2, that $BP_\alpha \in (\mathrm{CR})$ or $AP_\alpha \in (\mathrm{CR})$ and

$$\|(AP_\alpha)^\dagger\| \leqq \|B^\dagger\| \, \|(B^\dagger BP_\alpha)^\dagger\| \leqq \|B^\dagger\| \, .$$

Hence, by Proposition 2.5 we have $(AP_\alpha)^\dagger \to (AP)^\dagger$ or $(AP_\alpha)(AP_\alpha)^\dagger \to (AP)(AP)^\dagger$ (un) (resp. (st)), which is the desired. q.e.d.

3. Local continuity of subspace maps. Let $A \in (\mathrm{CR})$ and $Q = A^\dagger A$. Then, for a projection P in $B(H)$ we have $A^\dagger A(Q^\perp \vee P) = Q(Q^\perp \vee P) \in (\mathrm{CR})$, so that $A(Q^\perp \vee P) \in (\mathrm{CR})$ (say, by Lemma 1.2). Using this fact, we have the following:

LEMMA 3.1. *Let $A \in (\mathrm{CR})$ and $Q = A^\dagger A$. Then for $M \in C(H)$ we have $(AM)^- = A(Q^\perp \vee P_M)H$, or equivalently,*

$$\operatorname{proj} \phi_A(M) = \{A(Q^\perp \vee P_M)\}\{A(Q^\perp \vee P_M)\}^\dagger = A\{A(Q^\perp \vee P_K)\}^\dagger \, . \tag{3.1}$$

PROOF. Since $(AM)^- = (AP_MH)^- \subset \{A(Q^\perp \vee P_M)H\}^- = A(Q^\perp \vee P_M)H \subset$

$(AM)^{-}$, we have the first identity. The identities (3.1) is now clear. q.e.d.

To discuss the local continuity of a subspace map ϕ_A $(A \in (\mathrm{CR}))$, it is convenient to introduce the auxiliary functions ψ_A and η_Q $(Q = A^{\dagger}A)$ from $C(H)$ into $B(H)$, defined by

$$\psi_A(M) = \{A(Q^{\perp} \vee P_M)\}^{\dagger} \quad \text{and} \quad \eta_Q(M) = Q^{\perp} \vee P_M \,.$$

THEOREM 3.2. *Let* $A \in (\mathrm{CR})$, $Q = A^{\dagger}A$ *and* $M_0 \in C(H)$. *Then the following conditions are equivalent.*

(1) ϕ_A *is uniformly* (resp. *strongly*) *continuous at* M_0.

(2) ϕ_Q *is uniformly* (resp. *strongly*) *continuous at* M_0.

(3) ψ_A *is uniformly* (resp. *strongly*) *continuous at* M_0.

(4) η_Q *is uniformly* (resp. *strongly*) *continuous at* M_0.

PROOF. (Since the argument is quite parallel for the strong topology, we only give the proof for the uniform topology.)

(1) $\Leftrightarrow$ (3) By Lemma 3.1 we see $\mathrm{proj}\,\phi_A(M) = A\psi_A(M)$ and $\psi_A(M) = Q\psi_A(M) = A^{\dagger}\cdot\mathrm{proj}\,\phi_A(M)$. Those identities show the desired equivalence.

(2) $\Leftrightarrow$ (4) It suffices to note that $Q^{\perp} \vee P = Q(Q^{\perp} \vee P) + Q^{\perp} = \mathrm{proj}\,\phi_Q(PH) + Q^{\perp}$ for every projection P.

(2) $\Rightarrow$ (3) Let $\{M_\alpha\}$ be a net in $C(H)$ converging to $M_0 \in C(H)$ uniformly. Write $R_\alpha = Q(Q^{\perp} \vee P_\alpha)$ and $R_0 = Q(Q^{\perp} \vee P_0)$, where $P_\alpha = \mathrm{proj}\,M_\alpha$ and $P_0 = \mathrm{proj}\,M_0$. Then, since $\|(AR_\alpha)^{\dagger}\| \leqq \|A^{\dagger}\|$ (say, by (1.2)), we have $(AR_\alpha)^{\dagger} \to (AR_0)^{\dagger}$ (un) if $R_\alpha \to R_0$ (un) by Proposition 2.5. Hence the assumption (2) implies (3).

