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APPROXIMATION BY PARTIAL SUMS AND CESÀRO MEANS OF MULTIPLE ORTHOGONAL SERIES

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1. Introduction. Let $(X, \mathscr{F}, \mu)$ be an arbitrary positive measure space and $\{\varphi_{ik}(x): i, k = 1, 2, \cdots\}$ an orthonormal system on this space. We shall consider the double orthogonal series

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}\varphi_{ik}(x) , \tag{1.1}$$

where $\{a_{ik}: i, k=1, 2, \cdots\}$ is a double sequence of real numbers (coefficients), for which

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 < \infty . \tag{1.2}$$

By the well-known Riesz-Fischer theorem there exists a function $f(x) \in L^2(X, \mathscr{F}, \mu)$ such that series (1.1) is the generalized Fourier series of $f(x)$ with respect to the system $\{\varphi_{ik}(x)\}$. In particular, denoting by

$$s_{mn}(x) = \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik}\varphi_{ik}(x) \quad (m, n = 1, 2, \cdots)$$

the rectangular partial sums of (1.1), we have

$$\int\Big[f(x) - s_{mn}(x)\Big]^2 d\mu(x) = \left\{\sum_{i=1}^{m}\sum_{k=n+1}^{\infty} + \sum_{i=m+1}^{\infty}\sum_{k=1}^{\infty}\right\} a_{ik}^2 \to 0 \quad \text{as} \quad \min(m, n) \to \infty .$$

Here and in the sequel the integrals are taken over the whole space X. By the above relation, the rectangular partial sums $s_{mn}(x)$ of (1.1) converge to $f(x)$ in L^2-metric.

It is a fundamental fact that condition (1.2) itself does not ensure the pointwise convergence of $s_{mn}(x)$ to $f(x)$ almost everywhere on X (in abbreviation: a.e.).

The extension of the famous Rademacher-Menšov theorem proved by a number of authors (see, e.g. [1], [7] etc.) reads as follows: *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log(i + 1)]^2[\log(k + 1)]^2 < \infty , \tag{1.3}$$

then the rectangular partial sums $s_{mn}(x)$ *converge to* $f(x)$ *a.e. as* $\min(m, n) \to \infty$. (The logarithms are to the base 2.)

Hence one can deduce, as a simple consequence, the following statement: *If* $1 \leqq i_1 \leqq i_2 \leqq \cdots$ *and* $1 \leqq k_1 \leqq k_2 \leqq \cdots$ *are two sequences of integers, for which* $i_p \to \infty$ *as* $p \to \infty$, $k_q \to \infty$ *as* $q \to \infty$, *and*

$$\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\Bigl(\sum_{i=i_{p-1}+1}^{i_p}\ \sum_{k=k_{q-1}+1}^{k_q} a_{ik}^2\Bigr)[\log(p+1)]^2[\log(q+1)]^2 < \infty ,$$

where $i_0 = k_0 = 0$, *then the rectangular partial sums* $s_{i_p,k_q}(x)$ *of* (1.1) *converge to* $f(x)$ *a.e. as* $\min(p, q) \to \infty$. (The empty sums $\sum_{i=i_{p-1}+1}^{i_p}\sum_{k=k_{q-1}+1}^{k_q}$, with either $i_{p-1} = i_p$ or $k_{q-1} = k_q$ if any, are defined to be equal to 0.)

The special case $i_p = 2^{p-1}$ and $k_q = 2^{q-1}$ $(p, q = 1, 2, \cdots)$ is of particular interest: *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2[\log\log(i+3)]^2[\log\log(k+3)]^2 < \infty , \tag{1.4}$$

then the rectangular partial sums $s_{2^p,2^q}(x)$ *of* (1.1) *converge to* $f(x)$ *a.e. as* $\min(p, q) \to \infty$.

Denote by $\sigma_{mn}(x)$ the first arithmetic means of the rectangular partial sums:

$$\begin{aligned}\sigma_{mn}(x) &= m^{-1}n^{-1}\sum_{i=1}^{m}\sum_{k=1}^{n} s_{ik}(x)\\ &= \sum_{i=1}^{m}\sum_{k=1}^{n}\Bigl(1-\frac{i-1}{m}\Bigr)\Bigl(1-\frac{k-1}{n}\Bigr)a_{ik}\varphi_{ik}(x) \quad (m, n = 1, 2, \cdots) .\end{aligned}$$

The a.e. equiconvergence of the two double subsequences $\{s_{2^p,2^q}(x)\colon p, q = 0, 1, \cdots\}$ and $\{\sigma_{2^p,2^q}(x)\colon p, q = 0, 1, \cdots\}$ is no longer true, which is the case for (ordinary) single orthogonal series (see, e.g. [2, p. 118]). In spite of this fact, *under condition* (1.4) *the means* $\sigma_{mn}(x)$ *do converge to* $f(x)$ *a.e. as* $\min(m, n) \to \infty$ (see [5]).

2. The main results. Approximation by rectangular partial sums and their means. Let $\{\kappa(m, n)\colon m, n = 1, 2, \cdots\}$ and $\{\lambda(m, n)\colon m, n = 1, 2, \cdots\}$ be two double sequences of real numbers, $\lambda(m, n) \neq 0$ when both m and n are large enough. We write

$$\kappa(m, n) = o\{\lambda(m, n)\}$$

if

$$\kappa(m, n)/\lambda(m, n) \to 0 \quad \text{as} \quad \min(m, n) \to \infty$$

and there exists a constant C such that

$$|\kappa(m, n)| \leqq C|\lambda(m, n)| \quad (m, n = 1, 2, \cdots) .$$

In the sequel C, C_1, and C_2 denote positive constants, not necessarily the same at each occurrence. Furthermore, we set

$$\Delta_{10}\kappa(m, n) = \kappa(m, n) - \kappa(m-1, n) ,$$
$$\Delta_{01}\kappa(m, n) = \kappa(m, n) - \kappa(m, n-1) ,$$

and

$$\Delta_{11}\kappa(m, n) = \kappa(m, n) - \kappa(m-1, n) - \kappa(m, n-1) + \kappa(m-1, n-1)$$
$$(m, n = 1, 2, \cdots; \kappa(m, 0) = \kappa(0, n) = 0) .$$

In the introduction we have already mentioned that (1.3) and (1.4) are sufficient conditions for the a.e. convergence of $s_{mn}(x)$ and $\sigma_{mn}(x)$ to $f(x)$, respectively. Now the main point is that if we require the fulfilment of a stronger condition instead of (1.3) or (1.4), then we can even state an approximation rate for the deviations $s_{mn}(x) - f(x)$ and $\sigma_{mn}(x) - f(x)$, respectively. The results obtained can be considered as the extensions of the corresponding theorems of [6], [8] and [4] from single orthogonal series to double ones.

