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BILINEAR FOURIER MULTIPLIERS

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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Introduction. An m-linear operator $T_\sigma(f_1, f_2, \cdots, f_m)$ is said to be an m-linear Fourier multiplier with symbol $\sigma(\xi_1, \xi_2, \cdots, \xi_m)$, if it has the following form

$$T_\sigma(f_1, f_2, \cdots, f_m) = (2\pi)^{-nm}\int_{R^{nm}} e^{ix\cdot(\xi_1+\cdots+\xi_m)}\sigma(\xi_1, \cdots, \xi_m)\hat{f}_1(\xi_1)\cdots\hat{f}_m(\xi_m)d\xi_1\cdots d\xi_m ,$$

where $x, \xi_1, \cdots, \xi_m \in R^n$, $x\cdot y = x_1y_1 + \cdots + x_ny_n$ and $\hat{f}$ is the Fourier transform of f, i.e. $\hat{f}(\zeta) = \int_{R^n} e^{-ix\cdot\zeta}f(x)dx$, which is denoted also by $\mathscr{F}f$. In [6] we showed the following: If $\sigma\in C^{2nm+1}(R^{nm}\setminus\{0\})$ and $|\partial_\xi^\alpha\sigma(\xi)| \leqq C_\alpha|\xi|^{-|\alpha|}$, $\xi\neq 0$, $|\alpha|\leqq 2nm+1$, then for $p_j\in[1, \infty]$ $(1\leqq j\leqq m)$ and $0\leqq 1/p = 1/p_1 + \cdots + 1/p_m \leqq 1$ it holds that

$$(0.1)\qquad \|T_\sigma(f_1, \cdots, f_m)\|_p \leqq C(p_1, \cdots, p_m)\|f_1\|_{p_1}\cdots\|f_m\|_{p_m} ,$$

$(f_j\in\mathscr{S}, j = 1, \cdots, m)$, where $\|f_j\|_{p_j} = \|f_j\|_{H^1}$ and $f_j\in H^1_{00}$ if $p_j = 1$, and in case $p = \infty$ the norm on the left-hand side is the BMO norm, which is denoted by $\|f\|_*$. Here $\mathscr{S}$ is the Schwartz class of smooth and rapidly decreasing functions, and H^1_{00} is the space of all $f\in\mathscr{S}$ such that $\hat{f}$ has compact support bounded away from the origin. $\|f\|_p$ is the usual $L^p(R^n)$ norm $(1\leqq p\leqq\infty)$. See [7], [5] or [6] for the definitions of H^p spaces and BMO.

We say that T_σ has *Property* (C) if the above inequalities (0.1) hold for all $1\leqq p_j\leqq\infty$ and $0\leqq 1/p = 1/p_1 + \cdots + 1/p_m \leqq 1$. In this note we try to relax the assumption for σ to obtain Property (C) in the case $n = 1$ and $m = 2$. That is, we deal with bilinear Fourier multipliers with non-smooth symbols. Our symbols have singularities on finite pieces of rays issuing from the origin. Our prototype is Calderón's commutator K_1 in Section 2. It has singularities on the rays $\{\theta = \pm\pi/2\}$ and $\{\theta = (1\pm 2)\pi/4\}$. In Section 1 we deal with non-homogeneous symbols.

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In Section 2 we treat the concrete case of homogeneous symbols such as Calderón's commutator. The case of general homogeneous symbols is treated in Section 3. In Section 4 we will see that the assumptions in Section 3 are relaxed further, if we deal with the case $1 < p_1, p_2 \leqq \infty$ and $0 < 1/p = 1/p_1 + 1/p_2 \leqq 1$.

In the sequel, $\boldsymbol{R}^2$ denotes the two dimensional Euclidean space and (ξ, α) will denote a point in $\boldsymbol{R}^2$. (r, θ) will always denote the polar coordinate representation of the point (ξ, α). For a ray $\{\theta = \theta_0\} = \{(r, \theta); r > 0, \theta = \theta_0\}$ we call a set of the form $\{(r, \theta); r > 0, \theta_1 < \theta < \theta_2\} = \{\theta_1 < \theta < \theta_2\}$ a sector neighborhood of that ray, $(\theta_1 < \theta_0 < \theta_2)$. $\partial_\xi^k A = \partial^k A/\partial \xi^k$. $C^k(E)$ will denote the space of k-times continuously differentiable functions on the set E. For a function $f(t)$ on an interval $f(a + 0)$ will denote the right hand limit of f at $t = a$. The letter C will always denote a constant and does not necessarily denote the same one.

Finally we remark the following. In order to prove Property (C), it suffices to show (0.1) in three cases $(p_1 = 1, p_2 = \infty)$, $(p_1 = \infty, p_2 = 1)$ and $(p_1 = p_2 = \infty)$, by virtue of the multilinear interpolation theory.

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1. Operators with non-homogeneous symbol. Our first main result is the following.

THEOREM 1.1. *Let* $0 \leqq \theta_j < 2\pi$ *and* $b_j = \cot \theta_j$, $(j = 1, 2, \cdots, k)$. *Let* $\sigma \in C(\boldsymbol{R}^2\backslash\{0\})$ *satisfy the following conditions*:

(i) *For* $0 \leqq m + n \leqq 5$, $0 \leqq m, n \leqq 3$, $\partial_\xi^m \partial_\alpha^n \sigma(\xi, \alpha)$ *exist for* $\theta \neq \theta_j$ $(j = 1, 2, \cdots, k)$ *and*

$$|\partial_\xi^m \partial_\alpha^n \sigma(\xi, \alpha)| \leqq C(m, n)(|\xi| + |\alpha|)^{-m-n} . \tag{1.1}$$

(ii) *For* $0 \leqq m + n \leqq 4$, $0 \leqq m, n \leqq 3$, $\partial_\xi^m \partial_\alpha^n \sigma(\xi, \alpha \pm 0)$ *and* $\partial_\xi^m \partial_\alpha^n \sigma(\xi \pm 0, \alpha)$ *exist.*

(iii) *In case* $\theta_j = \pi/2$ *or* $3\pi/2$, $\partial_\xi \sigma(t, s)$ *is continuous at* $t = 0$ *for each* $(0, s) \in \{\theta = \theta_j\}$, *and*

$$\left|\frac{d^n}{ds^n} \partial_\xi^m \sigma(0 \pm 0, s)\right| \leqq C(m, n)|s|^{-m-n} , \qquad 0 \leqq m, n \leqq 2 . \tag{1.2}$$

(iii′) *In case* $\theta_j = 0$ *or* π, *conditions similar to* (iii) *are fulfilled, where the roles of* ξ *and* α *are interchanged.*

(iii″) *In case* $\theta_j \neq 0, \pi/2, \pi, 3\pi/2$, $\partial_\xi^m \sigma(b_j s + t, s)$ *and* $\partial_\xi^m \sigma(b_j s, s + t)$ *are continuous at* $t = 0$ *for each* $(b_j s, s) \in \{\theta = \theta_j\}$, $(m = 1, 2)$, *and*

$$\left|\frac{d}{ds}\partial_\xi^m\partial_\alpha^n\sigma(b_js, s \pm 0)\right| \leqq C(m, n)|s|^{-m-n-1}, \quad 0 \leqq m+n \leqq 3, \quad 0 \leqq m \leqq 3,\ 0 \leqq n \leqq 1. \tag{1.3}$$

Then, T_σ has Property (C).

