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THE LIMIT SET OF DEFORMATIONS OF SOME FUCHSIAN GROUPS

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1. Preliminaries. Let G and Λ be a non-elementary finitely generated Fuchsian group of the second kind acting on the upper half complex plane H and its limit set, respectively. The limit set Λ of G lies on the extended real axis $\hat{R}$ which is the boundary of H. We say that G has type $(g; n; m)$ if we obtain $S = H/G$ from a compact surface of genus g by removing $n\ (\geqq 0)$ points and $m\ (\geqq 0)$ conformal discs. Put $m_t(\delta, \Lambda) = \inf \sum_i \mathrm{dia}^t(I_i)$, where the infimum is taken over all coverings of Λ by sequences $\{I_i\}$ of intervals I_i on $\hat{R}$ with the spherical diameter $\mathrm{dia}(I_i)$ less than a given number $\delta > 0$. Further, put $m_t(\Lambda) = \lim_{\delta\to 0} m_t(\delta, \Lambda)$, which is called the t-dimensional Hausdorff measure of Λ. We call $d(\Lambda) = \inf\{t > 0; m_t(\Lambda) = 0\}$ the Hausdorff dimension of Λ ([1], [3]).

The first purpose of this paper is to let the Hausdorff dimension $d(\Lambda)$ increase by deformations of G without altering the type $(g; n; m)$. This was essentially done by Beardon [4] when H/G is a punctured surface and he also proved that the Hausdorff dimension of the limit set is less than 1 for any finitely generated Fuchsian group of the second kind. The second purpose is to show the existence of Fuchsian groups of type $(1; 0; 1)$ or $(0; 0; 3)$ such that the Hausdorff dimension of its limit set is equal to an arbitrary number $t \in (0, 1)$.

2. Statement of the main theorem. Let

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} \exp(\sqrt{-1}\pi/4 & 0 \\ 0 & \exp(-\sqrt{-1}\pi/4) \end{pmatrix}$$

be Möbius transformations acting on the extended complex plane and making the unit disc Δ invariant, where $a > b > 1$ and $a^2 - b^2 = 1$. Denote by c_1, c_1', c_2 and c_2' the isometric circles of the Möbius transformations A^{-1}, A, $MA^{-1}M^{-1}$ and MAM^{-1}, respectively. We see that $c_2 = M(c_1)$, $c_2' = M^{-1}(c_1)$ and $\{c_i, c_i'\}_{i=1}^2$ are mutually disjoint circles and that each of these circles is orthogonal to the unit circle. Put $D_0 = \bigcap_{i=1}^2 \{\mathrm{ext}(c_i) \cap \mathrm{ext}(c_i')\}$, where $\mathrm{ext}(c)$ denotes the exterior of the circle c. Then the Schottky group Γ generated by A and MAM^{-1} is a Fuchsian group of type (1; 0; 1) acting on the unit disc $\varDelta$ and having a fundamental domain $B = D_0 \cap \varDelta$. Let $\{D_\nu\}$ be all of the equivalents of D_0 by Γ. Then D_ν clusters to a perfect non-dence set $\Lambda(\Gamma)$ on the unit circle. We call $\Lambda(\Gamma)$ the limit set of Γ. Now, let us consider the conjugate $T^{-1}\Gamma T$ of Γ, where

$$T = (-2\sqrt{-1})^{-1/2}\begin{pmatrix} -1 & \sqrt{-1} \\ 1 & \sqrt{-1} \end{pmatrix},$$

is a Möbius transformation. Then $T^{-1}\Gamma T$ is a Fuchsian group acting on the upper half complex plane H and is generated by h and EhE^{-1}, where for $\lambda = a - b$, h and E are defined by