(3) $\Rightarrow$ (2) Note $\|AR_\alpha\| \leqq \|A\|$. Hence we have, by Remark after Lemma 2.4, that $AR_\alpha = (AR_\alpha)^{\dagger\dagger} \to (AR_0)^{\dagger\dagger} = AR_0$ (un) if $(AR_\alpha)^{\dagger} \to (AR_0)^{\dagger}$ (un). Hence, if we assume (3) we have $R_\alpha = A^{\dagger}\cdot AR_\alpha \to A^{\dagger}\cdot AR_0 = R_0$ (un), which implies (2). q.e.d.

REMARK. Define $\liminf_\alpha M_\alpha = \{x; \mathrm{dist}(x, M_\alpha) \to 0\}$ for a net $\{M_\alpha\}$ in $C(H)$. Suppose $M_\alpha \to M \in C(H)$ strongly. Then we can prove

$$\liminf_{\alpha} \phi_A(M_\alpha) \supset \phi_A(M)$$

(without the restriction $A \in (\mathrm{CR})$). This relation says that ϕ_A is lower semicontinuous at M with respect to the strong topology.

To seek more precise conditions for the local continuity of subspace maps, we provide the following result.

LEMMA 3.3. *Let P and Q be projections satisfying the three conditions*;

(1) $\|PQ^{\perp}\| = 1$,

(2) $P^{\perp}H + QH \neq H$,

(3) $P^{\perp} \wedge Q \neq 0$, *i.e.*, $P^{\perp}H \cap QH \neq \{0\}$.

Then, ϕ_Q is not uniformly (strongly) continuous at PH.

PROOF. By (1) there exists a sequence $\{x_n\}$ in H such that $\|x_n\| = 1$ and $\|PQ^{\perp}x_n\| \to 1$. We easily see that $Px_n - x_n \to 0$ and $Q^{\perp}x_n - x_n \to 0$. Since $P^{\perp}H + QH$ is nowhere dense in H by (2), we may assume that for all n, $x_n \notin P^{\perp}H + QH$, or equivalently, $Px_n \notin PQH$. Put

$$y_n = Px_n/\|Px_n\|\ , \qquad z_n = Q^{\perp}x_n/\|Q^{\perp}x_n\|$$

and choose $w \in P^{\perp}H \cap QH$ with $\|w\| = 1$. By using those elements we define

$$U_n = y_n \otimes y_n\ , \qquad R_n = (a_n z_n + b_n w) \otimes (a_n z_n + b_n w)\ ,$$

where $a_n = \cos(1/n)$, $b_n = \sin(1/n)$ and $y \otimes y$ $(y \in H)$ is an operator such that $(y \otimes y)x = (x, y)y$ for $x \in H$. Clearly, they are projections and $U_n - R_n \to 0$ (un). For each n, the operator $V_n := P - U_n$ $(=P(1 - U_n))$ is also a projection and $\|V_nR_n\| = \|P(1-U_n)R_n\| \leqq \|R_n - U_n\| \to 0$. Hence, we may assume $\|V_nR_n(V_n^{\perp} \vee R_n^{\perp})\| < 1$ for all n. By Lemma 1.3 we then have $S_n := V_n + R_n \in (\mathrm{CR})$ and

$$\|S_n^{\dagger}\| \leqq (1 - \|V_nR_n(V_n^{\perp} \vee R_n^{\perp})\|)^{-2} \leqq (1 - \|V_nR_n\|)^{-2} \quad (\to 1)\ .$$

This says that $\{\|S_n^{\dagger}\|\}$ is bounded. Hence, since $S_n \to P$ (un), we see $S_nS_n^{\dagger} \to P$ (un) by Lemma 2.3. Put $P_n = S_nS_n^{\dagger}$. Now, what we want to show is thst $\phi_Q(P_nH)$ does not converge to $\phi_Q(PH)$ uniformly. Since w is orthogonal to $\phi_Q(PH)$, it suffices to show

$$\phi_Q(P_nH) = \phi_Q(PH) + [w]\ , \tag{3.2}$$

where $[w]$ is the linear space generated by w. To this end, let $u \in \ker S_nQ$ or $S_nQu = 0$. Then we have

$$PQu - (Qu, y_n)y_n + (Qu, a_nz_n + b_nw)(a_nz_n + b_nw) = 0\ .$$

Since $z_n, y_n \in PH$ and $w \in P^{\perp}H$, we see $(Qu, a_nz_n + b_nw) = 0$, so that $PQu = (Qu, y_n)y_n$. Recall $y_n \notin PQH$. Hence $PQu = 0$, i.e., $u \in \ker PQ$. This implies