Before stating our main results, let us introduce one more notation. Let $\alpha > 1$ be a given number and denote by Λ_α the class of those nondecreasing sequences $\{\lambda(m): m = 1, 2, \cdots\}$ of positive numbers, for which

$$1 < C_1 \leqq \lambda(2^{m+1})/\lambda(2^m) \leqq C_2 < \alpha \tag{2.1}$$

for all m large enough, say for $m \geqq m_0$, where m_0 may depend on $\{\lambda(m)\}$. For example, $\lambda(m) = m^{\gamma_1}[\log(m+1)]^{\gamma_2}[\log\log(m+3)]^{\gamma_3}$ is in Λ_α if $\gamma_1 > 0$ and $\alpha > 2^{\gamma_1}$, while γ_2 and γ_3 are arbitrary numbers.

THEOREM 1. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2\lambda_1^2(i)\lambda_2^2(k) < \infty , \tag{2.2}$$

where both $\{\lambda_1(i)\}$ *and* $\{\lambda_2(k)\}$ *belong to* Λ_2, *then*

$$\sigma_{mn}(x) - f(x) = o_x\{\lambda_1^{-1}(m) + \lambda_2^{-1}(n)\} \quad a.e. , \tag{2.3}$$

and there exists a function $g(x) \in L^2(X, \mathscr{F}, \mu)$ *such that*

$$\min\{\lambda_1(m), \lambda_2(n)\}\,|\sigma_{mn}(x) - f(x)| \leqq g(x) \quad a.e. \quad (m, n = 1, 2, \cdots) . \tag{2.4}$$

For single orthogonal series a similar theorem with $\lambda_1(i) = i^\gamma$, $0 < \gamma < 1$, was proved by Leindler [6].

Assuming that (m, n) tends restrictedly to ∞, one can obtain essentially the same rate of approximation under a weaker assumption.

THEOREM 2. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2\lambda^2(\max(i, k)) < \infty , \tag{2.5}$$

where $\{\lambda(m)\} \in \Lambda_2$, *then for every* $\theta > 1$ *we have*

$$\max_{n;\theta^{-1}\leqq n/m\leqq\theta} |\sigma_{mn}(x) - f(x)| = o_x\{\lambda^{-1}(m)\} \quad a.e. , \tag{2.6}$$

and there exists a function $g(x) \in L^2(X, \mathscr{F}, \mu)$ *such that*

$$\lambda(m) \max_{n:\theta^{-1}\leqq n/m\leqq\theta} |\sigma_{mn}(x) - f(x)| \leqq g(x) \quad a.e. \quad (m = 1, 2, \cdots) . \tag{2.7}$$

It is a trivial observation that (2.6) implies that

$$m^{-1}n^{-1}\sum_{i=1}^{m}\sum_{k=1}^{n}(s_{ik}(x) - f(x)) = o_x\{\lambda^{-1}(m)\} \quad \text{a.e.} ,$$

provided $\theta^{-1} \leqq n/m \leqq \theta$. The following theorem indicates that the mean value of $s_{ik}(x) - f(x)$ is of $o_x\{\lambda^{-1}(m)\}$, not because of the cancellation of positive and negative terms, but because the indices (i, k) for which $|s_{ik}(x) - f(x)|$ is not small are sparse.

THEOREM 3. *If condition* (2.5) *is satisfied with* $\{\lambda(m)\} \in \Lambda_2$ *and* $\{m\lambda^{-1}(m)\}$ *is nondecreasing, then for every* $\theta > 1$ *we have*

$$m^{-2}\sum_{i=1}^{m}\sum_{k=\theta^{-1}i}^{\theta i}[s_{ik}(x) - f(x)]^2 = o_x\{\lambda^{-2}(m)\} \quad a.e. \tag{2.8}$$

If (2.5) *is satisfied with* $\{\lambda(m)\} \in \Lambda_{\sqrt{2}}$ *and* $\{m\lambda^{-2}(m)\}$ *is nondecreasing, then for every* $\theta > 1$

$$m^{-1}\sum_{i=1}^{m} i^{-1}\sum_{k=\theta^{-1}i}^{\theta i}[s_{ik}(x) - f(x)]^2 = o_x\{\lambda^{-2}(m)\} \quad a.e. \tag{2.9}$$

Furthermore, there exists a function $h(x) \in L^1(X, \mathscr{F}, \mu)$ *such that the left-hand sides of* (2.8) *and* (2.9) *both multiplied by* $\lambda^2(m)$ *do not exceed* $h(x)$ *a.e.* $(m = 1, 2, \cdots)$.

Here and in the sequel by $\sum_{k=\theta^{-1}i}^{\theta i}$ we mean that the summation is extended over all integers k, for which $\theta^{-1} \leqq k/i \leqq \theta$.

We note that for single orthogonal series a similar theorem with $\lambda(m) = m^\gamma$, $0 < \gamma < 1/2$, was proved by Sunouchi [8].

We make four further remarks.

1° Following Alexits [3], this type of approximation is called strong approximation. In particular, from (2.8) and (2.9) it follows that

$$m^{-2}\sum_{i=1}^{m}\sum_{k=\theta^{-1}i}^{\theta i}|s_{ik}(x) - f(x)| = o_x\{\lambda^{-1}(m)\} \quad \text{a.e.}$$

and

$$m^{-1}\sum_{i=1}^{m} i^{-1} \sum_{k=\theta^{-1}i}^{\theta i} |s_{ik}(x) - f(x)| = o_x\{\lambda^{-1}(m)\} \quad \text{a.e.}\,, \tag{2.10}$$

respectively.