$$h = T^{-1}AT = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad E = T^{-1}MT = \sqrt{2}^{-1}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Since $a > b > 1$ and $a^2 - b^2 = 1$, we have $0 < \lambda < \sqrt{2} - 1$. Clearly, $T^{-1}\Gamma T$ is determined by $\lambda \in (0, \sqrt{2} - 1)$ and we put $T^{-1}\Gamma T = \Gamma_\lambda$. The domain $T^{-1}(B) = T^{-1}(D_0) \cap H$ is a fundamental domain of the Fuchsian group Γ_λ and we see easily that $T^{-1}(D_0)$ is bounded by four circles $|z| = \lambda$, $|z| = \lambda^{-1}$, $|z \pm (1 + \lambda^2)(1 - \lambda^2)^{-1}| = 2\lambda(1 - \lambda^2)^{-1}$ which are orthogonal to the real axis R and are denoted by $\bar{c}_1$, $\bar{c}_1'$, $\bar{c}_2$, $\bar{c}_2'$, respectively. Clearly

$$E(\bar{c}_1) = \bar{c}_2 \quad \text{and} \quad E^{-1}(\bar{c}_1) = \bar{c}_2' . \tag{2.1}$$

Hence, for $\lambda \in (0, \sqrt{2} - 1)$, Γ_λ is a Fuchsian group of the second kind of type (1; 0; 1) and with the limit set $\Lambda(\Gamma_\lambda) = T^{-1}(\Lambda(\Gamma))$. The first purpose of this paper is to prove the following:

THEOREM 1. *Under the above situation,* $d(\Lambda(\Gamma_\lambda))$ *tends to* 1 *as* $\lambda \in (0, \sqrt{2} - 1)$ *tends to* $\sqrt{2} - 1$.

In the following two sections § 3 and § 4, we prove some preparatory lemmas for the proof of Theorem 1, which we give in § 5. In § 6, we state an application of the theorem.

3. General Cantor sets. Let I be a closed interval on the real axis R of the complex plane. We take $k\,(\geqq 2)$ disjoint closed intervals $I(i_1)$ $(i_1 = 1, 2, \cdots, k)$ in I and k disjoint closed intervals $I(i_1 i_2)$ $(i_2 = 1, 2, \cdots, k)$ in $I(i_1)$ and proceed similarly. Then, after n steps, we obtain k^n closed intervals $I(i_1\, i_2 \cdots i_n)$ $(i_1, \cdots, i_n = 1, 2, \cdots, k)$ such that $I(i_1\, i_2 \cdots i_n\, i_{n+1}) \subset I(i_1\, i_2 \cdots i_n)$ $(i_{n+1} = 1, 2, \cdots, k)$. We put

$$C = \bigcap_{n=1}^{\infty} \bigcup_{i_1, \cdots, i_n = 1}^{k} I(i_1\, i_2 \cdots i_n) . \tag{3.1}$$

DEFINITION 1. The set C constructed above is said to be a general Cantor set if it satisfies the following conditions:

(a) There exists a constant $A \in (0, 1)$ such that

$$|I(i_1 i_2 \cdots i_n i_{n+1})| \geqq A|I(i_1 i_2 \cdots i_n)| \quad (i_{n+1} = 1, 2, \cdots, k) ,$$

where $|J|$ is the length of an interval J.

(b) There is constant $B \in (0, 1)$ such that

$$\rho(I(i_1 i_2 \cdots i_n s), I(i_1 i_2 \cdots i_n t)) \geqq B|I(i_1 i_2 \cdots i_n)| ,$$

where $s \neq t$ and $\rho(J_1, J_2) = \inf\{|x - y|; x \in J_1, y \in J_2\}$. Here a closed interval $I(i_1 i_2 \cdots i_n)$ $(i_1, i_2, \cdots, i_n = 1, 2, \cdots, k)$ is said to be a fundamental interval for the set C.

DEFINITION 2. The set $\mathscr{F} = \{I_1^*, I_2^*, \cdots, I_p^*\}$ is called a fundamental system of a given general Cantor set C, if it satisfies the following conditions:

(a) I_i^* is a fundamental interval for C $(1 \leqq i \leqq p)$.

(b) $I_i^* \cap I_j^* = \emptyset$ $(i \neq j, 1 \leqq i, j \leqq p)$.

(c) $\bigcup_{i=1}^p I_i^* \supset C$.