$$(QPH)^{-} \subset (QS_nH)^{-} \quad (=(QP_nH)^{-})\ . \tag{3.3}$$

Moreover, we see, by a simple computation, $QS_nw = b_n^2 w$ or

$$w \in QS_nH\ . \tag{3.4}$$

Hence we have

$$(QS_nH)^- \subset \{Q(V_n + R_n)H\}^- \subset \{QP(1 - U_n)H\}^- + (QR_nH)^- \subset (QPH)^- + [w] \subset (QS_nH)^- ,$$

which implies (3.2). For the strong continuity, note that the convergence of $\{S_n\}$ (and hence $\{P_n\}$) is strong by the construction of S_n, so that the identity (3.2) also shows the discontinuity of ϕ_Q at PH. q.e.d.

COROLLARY 3.4. *Let P and Q be projections with $P \wedge Q^\perp \neq 0$ and $P^\perp \wedge Q \neq 0$. Then ϕ_Q is not uniformly (strongly) continuous at PH.*

PROOF. We have $\|PQ^\perp x\| = \|x\|$ for $x \in (P \wedge Q^\perp)H$, i.e., $\|PQ^\perp\| = 1$. We also have $P^\perp H + QH \subset (P \wedge Q^\perp)^\perp H \neq H$. q.e.d.

COROLLARY 3.5. *Let P and Q be projections with $PQ \notin$ (CR) and $P^\perp \wedge Q \neq 0$. Then ϕ_Q is not uniformly (strongly) continuous at PH.*

PROOF. By Lemma 1.3 we see that $P^\perp H + QH$ is not closed, so that we have (2) of Lemma 3.3. Again, by Lemma 1.3 we have $1 \geqq \|PQ^\perp\| \geqq \|PQ^\perp(P^\perp \vee Q)\| = 1$, which implies (1) of Lemma 3.3. q.e.d.

For the subspace map of a general operator we have:

PROPOSITION 3.6. *Let $A \in B(H)$ and $Q = \text{proj}\,(A^*H)^-$.* If *we add*

(4) $$A \in (\text{CR}) \quad or$$

(4′) $$(P^\perp \wedge Q)A^*A = 0$$

to the conditions (1)-(3) *in Lemma* 3.3, *then ϕ_A is not uniformly (strongly) continuous at PH.*

PROOF. We use the same notations as in Lemma 3.3. By (3.3), (3.4) and the obvious identity $AQ = A$, we have $(APH)^- \subset (AP_nH)^-$ and $Aw \in AP_nH$. Hence we have

$$(AP_nH)^- = (APH)^- + [Aw] .$$

Now, to see the discontinuity of ϕ_A at PH, it suffices to show that $Aw \notin (APH)^-$. First, (4) implies this relation. For otherwise $Aw \in (APH)^- = A(Q^\perp \vee P)H$, so that $w = A^\dagger Aw \in Q(Q^\perp \vee P)H \subset (P^\perp \wedge Q)^\perp H$. This is a contradiction. Next, (4′) implies that Aw is orthogonal to $(APH)^-$, because $(Aw, APu) = (w, (P^\perp \wedge Q)A^*APu) = 0$ for $u \in H$. q.e.d.

With a norm inequality we give an equivalent condition for the uniform continuity of a subspace map at a point.

THEOREM 3.7. *Let $A \in$ (CR) and $M \in C(H)$. Write $Q = A^\dagger A$ and $P = P_M$. Then the condition*

(3.5) $$\min\{\|PQ^\perp\|, \|P^\perp Q\|\} < 1$$

implies that ϕ_A is uniformly continuous at M. Conversely, if we assume $AP \in$ (CR) then the uniform continuity of ϕ_A at M implies (3.5).