For example, the latter relation can be shown by making use of the Cauchy inequality in the following setting:

$$\begin{aligned}&\Big[m^{-1}\sum_{i=1}^{m}\sum_{k=\theta^{-1}i}^{\theta i} i^{-1/2}(i^{-1/2}|s_{ik}(x) - f(x)|)\Big]^2\\ &\leqq m^{-2}\Big(\sum_{i=1}^{m}\sum_{k=\theta^{-1}i}^{\theta i} i^{-1}\Big)\Big(\sum_{i=1}^{m}\sum_{k=\theta^{-1}i}^{\theta i} i^{-1}[s_{ik}(x) - f(x)]^2\Big)\\ &\leqq (\theta - \theta^{-1} + 1)m^{-1}\sum_{i=1}^{m} i^{-1}\sum_{k=\theta^{-1}i}^{\theta i}[s_{ik}(x) - f(x)]^2\,.\end{aligned}$$

Now if we apply (2.9), then we obtain (2.10).

2° Slightly modifying the proof of Theorem 3, one can conclude the following result, too, which corresponds to the special case $\lambda(m) \equiv 1$.

THEOREM 4. *If condition* (1.2) *is satisfied and the Cesàro means* $\sigma_{mn}(x)$ *converge to* $f(x)$ *a.e. as* $\min(m, n) \to \infty$, *then for every* $\theta > 1$ *the left-hand side of* (2.9) *is* $o_x\{1\}$ *a.e. and does not exceed a function* $h(x) \in L^1(X, \mathscr{F}, \mu)$ *a.e.* ($m = 1, 2, \cdots$).

For single orthogonal series the corresponding theorem was proved by Borgen [4].

3° It is an open question whether statement (2.9) can be strengthened into the following stronger one:

$$m^{-1}\sum_{i=1}^{m}\Big\{\max_{k:\theta^{-1}\leqq k/i\leqq\theta}[s_{ik}(x) - f(x)]^2\Big\} = o_x\{\lambda^{-2}(m)\} \quad \text{a.e.}$$

Our conjecture is that the answer lies in the negative.

4° It is also an open question whether one can deduce the following strong approximation type result starting with Theorem 1: If condition (2.2) is satisfied with $\lambda_1(m) = \lambda_2(m) = \lambda(m) \in \Lambda_2$ and $\{m\lambda^{-1}(m)\}$ is nondecreasing, then the relation

$$m^{-1}n^{-1}\sum_{i=1}^{m}\sum_{k=1}^{n}[s_{ik}(x) - f(x)]^2 = o_x\{\lambda^{-2}(m) + \lambda^{-2}(n)\} \quad \text{a.e.}$$

holds true.

3. Approximation by special partial sums and their means. We fix an (ordinary) nondecreasing sequence $Q = \{Q_r: r = 1, 2, \cdots\}$ of finite sets in $N^2 = \{(i, k): i, k = 1, 2, \cdots\}$ such that

$$\bigcup_{r=1}^{\infty} Q_r = N^2 .$$

In this section our goal is to study the approximation properties, while using the sums

$$s_r(Q; x) = \sum_{(i,k) \in Q_r} a_{ik}\varphi_{ik}(x) \quad (r = 1, 2, \cdots) .$$

These sums can be regarded as a single sequence of certain partial sums of (1.1), which are generated by Q.

The most important special cases are those when the Q_r are either rectangles or (quarter) circles in N^2:

(i) The case

$$Q_r = \{(i, k) \in N^2 \colon i \leqq m_r \quad \text{and} \quad k \leqq n_r\} \quad (r = 1, 2, \cdots) ,$$

where $1 \leqq m_1 \leqq m_2 \leqq \cdots$ and $1 \leqq n_1 \leqq n_2 \leqq \cdots$ are two sequences of integers, both tending to $+\infty$, provides a single subsequence $\{s_{m_r,n_r}(x) \colon r = 1, 2, \cdots\}$ of the double sequence $\{s_{mn}(x) \colon m, n = 1, 2, \cdots\}$ of the rectangular partial sums. In particular, the case $m_r = n_r = r$ $(r = 1, 2, \cdots)$ gives the so-called square partial sums $s_{rr}(x)$ of (1.1).

(ii) The case

$$Q_r = \{(i, k) \in N^2 \colon i^2 + k^2 \leqq r^2\} \quad (r = 1, 2, \cdots)$$

provides for the spherical partial sums of (1.1).

Denote by $\sigma_r(Q; x)$ the first arithmetic means of the partial sums $s_r(Q; x)$:

$$\begin{aligned}\sigma_r(Q; x) &= r^{-1} \sum_{\rho=1}^{r} s_\rho(Q; x) \\ &= \sum_{\rho=1}^{r} \left(1 - \frac{\rho - 1}{r}\right) \sum_{(i,k) \in Q_\rho \setminus Q_{\rho-1}} a_{ik}\varphi_{ik}(x) \quad (r = 1, 2, \cdots) ,\end{aligned}$$

where we set $Q_0 = \varnothing$.

The results of [6], [8] and [4] pertaining to single orthogonal series can be extended to this case as follows.