Moreover, if $\sigma(0, \alpha) = 0$, then for $1 \leqq p \leqq \infty$

$$\| T_\sigma(f, g)\|_p \leqq C_p \| f \|_* \| g \|_p, \qquad f \in \mathrm{BMO},\ g \in L^p, \tag{1.4}$$

where $p = *$ on the left-hand side and $g \in L^\infty \cap L^2$ or $f \in \mathrm{BMO} \cap L^2$ when $p = \infty$, and in case $p = 1$ the norm of g is the H^1 norm and $g \in H^1$.

We shall prove the theorem in the following way. First, using a partition of unity of the unit circle, we decompose the symbol σ into a finite sum of such symbols that each of them has support in a small sector neighborhood and satisfies the conditions in Theorem 1.1 with $k = 1$.

So, we begin with the following elementary lemma.

LEMMA 1.2. *Let $0 < \theta_1 < \pi/2 < \theta_2 < \pi$. Let $\sigma \in C(\boldsymbol{R}^2\setminus\{0\})$ be such that* $\operatorname{supp}\sigma \subset S(\theta_1, \theta_2)$ *and σ satisfies the conditions* (i), (ii), (iii) *in Theorem* 1.1 *under the assumption $k = 1$ and $\theta_k = \pi/2$. Let $\phi, \psi \in \mathscr{S}$ be such that $\hat{\phi}$ and $\hat{\psi}$ have compact support and* $\operatorname{supp}\hat{\psi} \subset (0, \infty)$. *Then, it holds that*

$$|\mathscr{F}(\sigma(\xi/t, \alpha/t)\hat{\phi}(\xi)\hat{\psi}(\alpha))(u, v)| \leqq C(1 + |u| + |v|)^{-1} \times (1 + |u|)^{-2}(1 + |v|)^{-2}, \tag{1.5}$$

where C is independent of $t > 0$.

PROOF. For the sake of simplicity we prove the assertion in the case $t = 1$. Put $A(\xi, \alpha) = \sigma(\xi, \alpha)\hat{\phi}(\xi)\hat{\psi}(\alpha)$ and $F(u, v) = \mathscr{F}A(u, v)$. Then integrating $A(\xi, \alpha)e^{-i\xi u}$ by parts three times with respect to ξ, we have by the continuity of $\partial_\xi^m\sigma(\xi, \alpha)$, $(m = 0, 1)$,

$$\begin{aligned}\int_{-\infty}^{\infty} A(\xi, \alpha)e^{-i\xi u}d\xi \\ = -iu^{-3}[\partial_\xi^2A(0-0, \alpha) - \partial_\xi^2A(0+0, \alpha)] + iu^{-3}\int_0^\infty \partial_\xi^3A(\xi, \alpha)e^{-i\xi u}d\xi .\end{aligned} \tag{1.6}$$

Hence, integration by parts with respect to α yields

$$\begin{aligned}(1 + iu^3)F(u, v) &= -v^{-2}\int_0^\infty\int_{-\infty}^\infty [\partial_\alpha^2A(\xi, \alpha) - \partial_\xi^3\partial_\alpha^2A(\xi, \alpha)]e^{-i(\xi u+\alpha v)}d\xi d\alpha \\ &\quad - v^{-2}\int_0^\infty \frac{d^2}{d\alpha^2}[\partial_\xi^2A(0-0, \alpha) - \partial_\xi^2A(0+0, \alpha)]e^{-i\alpha v}d\alpha \\ &= -v^{-2}(I_1 + I_2) .\end{aligned}$$

By (1.1) and the support properties of $\hat{\phi}$ and $\hat{\psi}$, we get $|I_1| < \infty$ and by (1.2) we get $|I_2| < \infty$. Hence

$$|F(u, v)| \leqq C(1+|u|)^{-3}(1+|v|)^{-2}\,. \tag{1.7}$$

Next, integrating $A(\xi, \alpha)e^{-i\xi u}$ by parts twice with respect to ξ we have

$$\int_{-\infty}^{\infty} A(\xi, \alpha)e^{-i\xi u}d\xi = -u^{-2}\int_{-\infty}^{\infty} \partial_\xi^2 A(\xi, \alpha)e^{-i\xi u}d\xi\,. \tag{1.8}$$

So, integration by parts with respect to α implies

$$(1+u^2)F(u, v) = iv^{-3}\int_0^\infty\int_{-\infty}^\infty [\partial_\alpha^3 A(\xi, \alpha) - \partial_\xi^2\partial_\alpha^3 A(\xi, \alpha)]e^{-i(\xi u+\alpha v)}d\xi d\alpha\,.$$

Hence from (1.1) we obtain

$$|F(u, v)| \leqq C(1+|u|)^{-2}(1+|v|)^{-3}\,. \tag{1.9}$$

If $|u| < |v|$, then $1+|u|+|v| < 2(1+|v|)$, so from (1.9) we obtain (1.5). Similarly, if $|v| \leqq |u|$, using (1.7) we obtain (1.5). That the constant C in (1.5) does not depend on t can be easily checked by reconsidering the above proof and using (1.1) and (1.2). q.e.d.

LEMMA 1.3. *Let $0 < \theta_1 < \theta_0 < \theta_2 < \pi/2$. Let $\sigma \in C(\boldsymbol{R}^2\backslash\{0\})$ be such that $\operatorname{supp}\sigma \subset S(\theta_1, \theta_2)$ and σ satisfies conditions* (i), (ii) *and* (iii″) *in Theorem* 1.1 *under the assumption $k=1$ and $\theta_k = \theta_0$. Let $\phi, \psi \in \mathscr{S}$ be such that $\hat\phi$ and $\hat\psi$ have compact support in $(0, \infty)$. Then it holds that*

$$|\mathscr{F}(\sigma(\xi/t, \alpha/t)\hat\phi(\xi)\hat\psi(\alpha))(u, v)| \leqq C(1+|u|)^{-2} \times (1+|v|)^{-2}(1+|bu+v|)^{-1}\,, \tag{1.10}$$

where $b = \cot\theta_0$ and C is independent of $t > 0$.