The following lemmas are known.

LEMMA 1 ([3], [6]). *Let C be a general Cantor set constructed as in* (3.1). *Then* $\mathrm{Max}_{1 \leqq i_1, \cdots, i_n \leqq k} |I(i_1 i_2 \cdots i_n)|$ *tends to* 0 *as n tends to* ∞.

LEMMA 2 ([1], [3]). *Let C be a general Cantor set and suppose that $M_t(C)$ is defined as for $m_t(C)$ with an additional restriction that the covering $\{I_n\}$ is a fundamental system of C. Then*

$$M_t(C) \geqq m_t(C) \geqq B^t M_t(C) .$$

LEMMA 3 ([1], [3]). *Let C be a general Cantor set constructed as in* (3.1). *If, for all $n = 1, 2, \cdots$ and all $i_1, \cdots, i_n = 1, \cdots, k$,*

$$\sum_{j=1}^{k} |I(i_1 i_2 \cdots i_n j)|^t \geqq |I(i_1 i_2 \cdots i_n)|^t ,$$

then $d(C) \geqq t$.

4. A general Cantor set associated with $\Lambda(\Gamma_\lambda)$. Now we return to the Fuchsian group Γ_λ $(\lambda \in (0, \sqrt{2} - 1))$ introduced in § 2. We construct a general Cantor set L_λ associated with the limit set $\Lambda(\Gamma_\lambda)$ of Γ_λ. Let G_1 be the set consisting of Möbius transformations

$$(4.1)\qquad \begin{aligned} &(\mathrm{i}) && \gamma_j = (hE)^j h , && (1 \leqq j \leqq N) , \\ &(\mathrm{ii}) && \gamma_{j+N} = (hE^{-1})^j h , && (1 \leqq j \leqq N) , \\ &(\mathrm{iii}) && \gamma_{j+2N} = (hE)^j hE^{-1} , && (1 \leqq j \leqq N) , \\ &(\mathrm{iv}) && \gamma_{j+3N} = (hE^{-1})^j hE , && (1 \leqq j \leqq N) , \end{aligned}$$

together with an element $\gamma_{4N+1} = h$, where N is an integer determined later. We shall refer to these elements as being of types (i), (ii), (iii) and (iv), respectively. Having defined G_1, we define G_n for all positive integers n inductively by

$$G_{n+1} = \{UV;\, U \in G_n,\, V \in G_1\}$$

and further, put

$$L_\lambda = \bigcap_{k=1}^{\infty} \bigcup_{U \in G_k} U(\bar{I}_\lambda)\,, \tag{4.2}$$

where I_λ is the open interval $(-\lambda, \lambda)$ on the real axis R and $\bar{I}_\lambda$ denotes the closure of I_λ. We see easily $h(\bar{I}_\lambda) \subset I_\lambda$, $E(\bar{I}_\lambda) \cap E^{-1}(\bar{I}_\lambda) = \varnothing$, $(E(\bar{I}_\lambda) \cup E^{-1}(\bar{I}_\lambda)) \cap I_\lambda = \varnothing$ and $hE(\bar{I}_\lambda) \cup hE^{-1}(\bar{I}_\lambda) \subset I_\lambda$. Hence, if $U \in G_n$ and if V_1 and V_2 in G_1 are distinct, then $I_\lambda \supset V_1(\bar{I}_\lambda)$ and $UV_1(\bar{I}_\lambda) \cap UV_2(\bar{I}_\lambda) = \varnothing$. Let G_1^* be the set consisting of Möbius transformations h, hE and hE^{-1}. We define G_n^* for all positive integers n inductively by $G_{n+1}^* = \{UV;\, U \in G_n^*,\, V \in G_1\}$. It is easily seen that $\{\bigcap_{n=1}^{\infty} \bigcup_{U \in G_n} U(\bar{I}_\lambda)\} \subset \{\bigcap_{n=1}^{\infty} \bigcup_{U \in G_n^*} U(\bar{I}_\lambda)\}$. Recalling that $\Gamma = T\Gamma_\lambda T^{-1}$ is a Schottky group and noting

$$T\Big(\bigcap_{n=1}^{\infty} \bigcup_{U \in G_n^*} U(\bar{I}_\lambda)\Big) = \Lambda(\Gamma) \cap T(I_\lambda)\,,$$

we see

$$L_\lambda \subset \{\Lambda(\Gamma_\lambda) \cap I_\lambda\}\,. \tag{4.3}$$

We prove the following which gives a useful estimate later.