PROOF. Assume $\|PQ^{\perp}\| < 1$, and let $P_n := \mathrm{proj}\, M_n \to P$ (un) ($M_n \in C(H)$). Then, since $\|P_n Q^{\perp}(P_n^{\perp} \vee Q)\| \leqq \|P_n Q^{\perp}\| \to \|PQ^{\perp}\|$, we have $P_n Q \in$ (CR) for all sufficiently large n, by Lemma 1.3. Furthermore, we have

$$\|(P_n Q)^{\dagger}\| \leqq (1 - \|P_n Q^{\perp}(P_n^{\perp} \vee Q)\|)^{-2} \leqq (1 - \|P_n Q^{\perp}\|)^{-2} \to (1 - \|PQ^{\perp}\|)^{-2} .$$

Hence $\{\|(QP_n)^{\dagger}\|\}$ is bounded, so that $(QP_n)^{\dagger} \to (QP)$ or $(QP_n)(QP_n)^{\dagger} \to (OP)(QP)^{\dagger}$ (un). This implies the uniform continuity of ϕ_Q and hence of ϕ_A at M (say, by Theorem 3.2). Using the identity $\|P_n^{\perp} Q(P_n \vee Q^{\perp})\| = \|P_n Q^{\perp}(P_n^{\perp} \vee Q)\|$, we could obtain the same conclusion when we begin with the assumption $\|P^{\perp} Q\| < 1$ instead of $\|PQ^{\perp}\| < 1$. To see the latter half of the theorem, let ϕ_A (and hence ϕ_Q) be uniformly continuous at M. Then, by Corollary 3.4 we see that $P^{\perp} \wedge Q = 0$ or $P \wedge Q^{\perp} = 0$. If $P^{\perp} \wedge Q = 0$, then under the assumption $AP \in$ (CR) or equivalently $QP \in$ (CR)) we have $\|QP^{\perp}\| = \|QP^{\perp}(Q^{\perp} \vee P)\| < 1$ by Lemma 1.3. We can see $\|PQ^{\perp}\| < 1$ similarly, when $P \wedge Q^{\perp} = 0$. q.e.d.

The next result was shown by Longstaff [8, Theorem 1] without the assumption $A \in$ (CR).

COROLLARY 3.8. *Let A $(\neq 0) \in$ (CR). Then ϕ_A is uniformly continuous on $C(H)$, i.e., at every point $M \in C(H)$ if and only if A is left-invertible.*

PROOF. If A is not left-invertible, then $Q := A^{\dagger} A \neq 1$. Hence, putting $P = Q^{\perp}$, we see that the left hand side of (3.5) is equal to 1. The converse assertion is clear by (3.5). q.e.d.

4. **Lipschitz constants of subspace maps.** For $A \in$ (CR), define

$$C_A(H) = \{M \in C(H);\ P_M \text{ commutes with } A^{\dagger} A\} . \tag{4.1}$$

Then, since $A^{\dagger} A P_M (M \in C_A(H))$ is a projection we easily see that $AP_M \in$ (CR) (say, by Lemma 1.2) or $AM = (AM)^{-}$. If we restrict the map ϕ_A on $C_A(H)$, then since $\|(AP_M)^{\dagger}\| \leqq \|A^{\dagger}\|$ for $M \in C_A(H)$ (say, by (1.2)) we see by Corollary 2.2 that

$$\begin{aligned} \|\mathrm{proj}\, \phi_A(M) - \mathrm{proj}\, \phi_A(N)\| &= \|(AP_M)(AP_M)^{\dagger} - (AP_N)(AP_N)^{\dagger}\| \\ &\leqq \|A^{\dagger}\| \, \|A\| \, \|P_M - P_N\| . \qquad (M, N \in C_A(H)) \end{aligned}$$

In [2] we introduced the Lipschitz constant of ϕ_A by

$$\kappa_A = \sup\{\|\mathrm{proj}\, \phi_A(M) - \mathrm{proj}\, \phi_A(N)\| / \|P_M - P_N\|;\ M, N \in C_A(H),\ M \neq N\} ,$$

and proved that $\kappa_A = \|A\|/\gamma(A)$ when A is left-invertible [2, Theorem 3] (cf. [3, Theorem 3.1]). The following result shows that this identity is still true for every $A \in (\mathrm{CR})$.

PROPOSITION 4.1. *If* $A \in (\mathrm{CR})$, *then* $\kappa_A = \|A\|\,\|A^\dagger\|$.