PROOF. As in the proof of Lemma 1.2 we prove the assertion in the case $t=1$ and we let A and F be the same as before. By the continuity of $\partial_\xi^m\sigma(b\xi+\cdot, \xi)$, $(m=0, 1, 2)$, integrating $A(\xi, \alpha)e^{-i\xi u}$ by parts three times with respect to ξ we have

$$\int_0^\infty A(\xi, \alpha)e^{-i\xi u}d\xi = iu^{-3}\int_0^\infty \partial_\xi^3 A(\xi, \alpha)e^{-i\xi u}d\xi\,. \tag{1.11}$$

Hence, integration by parts with respect to α gives

$$\begin{aligned}
&(1+iu^3)F(u, v)\\
&\quad = -v^{-2}\int_0^\infty\int_0^\infty [\partial_\alpha^2 A(\xi, \alpha) - \partial_\xi^3\partial_\alpha^2 A(\xi, \alpha)]e^{-i(\xi u+\alpha v)}d\xi d\alpha\\
&\qquad - iv^{-1}\int_0^\infty [\partial_\xi^3 A(\xi, \xi/b-0) - \partial_\xi^3 A(\xi, \xi/b+0)]e^{-i\xi(u+v/b)}d\xi\\
&\qquad - v^{-2}\int_0^\infty [\partial_\xi^3\partial_\alpha A(\xi, \xi/b-0) - \partial_\xi^3\partial_\alpha A(\xi, \xi/b+0)]e^{-i\xi(u+v/b)}d\xi\\
&\qquad + v^{-2}\int_0^\infty [\partial_\alpha A(\xi, \xi/b-0)\partial_\alpha A(\xi, \xi/b+0)]e^{-i\xi(u+v/b)}d\xi
\end{aligned}$$

$$= I_1 + I_2 + I_3 + I_4 .$$

Then from (1.1) we obtain $|I_1| \leqq C|v|^{-2}$. From (1.3) we get $|I_3|$, $|I_4| \leqq C|v|^{-2}$. Integrating the integrand in I_2 by parts and using (1.3) we get $|I_2| \leqq C|v|^{-1}(1 + |bu + v|)^{-1}$. Summing up, we have

$$(1.12) \qquad |F(u, v)| \leqq C(1 + |u|)^{-3}(1 + |v|)^{-1}((1 + |v|)^{-1} + (1 + |bu + v|)^{-1}) .$$

Next, integration by parts with respect to ξ yields via (iii″)

$$F(u, v) = -u^{-2}\int_0^\infty\int_0^\infty \partial_\xi^2 A(\xi, \alpha)e^{-i(\xi u+\alpha v)}d\xi d\alpha = -u^{-2}J .$$

Hence, integrating the integrand by parts three times with respect to α, we get, on account of the continuity of $\partial_\xi^m\sigma(b\xi, \xi + \cdot)$, $(m = 0, 1, 2)$,

$$\begin{aligned} J &= iv^{-3}\int_0^\infty\int_0^\infty \partial_\xi^2\partial_\alpha^3 A(\xi, \alpha)e^{-i(\xi u+\alpha v)}d\xi d\alpha \\ &\quad + v^{-2}\int_0^\infty [\partial_\xi^2\partial_\alpha A(\xi, \xi/b - 0) - \partial_\xi^2\partial_\alpha A(\xi, \xi/b + 0)]e^{-i\xi(u+v/b)}d\xi \\ &\quad - iv^{-3}\int_0^\infty [\partial_\xi^2\partial_\alpha^2 A(\xi, \xi/b - 0) - \partial_\xi^2\partial_\alpha^2 A(\xi, \xi/b + 0)]e^{-i\xi(u+v/b)}d\xi \\ &= iv^{-3}J_1 + v^{-2}J_2 - iv^{-3}J_3 . \end{aligned}$$

Then by (1.1) we get $|J_1| \leqq C$ and by (1.3) we get $|J_3| \leqq C$. Integrating the integrand in J_2 by parts and using (1.3) we get $|J_2| \leqq C(1 + |bu + v|)^{-1}$. Integrating $A(\xi, \alpha)e^{-i(\xi u+\alpha v)}$ by parts three times with respect to α, we get the same representation for F as for J, where we replace $\partial_\xi^2 A$ by A. Hence, as for J we have $|F(u, v)| \leqq C(1 + |v|)^{-2}((1 + |v|)^{-1} + (1 + |bu + v|)^{-1})$. Summing up we obtain

$$(1.13) \qquad |F(u, v)| \leqq C(1 + |u|)^{-2}(1 + |v|)^{-2}((1 + |v|)^{-1} + (1 + |bu + v|)^{-1}) .$$

Now in order to show (1.10) we may assume $b = 1$. Then, if $2|u| < |v|$, we get $2(1 + |u + v|) < 3(1 + |v|)$. Hence using (1.13) we obtain (1.10). If $2|v| < |u|$, we get $1 + |v| < 1 + |u|$ and $|u|/2 < |u + v| < 3|u|/2$. Hence using (1.12) we get (1.10). If $|u| \leqq 2|v| \leqq 4|u|$, we have obviously (1.10). q.e.d.

We can now prove a special case of Theorem 1.1.

PROPOSITION 1.4. (a) *If σ satisfies the conditions in Lemma 1.2, then T_σ has Property (C) and it holds that for $1 \leqq p \leqq \infty$*

$$(1.4') \qquad \|T_\sigma(f, g)\|_p \leqq C_p\|f\|_p\|g\|_* , \qquad f \in L^p,\ g \in \mathrm{BMO} ,$$

where we use the convention in (1.4) in case $p = \infty$ or $p = 1$. Moreover *if* $\sigma(0, \alpha) = 0$, *then* (1.4) *holds.*

(b) *If* σ *satisfies the conditions in Lemma* 1.3, *then* T_σ *has Property* (*C*), *and* (1.4) *and* (1.4′) *hold.*

PROOF. (a) Let $\psi \in \mathscr{S}$ be such that $\operatorname{supp} \hat{\psi} \subset \{1/2 < \alpha < 2\}$ and $\int_0^\infty (\hat{\psi}(s))^2 s^{-1} ds = 1$ and let $\phi \in \mathscr{S}$ be such that $\hat{\phi}$ has compact support and $\hat{\phi}(\xi) = 1$ on $\{2 \cot \theta_2 < \xi < 2 \cot \theta_1\}$. Then applying Lemma 1.2 and putting $w(u, v) = (1 + |u|)^{-2}(1 + |v|)^{-2}(1 + |u| + |v|)^{-1}$ we obtain

$$b(t, u, v) = \mathscr{F}(\sigma(\xi/t, \alpha/t)\hat{\phi}(\xi)\hat{\psi}(\alpha))(u, v)/w(u, v) \in L^\infty(\boldsymbol{R}^2) \, , \tag{1.14}$$

and clearly

$$w(u, v)(1 + |u|)^{5/4}(1 + |v|)^{5/4} \in L^1(\boldsymbol{R}^2) \, . \tag{1.15}$$