THEOREM 5. *If*

$$\sum_{r=1}^{\infty} \Bigl(\sum_{(i,k) \in Q_r \setminus Q_{r-1}} a_{ik}^2 \Bigr) \lambda^2(r) < \infty , \tag{3.1}$$

where $\{\lambda(r)\} \in \Lambda_2$, *then*

$$\sigma_r(Q; x) - f(x) = o_x\{\lambda^{-1}(r)\} \quad a.e. ,$$

and there exists a function $g(x) \in L^2(X, \mathscr{F}, \mu)$ *such that*

$$\lambda(r) \,|\, \sigma_r(Q; x) - f(x) \,| \leqq g(x) \quad a.e. \ (r = 1, 2, \cdots) .$$

THEOHEM 6. *If condition* (3.1) *is satisfied with* $\{\lambda(r)\} \in \Lambda_{\sqrt{2}}$ *and* $\{r\lambda^{-2}(r)\}$ *is nondecreasing, then*

$$r^{-1} \sum_{\rho=1}^{r} [s_\rho(Q; x) - f(x)]^2 = o_x\{\lambda^{-2}(r)\} \quad a.e. , \tag{3.2}$$

and there exists a function $h(x) \in L^1(X, \mathscr{F}, \mu)$ *such that*

$$\lambda^2(r) r^{-1} \sum_{\rho=1}^{r} [s_\rho(Q; x) - f(x)]^2 \leqq h(x) \quad a.e. \quad (r = 1, 2, \cdots) .$$

For the special case of square partial sums condition (3.1) is equivalent to condition (2.5), because in this case $(i, k) \in Q_r \backslash Q_{r-1}$ is equivalent to the fact that $\max(i, k) = r$ $(r = 1, 2, \cdots)$.

COROLLARY. *If condition* (2.5) *is satisfied with* $\{\lambda(m)\} \in \Lambda_{\sqrt{2}}$ *and* $\{m\lambda^{-2}(m)\}$ *is nondecreasing, then*

$$m^{-1} \sum_{i=1}^{m} [s_{ii}(x) - f(x)]^2 = o_x\{\lambda^{-2}(m)\} \quad a.e. , \tag{3.3}$$

and there exists a function $h(x) \in L^1(X, \mathscr{F}, \mu)$ *such that*

$$\lambda^2(m) m^{-1} \sum_{i=1}^{m} [s_{ii}(x) - f(x)]^2 \leqq h(x) \quad a.e. \ (m = 1, 2, \cdots) .$$

It is instructive to compare conclusions (3.3) and (2.9) (formally writing $\theta = 1$ in Theorem 3, one gets weaker statements).

THEOREM 7. *If condition* (1.2) *is satisfied and* $\sigma_r(Q; x)$ *converges to* $f(x)$ *a.e., then the left-hand side of* (3.2) *is* $o_x\{1\}$ *a.e. and does not exceed a function* $h(x) \in L^1(X, \mathscr{F}, \mu)$ *a.e.* $(r = 1, 2, \cdots)$.

4. Proof of Theorem 1. The statement of Theorem 1 will be an immediate consequence of the following four lemmas.

LEMMA 1. *If* $\{\lambda(m)\} \in \Lambda_2$, *then*

$$\sum_{m=p}^{\infty} \lambda^{-1}(2^m) \leqq C\lambda^{-1}(2^p) , \tag{4.1}$$

$$\sum_{m=p}^{\infty} \lambda^2(2^m) 2^{-2m} \leqq C\lambda^2(2^p) 2^{-2p} \tag{4.2}$$

and

$$\sum_{m=0}^{p} \lambda^2(2^m) \leqq C\lambda^2(2^p) \quad (p = 0, 1, \cdots) . \tag{4.3}$$

In particular, (4.1) *implies that*

(4.4) $$\sum_{m=0}^{\infty} \lambda^{-2}(2^m) < \infty ,$$

while (4.2) *implies that*

(4.5) $$\sum_{m=i}^{\infty} \lambda^2(m)m^{-3} \leqq C\lambda^2(i)i^{-2} \quad (i = 1, 2, \cdots) .$$

PROOF. It can be done in routine ways. The left inequality in (2.1) yields (4.1) and (4.3), while the right inequality in (2.1) yields (4.2). We do not enter into details. □

In the following lemmas we only assume that $\{\lambda_1(m)\}$ and $\{\lambda_2(n)\}$ are nondecreasing sequences of positive numbers, possessing one or two properties of (4.1)–(4.5).

LEMMA 2. *Under condition* (2.2) *with such* $\{\lambda_1(m)\}$ *and* $\{\lambda_2(n)\}$ *that satisfy* (4.1), *we have*

$$f(x) - s_{2^p,2^q}(x) = o_x\{\lambda_1^{-1}(2^p) + \lambda_2^{-1}(2^q)\} \quad a.e. ,$$

and there exists $g(x) \in L^2$ *such that*

$$\min\{\lambda_1(2^p), \lambda_2(2^q)\}\,|f(x) - s_{2^p,2^q}(x)| \leqq g(x) \quad a.e. \ (p, q = 0, 1, \cdots) .$$

PROOF. Without loss of generality we may suppose that $a_{i1} = a_{1k} = 0$ $(i, k = 1, 2, \cdots)$. By (2.2),

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_1^2(2^m)\lambda_2^2(2^n) \int \left[\sum_{i=2^m+1}^{2^{m+1}} \sum_{k=2^n+1}^{2^{n+1}} a_{ik}\varphi_{ik}(x) \right]^2 d\mu(x) < \infty ,$$

whence B. Levi's theorem implies that

(4.6) $$\lambda_1(2^m)\lambda_2(2^n) \sum_{i=2^m+1}^{2^{m+1}} \sum_{k=2^n+1}^{2^{n+1}} a_{ik}\varphi_{ik}(x) \to 0 \quad \text{a.e.} \quad \text{as} \quad \max(m, n) \to \infty .$$

Furthermore, defining

$$g_1^2(x) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_1^2(2^m)\lambda_2^2(2^n) \left[\sum_{i=2^m+1}^{2^{m+1}} \sum_{k=2^n+1}^{2^{n+1}} a_{ik}\varphi_{ik}(x) \right]^2 ,$$

we have $g_1(x) \in L^2$. It is clear that

(4.7) $$\lambda_1(2^m)\lambda_2(2^n) \left| \sum_{i=2^m+1}^{2^{m+1}} \sum_{k=2^n+1}^{2^{n+1}} a_{ik}\varphi_{ik}(x) \right| \leqq g_1(x) \quad \text{a.e.} \quad (m, n = 0, 1, \cdots) .$$