LEMMA 4. *Let J be any sub-interval of I_λ and let $U \in \tilde{G} = \bigcup_{k=1}^{\infty} G_k$. Then*

$$|J|/2 \leqq |U(J)|\,|U(I_\lambda)|^{-1} \leqq |J|(1+\lambda)(1-\lambda)^{-1}\lambda^{-1}/2\,.$$

Furthermore, if $V \in G_1$, then

$$|UV(I_\lambda)| \leqq |U(I_\lambda)|/2\,.$$

PROOF. We can write $U \in \tilde{G}$ in the form

$$U = V_n \cdots V_2 V_1 (V_i \in G_1,\ i = 1, 2, \cdots, n)$$

for a suitable n. For intervals (λ, ∞), $(-\infty, -\lambda)$ etc. on the real axis R, we see $h^{-1}((\lambda, \infty)) \subset (\lambda^{-1}, \infty)$, $h^{-1}((-\infty, -\lambda)) \subset (-\infty, -\lambda^{-1})$, $E^{-1}((\lambda^{-1}, \infty)) \subset (\lambda, 1)$, $E((-\infty, -\lambda^{-1})) \subset (-1, -\lambda)$, $E((\lambda^{-1}, \infty)) \subset E^{-1}(I_\lambda)$ and $E^{-1}((-\infty, -\lambda^{-1})) \subset E(I_\lambda)$. Using these and putting $Q = E(I_\lambda) \cup E^{-1}(I_\lambda) \cup E^2(I_\lambda)$, we have $V^{-1}(Q) \subset Q$ for any $V \in G_1$. Moreover, if $V \in G_1$ is of type (iii), the $V^{-1} = EV_1^{-1}$ for some $V_1 \in G_1$ of type (i) and $V^{-1}((\lambda, \infty)) \subset E((\lambda^{-1}, \infty)) \subset (-(1+\lambda)(1-\lambda)^{-1}, -1)$. Noting those facts, we see that, if V_1 is identical with h or is of type (i) or (ii), then $U^{-1}(\infty) \in E^2(I_\lambda)$ and we have

$|U^{-1}(\infty)| > \lambda^{-1} > 1$. If V_1 is of type (iii), then $U^{-1}(\infty) \in V_1^{-1}(Q) \subset (-\infty, -1) \cap E^{-1}(I_\lambda)$ and thus we have $U^{-1}(\infty) < -1$. Similarly, if V_1 is of type (iv), then $U^{-1}(\infty) \in V_1^{-1}(Q) \subset (1, \infty) \cap E(I_\lambda)$ and therefore we have $U^{-1}(\infty) > 1$. Hence in all cases we have $|U^{-1}(\infty)| > 1$. We now denote by $J = (\alpha, \beta)$ the interval on the real axis and put $\zeta = U^{-1}(\infty)$. Then we see

$$|U(J)| \cdot |U(I_\lambda)|^{-1} = \left(\int_\alpha^\beta |U'(x)|\,dx\right)\left(\int_{-\lambda}^{\lambda} |U'(x)|\,dx\right)^{-1}$$
$$= (|J|/2)\lambda^{-1}(\zeta + \lambda)(\zeta - \lambda)(\zeta - \alpha)^{-1}(\zeta - \beta)^{-1}\,.$$

In the case $\zeta > 1$, we have

$$\text{(4.4)} \qquad |J|/2 \leqq (|J|/2)\lambda^{-1}(1 - \lambda)(1 + \lambda)^{-1} \leqq |U(J)|\,|U(I_\lambda)|^{-1} \leqq (|J|/2)\lambda^{-1}(1 + \lambda)(1 - \lambda)^{-1}$$

from the assumption $J \subset I_\lambda$. In the case $\zeta < -1$, a similar argument gives the same inequalities and completes the proof of the first part of our lemma.