PROOF. Let $A = V|A|$ be the polar decomposition of A with a partial isometry V which satisfies $V^*V = A^\dagger A$. Then, since $|A|^\dagger|A| = V^*V$, we see that $|A|P_L \in (\mathrm{CR})$ for any $L \in C_A(H)$ and

$$(3.6)\qquad (|A|P_L)^\dagger = (|A|P_L)^\dagger V^*V\,.$$

Hence, $AL = V|A|P_LH = V(|A|P_L)(|A|P_L)^\dagger H = V(|A|P_L)(|A|P_L)^\dagger V^*H$, or

$$\operatorname{proj} AL = V(|A|P_L)(|A|P_L)^\dagger V^*\,.$$

Hence, using the identity $|A| = V^*V|A|$ and (3.6), we have, for M, $N \in C_A(H)$,

$$\begin{aligned}\|\operatorname{proj} AM - \operatorname{proj} AN\| &= \|V\{(|A|P_M)(|A|P_M)^\dagger - (|A|P_N)(|A|P_N)^\dagger\}V^*\|\\ &= \|(|A|P_M)(|A|P_M)^\dagger - (|A|P_N)(|A|P_N)^\dagger\|\,.\end{aligned}$$

Clearly, this shows $\kappa_A = \kappa_{|A|}$. On the other hand, from the first paragraph of this section we easily see that $\kappa_A \leqq \|A\|\,\|A^\dagger\|$. Hence it suffices to show that the supremum κ_A attains $\|A\|\,\|A^\dagger\|$. Now, let $|A| = B \oplus 0$ be the direct sum representation of $|A|$ with respect to the orthogonal decomposition $(\ker A)^\perp \oplus \ker A$ of H. Then B is a non-negative invertible operator on $K := (\ker A)^\perp$. Since $A^\dagger A$ has the representation $1 \oplus 0$, we see that every operator $E \oplus 0$ with a projection E on K is in $C_A(H)$. Hence our problem is reduced to computing κ_B $(\leqq \kappa_A)$ on $C_B(K)$. But then $\|B\| = \|A\|$, and $\gamma(B)^{-1} = \|B^{-1}\| = \|\,|A|^\dagger\| = \|A^\dagger\|$ (say, by Lemma 1.1), so that we obtain $\kappa_B = \|B\|/\gamma(B) = \|A\|\,\|A^\dagger\|$.

q.e.d.

5. Transforms of families of closed linear subspaces. In this section we shall discuss some behavior of a subspace map ϕ_A $(A \in (\mathrm{CR}))$ on the set $C_A(H)$ defined by (4.1). The following result extends [8, Theorem 2].

THEOREM 5.1. *Let* $A \in (\mathrm{CR})$. *If* $\mathscr{F}$ *is a uniformly* (resp. *strongly, weakly*) *closed subset of* $C_A(H)$ *and* $P_M \leqq A^\dagger A$ (*i.e.*, $M \subset (\ker A)^\perp$) *for all* $M \in \mathscr{F}$, *then the image* $\phi_A(\mathscr{F})$ *is also uniformly* (resp. *strongly, weakly*) *closed.*

PROOF. Let $\{M_\alpha\}$ be a net in $\mathscr{F}$ and $AM_\alpha \to N \in C_A(H)$ uniformly (resp. strongly). ($C_A(H)$ is uniformly and strongly closed.) Write $P_\alpha =$

proj M_α. Then $(AP_\alpha)(AP_\alpha)^\dagger \to P_N$ (un) (resp. (st)). Hence, noting $A^\dagger AP_\alpha = P_\alpha$, we have $(AP_\alpha)^\dagger = A^\dagger\cdot(AP_\alpha)(AP_\alpha)^\dagger \to A^\dagger P_N$ (un) (resp. (st)). Since $\|AP_\alpha\| \leqq \|A\|$, we see, by Remark after Lemma 2.4, that

$$AP_\alpha = (AP_\alpha)^{\dagger\dagger} \to (A^\dagger P_N)^\dagger \text{ (un) (resp. (st))} .$$

Hence, $P_\alpha \to A^\dagger(A^\dagger P_N)^\dagger$ (un) (resp. (st)), so that $M{:} = A^\dagger(A^\dagger P_N)^\dagger H \in \mathscr{F}$. Hence, by the uniform (resp. strong) continuity of ϕ_A (say, directly by Proposition 2.5), we obtain that $N = AM \in \phi_A(\mathscr{F})$, which implies the uniform (resp. strong) closedness of $\phi_A(\mathscr{F})$. The weak closedness of $\phi_A(\mathscr{F})$ can be now obtained by (argument similar to that in [8]) using the weak compactness of any ball $\{T \in B(H){:}\ \|T\| \leqq C\}$ for $C > 0$. q.e.d.