Now we follow Coifman and Meyer [4, pp. 154–155]. By (1.15)

$$\sigma(\xi, \alpha)\hat{\phi}(t\xi)\hat{\psi}(t\alpha) = (2\pi)^{-2} \iint w(u, v) b(t, u, v) e^{-it(\xi u + \alpha v)} du dv \, . \tag{1.16}$$

Multiplying (1.16) by $\hat{\psi}(t\alpha)\hat{\phi}(t\xi)t^{-1}$ and integrating it with respect to t we obtain

$$\sigma(\xi, \alpha) = (2\pi)^{-2} \int_0^\infty \iint w(u, v) b(t, u, v) \hat{\psi}(t\alpha)\hat{\phi}(t\xi) e^{-it(\xi u + \alpha v)} du dv \frac{dt}{t} \, . \tag{1.17}$$

This is easily checked by using $\int_0^\infty (\hat{\psi}(t))^2 t^{-1} dt = 1$ and properties of the supports of σ, ϕ and ψ. Hence for $f, g \in \mathscr{S}$ we get via Fubini's theorem

$$\begin{aligned} (1.18) \quad & T_\sigma(f, g)(x) \\ &= \iint w(u, v) du dv \int_0^\infty \left[\iint e^{-ix(\xi + \alpha)} e^{i\alpha t v} \hat{\psi}(\alpha t) \hat{g}(\alpha) e^{it\xi u} \hat{\phi}(\xi t) \hat{f}(\xi) d\xi d\alpha \right] b(t, u, v) t^{-1} dt \\ &= \iint w(u, v) du dv \int_0^\infty (\psi_t^v * g)(\phi_t^u * f) b(t, u, v) t^{-1} dt \, , \end{aligned}$$

where $\phi^u(x) = \phi(x + u)$ and $\phi_t(x) = t^{-1}\phi(x/t)$. Then one can easily get

$$|\phi^u(x)| \leqq C \frac{(1 + |u|)^{5/4}}{(1 + |x|)^{5/4}} \, , \qquad |\psi^v(x)| \leqq C \frac{(1 + |v|)^{5/4}}{(1 + |x|)^{5/4}} \, . \tag{1.19}$$

Since $\mathscr{F}(\phi^u)(\xi) = e^{i\xi u}\hat{\phi}(\xi)$, $\mathscr{F}(\psi^v)(\alpha) = e^{i\alpha v}\hat{\psi}(\alpha)$, they fulfill the assumptions in Theorems 1, 2 and 3 in Yabuta [6]. Thus, if we apply those theorems to ϕ^u and ψ^v, then the proofs there imply that the operator norm of the operator

$$(f, g) \mapsto \int_0^\infty (\psi_t^v * g)(\phi_t^u * f) b(t, u, v) t^{-1} dt$$

is bounded by $C(1 + |u|)^{5/4}(1 + |v|)^{5/4}$. In fact we have only to use

Propositions 4.1 and 4.2 there and the bilinear interpolation theorem. Therefore, applying those theorems and using (1.15) we see that T_σ has Property (C). Remark 1 to those theorems gives (1.4′). In case $\sigma(0, \alpha) = 0$, we follow again Coifman and Meyer [4, p. 155]. Then, on account of the support properties of ϕ, ψ, σ, we have

$$T_\sigma(f, g) = \iint w(u, v)dudv \int_0^\infty (\psi_t^v * g)[(\phi_t^u - \phi_t) * f]b(t, u, v)t^{-1}dt . \tag{1.20}$$

Since $\int(\phi^u(x) - \phi(x))dx = 0$, we have (1.4) again by Remark 1 to Theorems 1, 2. 3 in [6].

(b) We take ψ as in the case (a). Let $\phi \in \mathscr{S}$ be such that $\hat{\phi}$ has compact support in $(0, \infty)$ and $\hat{\phi}(\xi) = 1$ on $\{2^{-1} \cot \theta_2 < \xi < 2 \cot \theta_1\}$. Set $w(u, v) = (1 + |u|)^{-2}(1 + |v|)^{-2}(1 + |bu + v|)^{-1}$. Then the proof proceeds as in the case (a), if we can show (1.15). It is checked as follows. To show it we may assume $b = -1$. Then we have

$$\begin{aligned}\iint_{0<u<2v<4u} & w(u, v)[(1 + |u|)(1 + |v|)]^{5/4}dudv \\ &\leqq C \int_0^\infty (1 + |u|)^{-3/2}du\left[\int_0^{u/2} (1 + v)^{-1}dv + \int_0^u (1 + v)^{-1}dv\right] \\ &\leqq C' \int_0^\infty (1 + u)^{-3/2} \log (1 + u)du < +\infty .\end{aligned}$$

The same estimate holds for the integral on the set $\{4u < 2v < u < 0\}$. The integrals on the other regions can be estimated more easily. q.e.d.

PROOF OF THEOREM 1.1. As is noted just after the statement of the theorem, we decompose σ and apply Proposition 1.4 and its obvious variant. Then the desired assertions follow.

REMARK 1. As is easily seen, the condition (iii″) in Theorem 1.1 can be replaced by a similar one, where the roles of ξ and α are interchanged.

REMARK 2. If $p = q = \infty$ in Theorem 1.1, we have $\| T_\sigma(f, g)\|_* \leqq C\| f \|_\infty \| g \|_\infty$ for f or $g \in L^\infty \cap L^2$.

In order to deduce a few consequences from Theorem 1.1, we need a lemma. We recall that the Hilbert transform Hf of f is given by $\mathscr{F}(Hf)(\xi) = (-i \operatorname{sgn} \xi)\hat{f}(\xi)$.

LEMMA 1.5. *For a symbol σ let us put $\sigma'(\xi, \alpha) = \sigma(\xi, \alpha) \operatorname{sgn} \xi$. Let $1 < p < \infty$, $1 \leqq q \leqq \infty$ and $1/r = 1/p + 1/q$. Then we have:*

(i) *If T_σ is a bounded bilinear operator from $H^1 \times L^\infty$ into L^1, so is $T_{\sigma'}$.*

(ii) *If* T_σ *is a bounded bilinear operator from* $L^p \times L^q$ *into* L^r, *so is* $T_{\sigma'}$.

(iii) *If* T_σ *is a bounded bilinear operator from* BMO $\times H^1$ *into* L^1, *so is* $T_{\sigma'}$.

(iv) *If* T_σ *is a bounded bilinear operator from* $(\mathrm{BMO} \cap L^2) \times L^\infty$ *or* $\mathrm{BMO} \times (L^\infty \cap L^2)$ $(\subset \mathrm{BMO} \times L^\infty)$ *into* BMO, *so is* $T_{\sigma'}$.

In (i), (ii), (iv) *the space* L^∞ *may be replaced by* BMO.