Finally, we have $hEh(I_\lambda) = (\lambda^2(1 - \lambda^3)(1 + \lambda^3)^{-1}, \lambda^2(1 + \lambda^3)(1 - \lambda^3)^{-1})$, $h(I_\lambda) = (-\lambda^3, \lambda^3)$ and since $\lambda \in (0, \sqrt{2} - 1)$,

$$\operatorname*{Max}_{V \in G_1} |V(I_\lambda)| \leqq 2\lambda^3(1 - \lambda^2)^{-1}\,.$$

Applying (4.4) with $J = V(I_\lambda)$ we get $|UV(I_\lambda)| \leqq |U(I_\lambda)|/2$, which is the second part of our lemma. q.e.d.

By using Lemma 4, we show the following.

LEMMA 5. *The set L_λ in* (4.2) *is a general Cantor set on the real axis.*

PROOF. Let $I = \bar{I}_\lambda$ and $\gamma_j(\bar{I}_\lambda) = I(j)$ for $\gamma_j \in G_1$ $(1 \leqq j \leqq 4N + 1)$ in (4.1). We can take $k = 4N + 1$ disjoint closed intervals $I(i)$ $(i = 1, 2, \cdots, k)$ in I and k disjoint closed intervals $\gamma_i\gamma_j(\bar{I}_\lambda) = I(i, j)$ $(j = 1, 2, \cdots, k)$ in $I(i)$ for $\gamma_i\gamma_j \in G_2$. Proceeding similarly, we have inductively

$$\{UV(\bar{I}_\lambda);\ V \in G_1\} = \{I(i_1 i_2 \cdots i_n j);\ 1 \leqq j \leqq k\}$$

for $U \in G_n$. Then, applying the first inequality of Lemma 4 to $J = V(I_\lambda)$ $(V \in G_1)$ and $U \in \tilde{G}$, we have $|UV(\bar{I}_\lambda)| \geqq A|U(\bar{I}_\lambda)|$ for the constant $A = \operatorname{Min}_{V \in G_1} |V(I_\lambda)|/2$. The set $I \backslash \bigcup_{V \in G_1} V(\bar{I}_\lambda)$ consists of a finite number of open arcs J_i. If V_1, V_2 are distinct elements of G_1, then there exists a subarc J of I_λ lying between $V_1(I_\lambda)$ and $V_2(I_\lambda)$ with $|J|/2 \geqq \operatorname{Min}|J_i|/2 > 0$. As $U(J)$ lies between $UV_1(I_\lambda)$ and $UV_2(I_\lambda)$, Lemma 4 implies

$$\rho(UV_1(\bar{I}_\lambda), UV_2(\bar{I}_\lambda)) \geqq |U(J)| \geqq |J|\,|U(I_\lambda)|/2 \geqq B|U(\bar{I}_\lambda)|$$

for $B = \mathrm{Min}_i |J_i|/2$. Thus, by Definition 1 we see that L_λ is a general Cantor set. q.e.d.

Next we show the following lemma.

LEMMA 6. *Let k be any integer greater than 1 and let $a_1, a_2, \cdots, a_k$ and s be the positive numbers satisfying $0 \leqq a_j \leqq a < 1$ $(1 \leqq j \leqq k)$ and $0 \leqq s \leqq a_1 + a_2 + \cdots + a_k < 1$. Then*

$$a_1^r + a_2^r + \cdots + a_k^r \geqq 1 ,$$

where

$$r = 1 - (1-s)(1-a)^{-1} .$$

PROOF. Let $x \in (0, 1)$ be a number uniquely determined by $a_1^x + a_2^x + \cdots + a_k^x = 1$. The inequality $y^t - 1 \leqq t(y-1)$ holds for $y \geqq 0$ and $0 \leqq t \leqq 1$. Taking $y = a_j$ and $t = 1 - x$, we have $a_j^{1-x} - 1 \leqq (a_j - 1)(1-x) \leqq (a-1)(1-x)$, which shows

$$a_j \leqq \{1 - (1-x)(1-a)\} a_j^x , \quad (1 \leqq j \leqq k) .$$

Hence we have $s \leqq \sum_{j=1}^k a_j \leqq \{1 - (1-x)(1-a)\}$. q.e.d.