Now considering the representation

$$f(x) - s_{2^p,2^q}(x) = \left\{ \sum_{i=1}^{2^p} \sum_{k=2^q+1}^{\infty} + \sum_{i=2^p+1}^{\infty} \sum_{k=1}^{\infty} \right\} a_{ik}\varphi_{ik}(x)$$

$$= \left\{\sum_{m=0}^{p-1}\sum_{n=q}^{\infty} + \sum_{m=p}^{\infty}\sum_{n=0}^{\infty}\right\}\left(\sum_{i=2^m+1}^{2^{m+1}}\sum_{k=2^n+1}^{2^{n+1}} a_{ik}\varphi_{ik}(x)\right)$$

and making use of (4.1), (4.6) and (4.7), we can obtain both assertions of Lemma 2. □

LEMMA 3. *Under condition* (2.2) *with such* $\{\lambda_1(m)\}$ *and* $\{\lambda_2(n)\}$ *that satisfy* (4.2) *and* (4.4), *we have*

$$s_{2^p,2^q}(x) - \sigma_{2^p,2^q}(x) = o_x\{\lambda_1^{-1}(2^p) + \lambda_2^{-1}(2^q)\} \quad a.e. ,$$

and there exists $g(x) \in L^2$ *such that*

$$\min\{\lambda_1(2^p), \lambda_2(2^q)\}\,|s_{2^p,2^q}(x) - \sigma_{2^p,2^q}(x)| \leqq g(x) \quad a.e. \quad (p, q = 0, 1, \cdots) .$$

PROOF. We may again suppose that $a_{i1} = a_{1k} = 0$ $(i, k = 1, 2, \cdots)$. We begin with the representation

$$\begin{aligned} (4.8) \qquad & s_{2^p,2^q}(x) - \sigma_{2^p,2^q}(x) \\ &= \sum_{i=1}^{2^p}\sum_{k=1}^{2^q}\left[\frac{i-1}{2^p}\left(1 - \frac{k-1}{2^q}\right) + \frac{k-1}{2^q}\right]a_{ik}\varphi_{ik}(x) \\ &=: A_{pq}^{(1)}(x) + A_{pq}^{(2)}(x) . \end{aligned}$$

We treat $A_{pq}^{(1)}(x)$ in detail. Using the Cauchy inequality,

$$\begin{aligned} (4.9) \quad [A_{pq}^{(1)}(x)]^2 &\leqq \left[\sum_{n=0}^{q-1}\left|\sum_{i=2}^{2^p}\sum_{k=2^n+1}^{2^{n+1}}\frac{i-1}{2^p}\left(1 - \frac{k-1}{2^q}\right)a_{ik}\varphi_{ik}(x)\right|\right]^2 \\ &\leqq \sum_{n=0}^{q-1}\lambda_2^2(2^n)\left[\sum_{i=2}^{2^p}\sum_{k=2^n+1}^{2^{n+1}}\frac{i-1}{2^p}\left(1 - \frac{k-1}{2^q}\right)a_{ik}\varphi_{ik}(x)\right]^2 \sum_{n=0}^{q-1}\lambda_2^{-2}(2^n) . \end{aligned}$$

By (4.4), here the second factor is bounded in q. We set

$$g_2^2(x) := \sum_{p=0}^{\infty}\lambda_1^2(2^p)\sum_{n=0}^{\infty}\lambda_2^2(2^n)\left[\sum_{i=2}^{2^p}\sum_{k=2^n+1}^{2^{n+1}}\frac{i-1}{2^p}\left(1 - \frac{k-1}{2^q}\right)a_{ik}\varphi_{ik}(x)\right]^2 .$$

The termwise integrated series is

$$\begin{aligned} &\sum_{p=0}^{\infty}\lambda_1^2(2^p)\sum_{n=0}^{\infty}\lambda_2^2(2^n)\sum_{i=2}^{2^p}\sum_{k=2^n+1}^{2^{n+1}}\left[\frac{i-1}{2^p}\left(1 - \frac{k-1}{2^q}\right)\right]^2 a_{ik}^2 \\ &\leqq \sum_{p=0}^{\infty}\frac{\lambda_1^2(2^p)}{2^{2p}}\sum_{i=2}^{2^p}\sum_{k=2}^{\infty}(i-1)^2\lambda_2^2(k)a_{ik}^2 \\ &= \sum_{i=2}^{\infty}\sum_{k=2}^{\infty}a_{ik}^2(i-1)^2\lambda_2^2(k)\sum_{p:2^p \geqq i}2^{-2p}\lambda_1^2(2^p) < \infty , \end{aligned}$$

the last inequality is due to (2.2) and (4.2). Thus B. Levi's theorem implies that $g_2(x) \in L^2$ and

$$\lambda_1^2(2^p)\sum_{n=0}^{\infty}\lambda_2^2(2^n)\left[\sum_{i=2}^{2^p}\sum_{k=2^n+1}^{2^{n+1}}\frac{i-1}{2^p}\left(1-\frac{k-1}{2^q}\right)a_{ik}\varphi_{ik}(x)\right]^2\to 0$$

a.e. as $p\to\infty$.

Taking (4.9) into account, we obtain that

$$\sup_{q\geq 0} A_{pq}^{(1)}(x) = o_x\{\lambda_1^{-1}(2^p)\} \quad \text{a.e.}$$

and

$$\lambda_1(2^p)\sup_{q\geq 0} A_{pq}^{(1)}(x) \leqq Cg_2(x) \quad \text{a.e.} \quad (p=0,1,\cdots)\,.$$