PROOF. By the definition of the bilinear Fourier multiplier we have $T_{\sigma'}(f, g) = T_\sigma(Hf, g)$. As is well-known, the Hilbert transform preserves the spaces H^1, BMO and L^p, $(1 < p < \infty)$, and is bounded there. Therefore we have the desired assertions. q.e.d.

Now combining Theorem 1.1 and Lemma 1.5 we get:

COROLLARY 1.6. *Let* σ *be as in Theorem* 1.1. *Then, for the symbol* $\sigma'(\xi, \alpha) = \sigma(\xi, \alpha) \operatorname{sgn} \xi$, T_σ *and* $T_{\sigma'}$ *are bounded bilinear operators from* $H^1 \times L^\infty$ *into* L^1.

Moreover, if $\sigma(0, \alpha) = 0$, *then* T_σ *and* $T_{\sigma'}$ *are bounded bilinear operators from* BMO $\times H^1$ *into* L^1 *and* (1.4) *holds for both* T_σ *and* $T_{\sigma'}$.

2. Special cases of homogeneous symbols. We investigate here the continuity of the following bilinear operators with homogeneous symbols. For any real number s we set

$$K_s(f, g)(x) = p.v.\frac{1}{s\pi}\int_{-\infty}^{\infty} \frac{G(x) - G(x - s(x - y))}{(x - y)^2} f(y)dy ,$$

where G is a primitive of g. Then one can easily check for $f, g \in \mathcal{S}$, as in [4, p. 162], that $K_s(f, g) = -iT_{\sigma(K_s)}(f, g)$ with symbol $\sigma(K_s)$ given by

$$\sigma(K_s) = [1 - (1 - |\xi/(s\alpha)|)^+] \operatorname{sgn} \xi + (1 - |\xi/(s\alpha)|)^+ \operatorname{sgn}(s\alpha) .$$

By the argument in [4, Proposition 4, p. 160] one can show the operator $K_s(\cdot, g)$ is a Calderón-Zygmund operator in the sense of Coifman and Meyer, if g is a bounded function (cf. Section 4). Hence we get

LEMMA 2.1. *For any real* s, K_s *is a bounded bilinear operator from* $H^1 \times L^\infty$ *into* L^1 *and* $L^\infty \times L^\infty$ *into* BMO.

We also have the following:

LEMMA 2.2. *For any real* s, K_s *is a bounded bilinear operator from* $L^\infty \times H^1$ *into* L^1.

PROOF. Let $T = K_s$. One has only to show the following inequality for an H^1-atom g; $\operatorname{supp} g$ is contained in an interval (a, b), $\int_a^b g(x)dx = 0$

and $\|g\|_\infty \leqq 1/(b-a)$,

$$\|T(f, g)\|_1 \leqq C\|f\|_\infty \|g\|_{H^1} . \tag{2.1}$$

Now for an H^1-atom g, $T(\cdot, g)$ is a Calderón-Zygmund operator, and so $T(f, g)$ is BMO for $f \in L^\infty$. Hence, in order to prove (2.1) one has only to show

$$\left|\int_{-\infty}^{\infty} T(f, g)(x)h(x)dx\right| \leqq C\|f\|_\infty \|h\|_\infty \|g\|_{H^1} \qquad \text{for} \quad h \in \mathcal{S}. \tag{2.2}$$

For the sake of simplicity we show the inequality (2.2) in the case of $T = K_1$. The other cases are treated in a quite similar way. Let g be as above and $G(x) = \int_a^x g(t)dt$. Then, by virtue of the properties of Calderón-Zygmund's singular integrals and $\int_a^b g(x)dx = 0$,

$$\begin{aligned}
\pi \int_{-\infty}^{\infty} T(f, g)hdx &= \int_a^b \left(p.v. \int_a^b \frac{G(x) - G(y)}{(x-y)^2} h(x)dx\right) f(y)dy \\
&\quad + \int_a^b \left(\int_b^\infty f(y)(x-y)^{-2}dy\right) G(x)h(x)dx \\
&\quad + \text{similar three terms} \\
&= I_1 + I_2 + I_3 + I_4 + I_5 .
\end{aligned}$$

Then

$$\begin{aligned}
|I_1| &\leqq \left(\int_a^b |T(h\chi_{(a,b)}, g)|^2 dy\right)^{1/2} \left(\int_a^b |f(y)|^2 dy\right)^{1/2} \\
&\leqq C\|g\|_\infty \|h\chi_{(a,b)}\|_2 \|f\|_\infty (b-a)^{1/2} \\
&\leqq C\|h\|_\infty \|f\|_\infty \|g\|_\infty (b-a) , \\
|I_2| &\leqq \|f\|_\infty \|h\|_\infty \int_a^b |G(x)|(b-x)^{-1}dx \leqq \|h\|_\infty \|f\|_\infty \|g\|_\infty (b-a) .
\end{aligned}$$

Similar inequalities hold for I_j, $j = 3, 4, 5$. Since $\|g\|_\infty (b-a) \leqq C\|g\|_{H^1}$, we have the desired inequality (2.2). This completes the proof of our lemma.

Combining Lemmas 2.1, 2.2 and Theorem 1.1 we obtain the following lemma.

LEMMA 2.3. *Let* $0 \leqq \theta_0 < 2\pi$ *and* $\sigma(\xi, \alpha) \in C^5(\boldsymbol{R}^2 \backslash (\{\theta = \theta_0\} \cup \{0\}))$ *be a homogeneous function of degree zero and satisfy, in a sector neighborhood of the ray* $\{\theta = \theta_0\}$, *one of the following conditions.*

(i) *In case* $\sin\theta_0 \neq 0$ *there exist* $a, b, c \in \boldsymbol{R}$ *such that*

$$\sigma(\xi, \alpha) = \begin{cases} a\cot\theta - a\cot\theta_0 + c & \theta \leqq \theta_0 \\ b\cot\theta - b\cot\theta_0 + c & \theta > \theta_0 . \end{cases} \tag{2.3}$$

(ii) *In case* $\cos\theta_0 \neq 0$ *there exist* $a, b, c \in \boldsymbol{R}$ *such that* (2.3) *holds, where we replace* cot *by* tan.

Then T_σ *is a bounded bilinear operator from* $H^1 \times L^\infty$ *into* L^1, *from* $L^\infty \times H^1$ *into* L^1 *and from* $(L^\infty \cap L^2) \times L^\infty (\subset L^\infty \times L^\infty)$ *into* BMO.

PROOF. First, we shall show the first and second assertions for the case (i). We write $\sigma \in SL$ (resp. $\sigma \in SR$) if T_σ is a bounded bilinear operator from $H^1 \times L^\infty$ (resp. $L^\infty \times H^1$) into L^1.