5. **Proof of Theorem 1.** Now we are going to prove Theorem 1. As we have seen in (4.4) and in Lemma 5, the set L_λ in (4.2) is a general Cantor set contained in $\Lambda(\Gamma_\lambda) \cap I_\lambda$. So it is sufficient to show that the Hausdorff dimension $d(L_\lambda)$ of L_λ tends to 1 as λ tends to $\sqrt{2} - 1$. Put

$$F = I_\lambda \backslash \bigcup_{V \in G_1} V(\bar{I}_\lambda) ,$$

and

$$Y = I_\lambda \backslash \{h(\bar{I}_\lambda) \cup hE(\bar{I}_\lambda) \cup hE^{-1}(\bar{I}_\lambda)\} .$$

Then $F \supset Y$ and

$$F \backslash Y = hE(\bar{I}_\lambda) \cup hE^{-1}(\bar{I}_\lambda) \backslash \bigcup_{V \in G_1} V(\bar{I}_\lambda) = \{F \cap hE(\bar{I}_\lambda)\} \cup \{F \cap hE^{-1}(\bar{I}_\lambda)\} . \tag{5.1}$$

Now we have

$$F \cap (hE)^n(\bar{I}_\lambda) \backslash (hE)^n(Y) = (hE)^n(\bar{I}_\lambda) \backslash \bigcup_{V \in G_1} V(\bar{I}_\lambda) \backslash (hE)^n(Y) = (hE)^n(\bar{I}_\lambda \backslash Y) \backslash \bigcup_{V \in G_1} V(\bar{I}_\lambda) .$$

If we denote by $\{-\lambda, \lambda\}$ the set consisting of two points $-\lambda$ and λ on the real axis, then the right hand side of the above is equal to

$$(hE)^n\{h(\bar{I}_\lambda) \cup hE(\bar{I}_\lambda) \cup hE^{-1}(\bar{I}_\lambda)\} \cup (hE)^n(\{-\lambda, \lambda\})\setminus \bigcup_{V \in G_1} V(\bar{I}_\lambda)$$
$$= (hE)^{n+1}(\bar{I}_\lambda) \cup (hE)^n(\{-\lambda, \lambda\})\setminus \bigcup_{V \in G_1} V(\bar{I}_\lambda) .$$

This, together with the inclusion $F \supset (hE)^n(Y)$, gives

$$m_1(F \cap (hE)^n(\bar{I}_\lambda)) = m_1((hE)^n(Y)) + m_1((hE)^{n+1}(\bar{I}_\lambda) \cap F) ,$$

where $m_1(S)$ denotes the Lebesgue measure on the real axis. A similar equality holds with hE replaced by hE^{-1}. Using these two equalities for $n = 1, 2, \cdots, N-1$ and (5.1), we have

$$m_1(F) = m_1(Y) + \sum_{k=1}^{N-1} m_1((hE)^k(Y)) + m_1((hE)^N(\bar{I}_\lambda) \cap F)$$
$$+ \sum_{k=1}^{N-1} m_1((hE^{-1})^k(Y)) + m_1((hE^{-1})^N(\bar{I}_\lambda) \cap F) .$$

As both Y and I_λ are symmetric with respect to the imaginary axis, we have $m_1((hE)^k(Y)) = m_1((hE^{-1})^k(Y))$ and we see that a similar equality holds with I_λ replaced by Y. Thus

$$m_1(F) \leqq m_1(Y) + 2\cdot \sum_{k=1}^{N-1} m_1((hE)^k(Y)) + 2m_1((hE)^N(\bar{I}_\lambda)) . \tag{5.2}$$