We can similarly deduce that

$$\sup_{p\geq 0} A_{pq}^{(2)}(x) = o_x\{\lambda_2^{-1}(2^q)\} \quad \text{a.e.}$$

and

$$\lambda_2(2^q)\sup_{p\geq 0} A_{pq}^{(2)}(x) \leqq g_3(x) \quad \text{a.e.} \quad (q=0,1,\cdots)\,,$$

where $g_3(x)\in L^2$. □

LEMMA 4. *Under condition* (2.2) *with such* $\{\lambda_1(m)\}$ *and* $\{\lambda_2(n)\}$ *that satisfy* (4.4) *and* (4.5), *we have*

$$\text{(4.10)} \qquad \begin{aligned} M_{pq}(x) :&= \max_{2^q\leqq m\leqq 2^{p+1}}\ \max_{2^q\leqq n\leqq 2^{q+1}} |\sigma_{mn}(x)-\sigma_{2^p,2^q}(x)| \\ &= o_x\{\lambda_1^{-1}(2^p)+\lambda_2^{-1}(2^q)\} \quad \text{a.e.}\,, \end{aligned}$$

and there exists $g(x)\in L^2$ *such that*

$$\min\{\lambda_1(2^p),\lambda_2(2^q)\}M_{pq}(x)\leqq g(x) \quad \text{a.e.} \quad (p,q=0,1,\cdots)\,.$$

PROOF. Our starting point is that

$$\text{(4.11)} \quad \begin{aligned} M_{pq}(x) \leqq& \max_{2^p\leqq m\leqq 2^{p+1}}\ \max_{2^q\leqq n\leqq 2^{q+1}} \Big|\sigma_{mn}(x)-\sigma_{m,2^q}(x) \\ &-\sigma_{2^p,n}(x)+\sigma_{2^p,2^q}(x)\Big| + \max_{2^p< m\leqq 2^{p+1}}|\sigma_{m,2^q}(x)-\sigma_{2^p,2^q}(x)| \\ &+\max_{2^q<n\leqq 2^{q+1}}|\sigma_{2^p,n}(x)-\sigma_{2^p,2^q}(x)| =: M_{pq}^{(1)}(x)+M_{pq}^{(2)}(x)+M_{pq}^{(3)}(x)\,. \end{aligned}$$

Owing to the identity

$$\text{(4.12)} \qquad \sigma_{mn}-\sigma_{m,2^q}-\sigma_{2^p,n}+\sigma_{2^p,2^q} = \sum_{i=2^p+1}^{m}\sum_{k=2^q+1}^{n}\Delta_{11}\sigma_{ik}$$

and to the Cauchy inequality we have that

$$[M_{pq}^{(1)}(x)]^2 \leqq 2^{p+q}\sum_{i=2^p+1}^{2^{p+1}}\sum_{k=2^q+1}^{2^{q+1}}[\Delta_{11}\sigma_{ik}(x)]^2\,.$$

In order to show that

$$\lambda_1(2^p)\lambda_2(2^q)M_{pq}^{(1)}(x) \to 0 \quad \text{a.e.} \quad \text{as} \quad \max(p, q) \to \infty$$

and

$$\lambda_1(2^p)\lambda_2(2^q)M_{pq}^{(1)}(x) \leqq g_4(x)$$
$$:= \left\{\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} mn\lambda_1^2(m)\lambda_2^2(n)[\varDelta_{11}\sigma_{mn}(x)]^2\right\}^{1/2} \quad \text{a.e.} \quad (p, q = 0, 1, \cdots)$$

involving that $g_4(x) \in L^2$, we use the representation

$$\varDelta_{11}\sigma_{mn}(x) = \sum_{i=2}^{m}\sum_{k=2}^{n} \frac{(i-1)(k-1)}{(m-1)m(n-1)n} a_{ik}\varphi_{ik}(x) \quad (m, n \geqq 2) \tag{4.13}$$

and apply again B. Levi's theorem:

$$\int g_4^2(x)d\mu(x) = \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} mn\lambda_1^2(m)\lambda_2^2(n) \sum_{i=2}^{m}\sum_{k=2}^{n} \frac{(i-1)^2(k-1)^2}{(m-1)^2m^2(n-1)^2n^2} a_{ik}^2$$
$$\leqq \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{\lambda_1^2(m)\lambda_2^2(n)}{m^3n^3} \sum_{i=2}^{m}\sum_{k=2}^{n} i^2k^2a_{ik}^2$$
$$= \sum_{i=2}^{\infty}\sum_{k=2}^{\infty} a_{ik}^2 i^2k^2 \sum_{m=i}^{\infty}\sum_{n=k}^{\infty} \frac{\lambda_1^2(m)\lambda_2^2(n)}{m^3n^3} < \infty ,$$

the last inequality follows from (2.2) and (4.5).

To handle $M_{pq}^{(2)}(x)$, we use the identity

$$\sigma_{m,2^q} - \sigma_{2^p,2^q} = \sum_{i=2^p+1}^{m} \varDelta_{10}\sigma_{i,2^q} \tag{4.14}$$

and the representation

$$\varDelta_{10}\sigma_{m,2^q}(x) = \sum_{i=2}^{m}\sum_{k=1}^{2^q} \frac{i-1}{(m-1)m}\left(1 - \frac{k-1}{2^q}\right)a_{ik}\varphi_{ik}(x) \quad (m \geqq 2,\ q \geqq 0) . \tag{4.15}$$

By (4.14) and the Cauchy inequality,

$$[M_{pq}^{(2)}(x)]^2 \leqq 2^p \sum_{m=2^p+1}^{2^{p+1}} [\varDelta_{10}\sigma_{m,2^q}(x)]^2 \leqq 2^p \sum_{m=2^p+1}^{2^{p+1}} \left\{\sum_{n=0}^{q-1} \lambda_2^2(2^n)\right.$$
$$\left. \times \left[\sum_{i=2}^{m}\sum_{k=2^n+1}^{2^{n+1}} \frac{i-1}{(m-1)m}\left(1 - \frac{k-1}{2^q}\right)a_{ik}\varphi_{ik}(x)\right]^2 \sum_{n=0}^{q-1} \lambda_2^{-2}(2^n)\right\} .$$

Due to (4.4), the last factor on the right-hand side is again bounded in q. We are going to show that

$$\lambda_1(2^p) \sup_{q \geqq 0} M_{pq}^{(2)}(x) \to 0 \quad \text{a.e.} \quad \text{as} \quad p \to \infty$$

and

$$\lambda_1(2^p)\sup_{q\geq 0} M_{pq}^{(2)}(x) \leqq Cg_5(x) \quad \text{a.e.} \quad (p=0,1,\cdots),$$

where

$$g_5^2(x) := \sum_{m=2}^{\infty}\sum_{n=0}^{\infty} m\lambda_1^2(m)\lambda_2^2(2^n) \times\left[\sum_{i=2}^{m}\sum_{k=2^n+1}^{2^{n+1}} \frac{i-1}{(m-1)m}\left(1-\frac{k-1}{2^q}\right)a_{ik}\varphi_{ik}(x)\right]^2 .$$