(a) Case $\theta_0 = \pi/2$. By Theorem 1.1 we may assume that the support of σ is contained in the open upper half plane. Let $A(\xi, \alpha) \in C^\infty(\boldsymbol{R}^2\backslash\{0\})$ be a homogeneous function of degree zero, with support contained in the open upper half plane and such that $A(\xi, \alpha) = 1$ on the sector $\{\pi/4 < \theta < 3\pi/4\}$. Then from Theorem 1.1 and Corollary 1.6 it follows that $(\xi/\alpha)A(\xi, \alpha)$, $(\xi/\alpha)A(\xi, \alpha)\operatorname{sgn}\xi \in SL \cap SR$, because $L^\infty \subset$ BMO. Hence there exists a linear combination $\tau(\xi, \alpha)$ of the above two symbols such that $\sigma - \tau \in C^5(\boldsymbol{R}^2\backslash\{0\})$. Since $\sigma - \tau$ is homogeneous of degree zero, by Theorem 1.1 we have $\sigma - \tau \in SL \cap SR$, and hence $\sigma \in SL \cap SR$. The case $\theta_0 = 3\pi/2$ can be treated similarly.

(b) Case $0 < \theta_0 < \pi/2$. Let $A(\xi, \alpha) \in C^\infty(\boldsymbol{R}^2\backslash\{0\})$ be a homogeneous function of degree zero, with support contained in the open first quadrant, such that $A(\xi, \alpha) = 1$ on a sector neighborhood of the ray $\{\theta = \theta_0\}$. Put $s = \cot\theta_0$ and $P(\xi, \alpha) = \sigma(K_{-s})(\xi, \alpha)$. Then by Lemmas 2.1 and 2.2 we get $P \in SL \cap SR$. Now

$$P(\xi, \alpha) = \begin{cases} 1 & (-\pi/2 \leqq \theta \leqq \theta_0) \\ 2\xi/(s\alpha) - 1 & (\theta_0 < \theta \leqq \pi/2) \\ -1 & (\pi/2 \leqq \theta \leqq \pi + \theta_0) \\ -2\xi/(s\alpha) + 1 & (\pi + \theta_0 < \theta < 3\pi/2) \,. \end{cases}$$

Let $B(\xi, \alpha) \in C^\infty(\boldsymbol{R}^2\backslash\{0\})$ be a homogeneous function of degree zero such that its support is contained in the sectors $\{|\alpha/\xi| < \tan\min(\theta_0, \pi/2 - \theta_0) = M\}$ and $B(\xi, \alpha) = 1$ on the sectors $\{|\alpha/\xi| < M/2\}$. Let $E(\xi, \alpha) = B(\alpha, \xi)$. Then from the case (i) it follows that $PE \in SL \cap SR$, and hence we get $P(1 - E) \in SL \cap SR$. So, by Lemma 1.5 we get $P(1 - E)\operatorname{sgn}\xi \in SL$, and hence $P(1 - E)(1 + \operatorname{sgn}\xi) \in SL$. Thus, via Theorem 1.1 we get $AP \in SL$. We have already seen that $PE \in SR$. Since clearly $PB \in C^\infty(\boldsymbol{R}^2\backslash\{0\})$, we get $PB \in SR$ by Theorem 1.1. Hence $P(1 - E - B) \in SR$. So, by Lemma 1.5 we get $P(1-E-B)\operatorname{sgn}\alpha \in SR$ and so $P(1-E-B)(1 + \operatorname{sgn}\alpha) \in SR$. Since $P(1-E-B)(1+\operatorname{sgn}\alpha)(1-\operatorname{sgn}\xi) \in C^\infty(\boldsymbol{R}^2\backslash\{0\})$, it belongs to SR by Theorem 1.1. Thus, we get $P(1 - E - B) \times (1 + \operatorname{sgn}\alpha)(1 + \operatorname{sgn}\xi) \in SR$. Therefore via Theorem 1.1 we obtain $AP \in SR$. Summing up, we get $AP \in SL \cap SR$. We get $A(\xi, \alpha)\xi/\alpha \in SL \cap SR$ by Theorem 1.1, because it belongs to $C^\infty(\boldsymbol{R}^2 \backslash \{0\})$. Now, clearly there exists a linear combination $\tau(\xi, \alpha)$ of AP and $A(\xi, \alpha)\xi/\alpha$ such that $\sigma - \tau \in C^5(\boldsymbol{R}^2 \backslash \{0\})$. Hence on account of Theorem 1.1 we see that $\sigma \in SL \cap SR$. The other three cases can be treated in a quite similar way.

(c) The last assertion for the case (i) can be shown in a way similar to the above proof for $\sigma \in SL$, by using $L^\infty \subset \mathrm{BMO}$ in addition.

(d) Finally, changing the variables ξ and α, we can deduce the assertions for the case (ii). q.e.d.

3. General case of homogeneous symbols. If σ is homogeneous of degree zero, then we can drop the continuity assumptions on derivatives in Theorem 1.1.

THEOREM 3.1. *Let $\sigma(\xi, \alpha)$ be homogeneous of degree zero and assume $\sigma(\xi, \alpha) = \sigma(r, \theta)$ is continuous and piecewise C^5 with respect to θ. Then T_σ has Property (C).*

PROOF. By Theorem 1.1 we may assume $\sigma \in C^5(\boldsymbol{R}^2 \backslash (\{\theta = \theta_0\} \cup \{0\}))$ for some θ_0: $0 \leqq \theta_0 < 2\pi$. (i) Case $\theta_0 = \pi/2$. Let $A(\xi, \alpha) \in C^\infty(\boldsymbol{R}^2 \backslash (\{\theta = \pi/2\} \cup \{0\}))$ be a homogeneous function of degree zero such that, in a sector neighborhood of $\{\theta = \pi/2\}$, $A(\xi, \alpha) = 0$ $(\theta < \pi/2)$ and $= \xi/\alpha$ $(\pi/2 < \theta)$. Then, since $A(0, \alpha) = \partial_\xi A(0+0, \alpha) = 0$ and $\partial_\xi A(0-0, \alpha) = 1/\alpha$, there exists a constant c such that $\sigma + cA \in C^1(\boldsymbol{R}^2 \backslash \{0\})$. Since $\sigma + cA$ is homogeneous of degree zero, it satisfies the assumption in Theorem 1.1. Hence, from Theorem 1.1 and Lemma 2.3 we see that T_σ has Property (C).