If $\mathscr{A}$ is a subset of $B(H)$, then we write Lat $\mathscr{A}$ for the lattice of all $M \in C(H)$ invariant under every member of $\mathscr{A}$. For a subset $\mathscr{F}$ of $C(H)$ we denote by Alg $\mathscr{F}$ the algebra of all $T \in B(H)$ leaving every member of $\mathscr{F}$ invariant. We say that $\mathscr{F} \subset C(H)$ is reflexive if $\mathscr{F} =$ Lat Alg $\mathscr{F}$. Now, we give an extension of [8, Proposition 2].

PROPOSITION 5.2. *Let $A \in$ (CR), and let $\mathscr{F}$ be a subset of $C_A(H)$ with $A^\dagger AH \in \mathscr{F}$. Then $\phi_A(\text{Lat Alg}\,\mathscr{F}) \cup \{H\} = \text{Lat Alg}\,\phi_A(\mathscr{F})$. Hence, if $\mathscr{F}$ is reflexive then so is $\phi_A(\mathscr{F}) \cup \{H\}$.*

PROOF. Write $\mathscr{G} = \phi_A(\mathscr{F})$. First, in order to show $\phi_A(\text{Lat Alg}\,\mathscr{F}) \subset \text{Lat Alg}\,\mathscr{G}$, let $M = \text{Lat Alg}\,\mathscr{F}$. Then, for $T \in \text{Alg}\,\mathscr{G}$, we see $TAH \subset AH$, so that

$$AA^\dagger TA = TA . \tag{5.1}$$

Put $X = A^\dagger TA$. Then, for every $F \in \mathscr{F}$

$$XF = A^\dagger TAF = A^\dagger\cdot TAF \subset A^\dagger AF .$$

Hence, since P_F commutes with $A^\dagger A$, we have $XF \subset F$, which implies $X \in \text{Alg}\,\mathscr{F}$. Hence $XM \subset M$, or $A^\dagger TAM \subset M$. By (5.1) this relation yields

$$TAM = AA^\dagger TAM \subset AM .$$

Since $T \in \text{Alg}\,\mathscr{G}$ is arbitrary, this implies $AM \in \text{Lat Alg}\,\mathscr{G}$, which is the desired. Next, to show the opposite inclusion $\text{Lat Alg}\,\mathscr{G} \subset \phi_A(\text{Lat Alg}\,\mathscr{F}) \cup \{H\}$, let $N \in \text{Lat Alg}\,\mathscr{G}$ and $N \neq H$. Then $Y(1 - AA^\dagger) \in \text{Alg}\,\mathscr{G}$ for every $Y \in B(H)$. Hence $Y(1 - AA^\dagger)N \subset N$. Since Y is arbitrary and $N \neq H$, we easily see that $(1 - AA^\dagger)N = \{0\}$, or $N = AA^\dagger N$. Now, it suffices to show that $A^\dagger N \in \text{Lat Alg}\,\mathscr{F}$. For, if this is shown then $N = AA^\dagger N \in \phi_A(\text{Lat Alg}\,\mathscr{F})$ (which is the desired). Let $S \in \text{Alg}\,\mathscr{F}$, and put $R = ASA^\dagger$. Then, for any $G{:} = AF \in \mathscr{G}$ $(F \in \mathscr{F})$, we have

$$RG = ASA^{\dagger}G = ASA^{\dagger}AF \subset ASF \subset AF = G\ ,$$

that is, $R \in \mathrm{Alg}\,\mathscr{G}$. Hence we see $RN \subset N$, or $ASA^{\dagger}N \subset N$. Since the assumption $A^{\dagger}AH \in \mathscr{F}$ means $SA^{\dagger}A = A^{\dagger}ASA^{\dagger}A$, we have

$$SA^{\dagger}N = SA^{\dagger}A \cdot A^{\dagger}N = A^{\dagger}ASA^{\dagger}A \cdot A^{\dagger}N = A^{\dagger} \cdot ASA^{\dagger}N \subset A^{\dagger}N\ .$$

This implies $A^{\dagger}N \in \mathrm{Lat\,Alg}\,\mathscr{F}$, because $S \in \mathrm{Alg}\,\mathscr{F}$ is arbitrary. Finally, if $\mathscr{F}$ is reflexive, then

$$\mathscr{G} \cup \{H\} = \phi_A(\mathrm{Lat\,Alg}\,\mathscr{F}) \cup \{H\} = \mathrm{Lat\,Alg}\,\mathscr{G} \cup \{H\} = \mathrm{Lat\,Alg}\,(\mathscr{G} \cup \{H\})\ ,$$

so that $\mathscr{G} \cup \{H\}$ is reflexive. q.e.d.

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