To this effect,

$$\begin{aligned}\int g_5^2(x)d\mu(x) &= \sum_{m=2}^{\infty}\sum_{n=0}^{\infty} m\lambda_1^2(m)\lambda_2^2(2^n) \times \sum_{i=2}^{m}\sum_{k=2^n+1}^{2^{n+1}}\left[\frac{i-1}{(m-1)m}\left(1-\frac{k-1}{2^q}\right)\right]^2 a_{ik}^2 \\ &\leqq \sum_{m=2}^{\infty}\sum_{n=0}^{\infty}\lambda_1^2(m)\lambda_2^2(2^n)\sum_{i=2}^{m}\sum_{k=2^n+1}^{2^{n+1}} i^2m^{-3}a_{ik}^2 \\ &\leqq \sum_{m=2}^{\infty}\lambda_1^2(m)m^{-3}\sum_{i=2}^{m}\sum_{k=2}^{\infty} i^2\lambda_2^2(k)a_{ik}^2 \\ &= \sum_{i=2}^{\infty}\sum_{k=2}^{\infty} a_{ik}^2 i^2\lambda_2^2(k)\sum_{m=i}^{\infty}\lambda_1^2(m)m^{-3} < \infty ,\end{aligned}$$

the last inequality is by (2.2) and (4.5).

In the same way, one can find that

$$\lambda_2(2^q)\sup_{p\geq 0} M_{pq}^{(3)}(x) \to 0 \quad \text{a.e.} \quad \text{as} \quad q\to\infty$$

and

$$\lambda_2(2^q)\sup_{p\geq 0} M_{pq}^{(3)}(x) \leqq g_6(x) \quad \text{a.e.} \quad (q=0,1,\cdots)$$

with a suitable $g_6(x)\in L^2$. □

5. **Proof of Theorem 2.** The proof is based on Lemma 1 and the following three lemmas, corresponding to Lemmas 2-4.

In this section we again assume that $\{\lambda(m)\}$ is a nondecreasing sequence of positive numbers, possessing only some properties of (4.1)-(4.6).

LEMMA 5. *Under condition* (2.5) *with such a* $\{\lambda(m)\}$ *that satisfies* (4.3), *we have*

$$f(x) - s_{2^p,2^p}(x) = o_x\{\lambda^{-1}(2^p)\} \quad a.e. ,$$

and there exists $g(x)\in L^2$ *such that*

$$\lambda(2^p)\,|\,f(x) - s_{2^p,2^p}(x)\,| \leqq g(x) \quad a.e. \quad (p=0,1,\cdots) .$$

PROOF. We set

$$g_7^2(x) := \sum_{p=0}^{\infty} \lambda^2(2^p)[f(x) - s_{2^p,2^p}(x)]^2 .$$

After integrating,

$$\int g_7^2(x) d\mu(x) = \sum_{p=0}^{\infty} \lambda^2(2^p) \int [f(x) - s_{2^p,2^p}(x)]^2 d\mu(x)$$
$$= \sum_{p=0}^{\infty} \lambda^2(2^p) \left\{ \sum_{i=1}^{2^p} \sum_{k=2^p+1}^{\infty} + \sum_{i=2^p+1}^{\infty} \sum_{k=1}^{\infty} \right\} a_{ik}^2 = I_1 + I_2 , \quad \text{say} .$$

Simple calculations give:

$$I_1 := \sum_{p=0}^{\infty} \lambda^2(2^p) \sum_{i=1}^{2^p} \sum_{k=2^p+1}^{\infty} a_{ik}^2$$
$$= \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} a_{ik}^2 \sum_{p: i \leqq 2^p < k} \lambda^2(2^p) \leqq C \sum_{i=1}^{\infty} \sum_{k=i+1}^{\infty} a_{ik}^2 \lambda^2(k) < \infty$$

and

$$I_2 \leqq C \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \lambda^2(\max(i, k)) < \infty ,$$

where we used (2.5) and (4.3). An application of B. Levi's theorem provides the statements of Lemma 5. □

LEMMA 6. *Under condition* (2.5) *with such a* $\{\lambda(m)\}$ *that satisfies* (4.2), *we have*

$$s_{2^p,2^p}(x) - \sigma_{2^p,2^p}(x) = o_x\{\lambda^{-1}(2^p)\} \quad a.e. ,$$

and there exists $g(x) \in L^2$ *such that*

$$\lambda(2^p) | s_{2^p,2^p}(x) - \sigma_{2^p,2^p}(x) | \leqq g(x) \quad a.e. \quad (p = 0, 1, \cdots) .$$

PROOF. We set

$$g_8^2(x) := \sum_{p=0}^{\infty} \lambda^2(2^p)[s_{2^p,2^p}(x) - \sigma_{2^p,2^p}(x)]^2$$

and use representation (4.8) for $p = q$. After integrating,

$$\int g_8^2(x) d\mu(x) = \sum_{p=0}^{\infty} \lambda^2(2^p) \int \left[s_{2^p,2^p}(x) - \sigma_{2^p,2^p}(x) \right]^2 d\mu(x)$$
$$= \sum_{p=0}^{\infty} \lambda(2^p) \sum_{i=1}^{2^p} \sum_{k=1}^{2^p} \left[\frac{i-1}{2^p} \left(1 - \frac{k-1}{2^p} \right) + \frac{k-1}{2^p} \right]^2 a_{ik}^2 \leqq 2(I_3 + I_4) .$$

Here

$$I_3 := \sum_{p=0}^{\infty} \lambda^2(2^p) \sum_{i=1}^{2^p} \sum_{k=1}^{2^p} \left[\frac{i-1}{2^p} \left(1 - \frac{k-1}{2^p} \right) \right]^2 a_{ik}^2$$

$$\leqq \sum_{p=0}^{\infty} \lambda^2(2^p)2^{-2p} \sum_{i=2}^{2^p} \sum_{k=1}^{2^p} (i-1)^2 a_{ik}^2$$

$$= \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 (i-1)^2 \sum_{p:2^p \geqq \max(i,k)} \lambda^2(2^p)2^{-2p} < \infty ,$$

the last inequality follows from (2.5) and (4.2).