(ii) Case $0 < \theta_0 < \pi/2$. Let $A(\xi, \alpha) \in C^\infty(\boldsymbol{R}^2 \backslash \{0\})$ be a homogeneous function of degree zero such that its support is contained in the open first quadrant and $A(\xi, \alpha) = 1$ on a sector neighborhood of the ray $\{\theta = \theta_0\}$. Let $G(\xi, \alpha) = 0$ $(0 < \theta < \theta_0)$ and $= \xi/\alpha - \cot\theta_0$ $(\theta_0 < \theta < \pi/2)$. Let $J(\xi, \alpha) = 0$ $(0 < \theta < \theta_0)$ and $= \alpha/\xi - \tan\theta_0$ $(\theta_0 < \theta < \pi/2)$. Then, since AG and AJ satisfy the assumption in Lemma 2.3, T_{AG} and T_{AJ} have Property (C). Now put $s = \cot\theta_0$, $t = \tan\theta_0$, $A_1 = AG$ and $A_2 = A_1 + s^2 AJ$. Then we get $A_1(\xi, t\xi) = \partial_\xi A_1(\xi + 0, t\xi) = \partial_\xi^2 A_1(\xi \pm 0, t\xi) = 0$ and $\partial_\xi A_1(\xi - 0, t\xi) = s/\xi$. Furthermore, $A_2(\xi, t\xi) = \partial_\xi A_2(\xi \pm 0, t\xi) = \partial_\xi^2 A_2(\xi + 0, t\xi) = 0$ and $\partial_\xi^2 A_2(\xi - 0, t\xi) = 2s/\xi^2$. Therefore, using the homogeneity of σ, A_1 and A_2, we can find two constants c_1 and c_2 such that $\sigma + c_1 A_1 + c_2 A_2$ is in $C^1(\boldsymbol{R}^2 \backslash \{0\})$ and satisfies the assumptions (i), (ii), (iii$''$) in Theorem 1.1. Hence by Theorem 1.1 we see that T_σ has Property (C). The other cases are treated quite similarly. q.e.d.

As a consequence we have the following.

COROLLARY 3.2. *Let $\sigma(\xi, \alpha)$ be the same as in Theorem* 3.1. *Assume, moreover,* $\sigma(0, \alpha) = 0$. *Then it holds that*

$$\|T_\sigma(f, g)\|_p \leqq C_p \|f\|_* \|g\|_p \qquad (1 \leqq p \leqq \infty)\,, \tag{3.1}$$

where we use the convention in Theorem 1.1 *provided* $p = 1$ *or* $p = \infty$.

PROOF. Since $\sigma(0, \alpha) = 0$, both $\sigma(\xi, \alpha)$ and $\sigma(\xi, \alpha) \times \operatorname{sgn} \xi$ satisfy the hypothesis in Theorem 3.1. Hence we have $\|T_\sigma(f, g)\|_p$, $\|T_\sigma(Hf, g)\|_p \leqq C_p \|f\|_\infty \|g\|_p$. It is well-known that every BMO function can be written in the form $f = h + Hk$, where $h, k \in L^\infty(\boldsymbol{R})$ and $\|f\|_*$ is equivalent to $\|h\|_\infty + \|k\|_\infty$. From these we can easily deduce the inequality (3.1). Here Hf denotes the Hilbert transform of f. q.e.d.

4. **The case** $1 < p_1$, $p_2 \leqq \infty$ **and** $0 < 1/p_1 + 1/p_2 \leqq 1$. In this case we can further relax the hypothesis in Theorem 3.1.

THEOREM 4.1. *Let $\sigma(\xi, \alpha)$ be a homogeneous function of degree zero such that the restriction $\omega(\theta)$ of σ to the unit circle has the first derivative of bounded variation. Then, if $1 < p$, $q \leqq \infty$ and $0 < 1/r = 1/p + 1/q \leqq 1$, it holds that*

$$\|T_\sigma(f, g)\|_r \leqq C(p, q) \|f\|_p \|g\|_q \qquad (f \in L^p,\ g \in L^q)\,. \tag{4.1}$$

PROOF. Our method is the same as in Coifman and Meyer [4, pp. 162–163]. We only point out important check points.

(i) The case $1 < p < \infty$ and $q = \infty$. Let $\phi(\xi, \alpha) \in C^\infty(\boldsymbol{R}^2\backslash\{0\})$ be homogeneous of degree zero, and satisfy $\phi(0, \pm 1) = \sigma(0, \pm 1)$, $\phi(1, -1) = \sigma(1, -1)$ and $\phi(-1, 1) = \sigma(-1, 1)$. From Theorem 1.1 it follows that T_ϕ satisfies (4.1), and hence we may assume $\sigma(0, \pm 1) = \sigma(1, -1) = \sigma(-1, 1) = 0$. Now we put $t = \xi/(\xi + \alpha)$ and $h(t) = \sigma(\xi, \alpha)$ ($\operatorname{sgn}(\xi + \alpha) > 0$ and $\operatorname{sgn} \xi > 0$). Then, $h(t) = \sigma(t, 1 - t)$. We put $b(v) = h(e^v)$. Then by the homogeneity of σ, we have $b(v) = \sigma(1, (1 - e^v)/e^v) = \sigma(e^v/(1 - e^v), 1)$. Hence, $b'(v) = \partial_\xi \sigma(e^v/(1 - e^v), 1) e^v (1 - e^v)^{-2} = -\partial_\alpha \sigma(1, (1 - e^v)/e^v) e^{-v}$. Therefore, since $\omega'(\theta)$ is bounded, we have $b'(v) = O(e^{-|v|})$ as $v \to \pm\infty$. By the assumption $\sigma(0, 1) = \sigma(1, -1) = 0$ we get $b(-\infty) = b(+\infty) = 0$. Consequently, we have $b(v) = O(e^{-|v|})$ as $v \to \pm\infty$. Hence $\mathscr{F}b$ is bounded on $(-\infty, \infty)$. Next, since $\omega'(\theta)$ is of bounded variation, one can easily see that $\partial_\xi \sigma(e^v/(1 - e^v), 1)$ and $\partial_\alpha \sigma(1, (1 - e^v)/e^v)$ are of bounded variation on $(-\infty, -1]$ and $[-1, \infty)$, respectively. Hence, we see that $b'(v)$ is of bounded variation on $(-\infty, \infty)$. So, by integration by parts we have

$$\left| \int_{-\infty}^{\infty} h(e^v) e^{-i\gamma v} dv \right| \leqq C/(1 + \gamma^2)\,.$$

Combining similar considerations for the other three cases ($\operatorname{sgn}(\xi + \alpha) > 0$,