We first estimate $m_1(Y)$. Put $\varepsilon = (\lambda + \lambda^{-1})^2 - 8 > 0$. Then

$$\begin{aligned} m_1(Y) &= 2\cdot\{\lambda(1-\lambda^2) - ((1+\lambda)(1-\lambda)^{-1} - (1-\lambda)(1+\lambda)^{-1})\lambda^2\} \\ &= 2\lambda^3(1-\lambda^2)^{-1}\varepsilon < \varepsilon . \end{aligned} \tag{5.3}$$

Next we estimate $m_1((hE)^k(Y))$. Put

$$(hE)^k(z) = (a_k z + b_k)(c_k z + d_k)^{-1} , \quad a_k d_k - b_k c_k = 1 . \tag{5.4}$$

We easily see $hE(E(I_\lambda)) = \hat{R}\setminus I_\lambda \supset E(I_\lambda)$ and also have $(hE)^k(E(I_\lambda)) \supset \hat{R}\setminus I_\lambda$ inductively. Hence we can deduce that the pole of $(hE)^k(z)$ lies in $E(\bar{I}_\lambda)$. This implies that $|d_k| > \lambda|c_k|$. Therefore we have the following estimate

$$m_1((hE)^k(Y)) = \int_Y |c_k z + d_k|^{-2} dz \leqq (|d_k| - \lambda|c_k|)^{-2} m_1(Y) . \tag{5.5}$$

Next we compute c_k and d_k. For real numbers a_k, b_k, c_k, d_k in (5.4), we have

$$\begin{pmatrix} a_{k+1} & b_{k+1} \\ c_{k+1} & d_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \sqrt{2}^{-1} \begin{pmatrix} \lambda & \lambda \\ -\lambda^{-1} & \lambda^{-1} \end{pmatrix} .$$

By an elementary computation, we have

$$c_k = (p^k - q^k)\{(p-q)(\sqrt{2})^k\}^{-1} ,$$
$$d_k = \{p^{k+1} - q^{k+1} + \lambda(p^k - q^k)\}\{(p-q)(\sqrt{2})^k\}^{-1} ,$$

where $p=\{-(\lambda+\lambda^{-1})-\sqrt{(\lambda+\lambda^{-1})^2-8}\}/2$ and $q=\{-(\lambda+\lambda^{-1})+\sqrt{(\lambda+\lambda^{-1})^2-8}\}/2$. From $p+\lambda<q+\lambda<0$ and $|p|=(\lambda+\lambda^{-1}+\sqrt{\varepsilon})/2>\sqrt{2}>|q|=(\lambda+\lambda^{-1}-\sqrt{\varepsilon})/2>\lambda$, we see

$$(5.6)\qquad \begin{aligned}|p-q|(|d_k|-\lambda|c_k|) &= 2^{-k/2}\{|p^k(p+\lambda)-q^k(q+\lambda)|-|p^k-q^k|\}\\ &= 2^{-k/2}\{|p|^k(|p|-\lambda-1)+|q|^k(\lambda+1-|q|)\}\\ &> 2^{-k/2}\{|p|^k(|p|-\lambda-1)\}\,.\end{aligned}$$

Since $|p|-|q|=\sqrt{\varepsilon}$ and $-\lambda+\lambda^{-1}=\sqrt{\varepsilon+4}$, we have

$$(5.7)\qquad \begin{aligned}&\sum_{k=1}^{N-1}(|d_k|-\lambda|c_k|)^{-2}\\ &\leqq \sum_{k=1}^{\infty}2^k(|p|-|q|)^2\{|p|^k(|p|-\lambda-1)\}^{-2}\\ &= 2(|p|-|q|)^2(|p|-\lambda-1)^{-2}(|p|^2-2)^{-1}\\ &= 2\varepsilon\{(\sqrt{\varepsilon+4}-2+\sqrt{\varepsilon})/2\}^{-2}\{(\varepsilon+\sqrt{\varepsilon}\sqrt{8+\varepsilon})/2\}^{-1}\\ &< (\sqrt{\varepsilon+4}+2-\sqrt{\varepsilon})^2\cdot\varepsilon^{-1/2}\\ &< 16\varepsilon^{-1/2}\,.\end{aligned}$$