$\operatorname{sgn}\xi < 0)$, $(\operatorname{sgn}(\xi+\alpha) < 0,\ \operatorname{sgn}\xi > 0)$, $(\operatorname{sgn}(\xi+\alpha < 0),\ \operatorname{sgn}\xi < 0)$, we get

$$T_\sigma(f, g) = \int_{-\infty}^{\infty}\left(\int_{R^2} e^{ix(\xi+\alpha)}|\xi|^{i\gamma}|\xi+\alpha|^{-i\gamma}\sum_{j,k=1,-1} m_{jk}(\gamma)\right.$$
$$\left.\times(1 + j\operatorname{sgn}\xi)(1 + k\operatorname{sgn}(\xi+\alpha))\hat{f}(\xi)\hat{g}(\alpha)d\xi d\alpha\right)\frac{d\gamma}{1+\gamma^2},$$

for some $m_{jk} \in L^\infty(\boldsymbol{R})$. Hence,

$$T_\sigma(f, g) = \int_{-\infty}^{\infty}[m_1(\gamma)M_{-\gamma}\{g(x)(M_\gamma f)(x)\} + m_2(\gamma)HM_{-\gamma}\{g(x)(M_\gamma f)(x)\}$$
$$+ m_3(\gamma)M_{-\gamma}\{g(x)(HM_\gamma f)(x)\} + m_4(\gamma)HM_{-\gamma}\{g(x)(HM_\gamma f)(x)\}]\frac{d\gamma}{1+\gamma^2},$$

for some $m_j(\gamma) \in L^\infty(\boldsymbol{R})$. Here, H is the Hilbert transform and $(M_\gamma f)^\wedge(\xi) = |\xi|^{i\gamma}\hat{f}(\xi)$. It is known that $\|M_\gamma f\|_{H^p} \leqq C_p(1+|\gamma|)^{|1/p-1/2|}\|f\|_{H^p}$ $(0<p<\infty)$, (see Miyachi [5, p. 282]). It is well-known that $\|Hf\|_{H^p} \leqq C_p\|f\|_{H^p}$ $(0 < p < \infty)$. Hence, if $1 < p < \infty$ we have

$$\|T_\sigma(f, g)\|_p \leqq C\|f\|_p\|g\|_\infty .$$

(ii) The case $1 < p, q < \infty$ and $1 \leqq r < \infty$. We choose ϕ so that $\phi(0, \pm 1) = \sigma(0, \pm 1)$ and $\phi(\pm 1, 0) = \sigma(\pm 1, 0)$. Hence we assume $\sigma(0, \pm 1) = \sigma(\pm 1, 0) = 0$. Then, putting $t = \xi/\alpha$ and arguing as in the case (i) we get

$$T_\sigma(f, g) = \int_{-\infty}^{\infty}[m_1(\gamma)(M_\gamma f)(M_{-\gamma}g) + m_2(\gamma)(M_\gamma Hf)(M_{-\gamma}g)$$
$$+ m_3(\gamma)(M_\gamma Hf)(M_{-\gamma}g) + m_4(\gamma)(M_\gamma Hf)(M_{-\gamma}Hg)]\frac{d\gamma}{1+\gamma^2},$$

for some $m_j(\gamma) \in L^\infty(\boldsymbol{R})$. Hence we obtain

$$\|T_\sigma(f, g)\|_r \leqq C\|f\|_p\|g\|_q . \qquad \text{q.e.d.}$$

REMARK 1. The above proof shows that, if $\sigma(0, \pm 1) = \sigma(\pm 1, 0) = 0$, then

$$\|T_\sigma(f, g)\|_r \leqq C_{p,q}\|f\|_{H^p}\|g\|_{H^q} \qquad (0 < p, q < \infty, 1/r = 1/p + 1/q) . \tag{4.2}$$

This poses the following questions. (i) Does (4.2) hold for the symbols in Theorem 4.1? (ii) Does (4.2) hold for the symbols in Theorem 1.1? (iii) Does (4.2) hold for the bilinearization of Littlewood-Paley's g-function in [6]? If (ii) is affirmative, so is (i). And if (iii) is affirmative, so is (ii).

REMARK 2. The Calderón commutator K_1 in Section 2 satisfies (4.2) for $1 < p,\ q < \infty$ and $1/r = 1/p + 1/q$ [2, Theorem A]. Hence, this is also true for the symbols in Theorem 4.1 if $\sigma(0, 1) = \sigma(1, 0)$ and $\sigma(0, -1) = \sigma(-1, 0)$. If one could show (4.2) for the operator K_{-1} $(1 < p, q < \infty)$, then the operators in Theorem 4.1 would also satisfy (4.2) $(1 < p, q < \infty)$. However, the method in [2] does not work in this case. One would need more devices. Here H^p is the H^p space in the sense of Stein-Weiss (cf. [5]).

REMARK 3. Since the operator $(f, g) \mapsto H(fg)$ clearly satisfies the inequality (4.1), using Theorem 1.1, Lemma 1.5 and Theorem 3.1, one can show the following: If $\sigma(\xi, \alpha)$ is homogeneous of degree zero, continuous except at $\theta = \pm\pi/2,\ 3\pi/4,\ -\pi/4$, piecewise C^1 and with the second derivative bounded with respect to θ, then the inequality (4.1) holds.

REMARK 4. Theorem 4.1 for the case $1 < p, q, r < \infty$ is stated implicitly in Theorem I in Coifman and Meyer [9]. By our Theorem 4.1, their theorem remains varid for the case $1 < p, q \leqq \infty$ and $1 \leqq r < \infty$.

REMARK 5. For the operator p.v. $\int_{-\infty}^{\infty} [A(x + t) + A(x - t) - 2A(x)] \times t^{-2}(\operatorname{sgn} t)f(x - t)dt$ we have the same integral representation as in the proof of Theorem 4.1, where $(1 + \gamma^2)^{-1}$ is replaced by $(1 + \gamma^2)^{-1}\log(1 + |\gamma|)$, (see [4, p. 162]). Hence, the same assertion as in Theorem 4.1 remains valid for this operator.

Using the argument in the proof of Corollary 3.2 we get:

COROLLARY 4.2. *Let $\sigma(\xi, \alpha)$ be as in Theorem 4.1 or Remark 3. Assume, moreover, $\sigma(\xi, 0) = 0$. Then it follows that*

$$\| T_\sigma(f, g)\|_p \leqq C_p \| f \|_p \| g \|_* \qquad (1 < p < \infty) . \tag{4.3}$$

5. Examples. For a real number s we put

$$S_s(f, g)(x) = \text{p.v.}\frac{1}{s\pi}\int_{-\infty}^{\infty} \frac{G(x + s(x - y)) + G(x - s(x - y)) - 2G(x)}{(x - y)^2} f(y)dy ,$$

$$U_s(f, g)(x) = \text{p.v.}\frac{1}{s\pi}\int_{-\infty}^{\infty} \frac{G(x + s(x - y)) - G(x - s(x - y))}{(x - y)^2} f(y)dy ,$$

where G is a primitive of g. Then these are bilinear Fourier multipliers with the following symbols

$$\sigma(S_s) = -2i(1 - |\xi/(s\alpha)|)^+ \operatorname{sgn}(s\alpha) ,$$

$$\sigma(U_s) = 2i[1 - (1 - |\xi/(s\alpha)|)^+] \operatorname{sgn} \xi .$$

To these symbols we can apply Theorem 3.1. As applications of Corollary 4.2 we have

$$\| H(G(x)f'(x)) - G(x)Hf'(x) \|_p \leqq C_p \| f \|_p \| g \|_* ,$$

and

$$\| gHf - H(fg) \|_p \leqq C_p \| f \|_p \| g \|_*$$

(cf. Coifman-Meyer [8, pp. 105-107]).

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