Furthermore, we have from (5.6)

$$(5.8)\qquad m_1((hE)^N(\bar{I}_\lambda))<\varepsilon/2$$

for N sufficiently large. Hence (5.2), (5.3), (5.5), (5.7) and (5.8) imply

$$m_1(F)\leqq 2\varepsilon+32\varepsilon^{1/2}\,.$$

Since $F=I_\lambda\backslash\bigcup_{V\in G_1}V(\bar{I}_\lambda)$, we have

$$|U(\bar{I}_\lambda)|=m_1(U(F))+\sum_{V\in G_1}|UV(\bar{I}_\lambda)|$$

for any element $U\in\tilde{G}$ and also have

$$(5.9)\qquad \sum_{V\in G_1}|UV(\bar{I}_\lambda)||U(\bar{I}_\lambda)|^{-1}=1-m_1(U(F))|U(\bar{I}_\lambda)|^{-1}\,.$$

As F is a union of open intervals, we have from Lemma 4 that, if $\lambda>1/5$, then

$$m_1(U(F))|U(\bar{I}_\lambda)|^{-1}\leqq(1+\lambda)\{\lambda(1-\lambda)\}^{-1}m_1(F)/2\leqq 4m_1(F)\,.$$

From this inequality and (5.9), we have

$$(5.10)\qquad \sum_{V\in G_1}|UV(\bar{I}_\lambda)||U(\bar{I}_\lambda)|^{-1}\geqq 1-4m_1(F)\,.$$

We take the numbers $a_1, a_2, \cdots, a_k$ in Lemma 6 to be the ratios $|UV(\bar{I}_\lambda)||U(\bar{I}_\lambda)|^{-1}$, $U\in\tilde{G}$, $V\in G_1$. Putting $a=1/2$ and $s=1-4m_1(F)$ in Lemma 4 and noting Lemmas 1, 2 and 3 we have

$$d(\Lambda(\Gamma_\lambda)) \geqq d(L_\lambda) \geqq 1 - 8(2\varepsilon + 16\varepsilon^{1/2})$$

from (5.10). Thus the proof of Theorem 1 is complete.

6. Applications. Let M, A, h, E and $\{c_i, c'_i\}_{i=1}^2$ be as those previously described in § 2. The group Γ generated by A and MAM^{-1} is a Fuchsian group acting on the unit disc and is of type (1; 0; 1). Put $W_1 = E^{-1}h^{-1}$ and $W_2 = E^{-1}h$. Then the group G freely generated by W_1, W_2 has type (0; 0; 3). The fundamental system of $\Lambda(\Gamma)$ coincides with that of $\Lambda(G)$. It is easily seen that $d(\Lambda(\Gamma)) = d(\Lambda(G))$ by Lemma 2. Applying Theorem 4 stated in [2] and [5] and Theorem 1 in the present paper, we have the following whose proof may be omitted.

THEOREM 2. *Assume* $0 < t < 1$. *Then there are Fuchsian groups* G *of types* (0; 0; 3) *and* (1; 0; 1) *with* $d(\Lambda(G)) = t$.

As a direct result of this theorem, we have the following.

COROLLARY 1. *There exist two distinct Fuchsian groups* G_1 *and* G_2 *with* $d(\Lambda(G_1)) = d(\Lambda(G_2))$ *and with the same fundamental regions.*

Using the continuity argument in [5], we also have the following.

COROLLARY 2. *Let* Γ *be a Fuchsian group of type* $(g; 0; m)$ *with* $2g - 2 + m > 0$, $m > 0$. *Then there is a quasiconformal mapping* w_ε *of the extended complex plane onto itself such that* $d(\Lambda(w_\varepsilon \Gamma w_\varepsilon^{-1})) > 1 - \varepsilon$ *for any small positive number* ε.

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