An analogous estimate is valid for I_4. □

LEMMA 7. *Under condition* (2.5) *with such a* $\{\lambda(m)\}$ *that satisfies* (4.2), *for every* $\theta > 1$ *we have*

$$M_p(x) := \max_{2^p < m \leqq 2^{p+1}} \max_{\theta^{-1} m \leqq n \leqq \theta m} |\sigma_{mn}(x) - \sigma_{2^p,2^p}(x)| = o_x\{\lambda^{-1}(2^p)\} \quad a.e.$$

and there exists $g(x) \in L^2$ *such that*

$$\lambda(2^p) M_p(x) \leqq g(x) \quad a.e. \quad (p = 0, 1, \cdots) .$$

PROOF. It is clear that

$$M_p(x) \leqq \max_{2^p < m \leqq 2^{p+1}} \max_{\theta^{-1} 2^p < n \leqq 2^p} |\sigma_{mn}(x) - \sigma_{2^p,2^p}(x)|$$
$$+ \max_{2^p < m \leqq 2^{p+1}} \max_{2^p < n \leqq \theta 2^{p+1}} |\sigma_{mn}(x) - \sigma_{2^p,2^p}(x)| =: M_p^{(1)}(x) + M_p^{(2)}(x) .$$

We treat here, say $M_p^{(2)}(x)$. The treatment of $M_p^{(1)}(x)$ is quite similar.

Now we estimate $M_p^{(2)}(x)$ in the same manner as we estimated $M_{pq}(x)$ in the proof of Lemma 4 (cf. (4.11)):

$$M_p^{(2)}(x) \leqq \max_{2^p < m \leqq 2^{p+1}} \max_{2^p < n \leqq \theta 2^{p+1}} |\sigma_{mn}(x) - \sigma_{m,2^p}(x)$$
$$- \sigma_{2^p,n}(x) + \sigma_{2^p,2^p}(x)| + \max_{2^p < m \leqq 2^{p+1}} |\sigma_{m,2^p}(x) - \sigma_{2^p,2^p}(x)|$$
$$+ \max_{2^p < n \leqq \theta 2^{p+1}} |\sigma_{2^p,n}(x) - \sigma_{2^p,2^p}(x)| =: M_p^{(21)}(x) + M_p^{(22)}(x) + M_p^{(23)}(x) .$$

Representation (4.12) and the Cauchy inequality make it possible to conclude that

$$(5.1) \qquad [M_p^{(21)}(x)]^2 \leqq \left[\sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^p+1}^{\theta 2^{p+1}} |\Delta_{11} \sigma_{mn}(x)| \right]^2$$
$$\leqq (2\theta - 1) 2^{2p} \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^p+1}^{\theta 2^{p+1}} [\Delta_{11} \sigma_{mn}(x)]^2 .$$

Setting

$$g_9^2(x) := \sum_{p=0}^{\infty} 2^{2p} \lambda^2(2^p) \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^p+1}^{\theta 2^{p+1}} [\Delta_{11} \sigma_{mn}(x)]^2 ,$$

we shall show that $g_9(x) \in L^2$. By (4.13),

$$\int g_9^2(x)d\mu(x)$$
$$= \sum_{p=0}^{\infty} 2^{2p}\lambda^2(2^p) \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^p+1}^{\theta 2^{p+1}} \sum_{i=2}^{m} \sum_{k=2}^{n} \frac{(i-1)^2(k-1)^2}{(m-1)^2 m^2 (n-1)^2 n^2} a_{ik}^2$$
$$\leqq \sum_{p=0}^{\infty} \lambda^2(2^p)2^{-6p} \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^p+1}^{\theta 2^{p+1}} \sum_{i=2}^{m} \sum_{k=2}^{n} i^2k^2 a_{ik}^2$$
$$\leqq (2\theta - 1) \sum_{p=0}^{\infty} \lambda^2(2^p)2^{-4p} \sum_{i=2}^{2^{p+1}} \sum_{k=2}^{\theta 2^{p+1}} i^2k^2 a_{ik}^2$$
$$= (2\theta - 1) \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 i^2 k^2 \sum_{p:2^{p+1} \geqq \max(i, \theta^{-1}k)} \lambda^2(2^p)2^{-4p} .$$

Denote by $p_0 = p_0(i, k, \theta)$ the integer, for which

$$2^{p_0} < \max(i, \theta^{-1}k) \leqq 2^{p_0+1} .$$

Then, by (4.2),

$$\sum_{p:2^{p+1} \geqq \max(i, \theta^{-1}k)} \lambda^2(2^p)2^{-4p} = \sum_{p=p_0}^{\infty} \lambda^2(2^p)2^{-4p} \tag{5.2}$$
$$\leqq 2^{-2p_0} \sum_{p=p_0}^{\infty} \lambda^2(2^p)2^{-2p} \leqq C\lambda^2(2^{p_0})2^{-4p_0}$$
$$\leqq 16C\theta^4\lambda^2(\max(i, k))(\max(i, k))^{-4} .$$

To sum up the reasonings above, we can see that

$$\int g_9^2(x)d\mu(x) \leqq C \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} a_{ik}^2 \lambda^2(\max(i, k)) < \infty .$$

From (5.1) and the definition of $g_9(x)$ it immediately follows that

$$M_p^{(21)}(x) = o_x\{\lambda^{-1}(2^p)\} \quad \text{a.e} .$$

and

$$\lambda(2^p)M_p^{(21)}(x) \leqq g_9(x) \quad \text{a.e.} \quad (p = 0, 1, \cdots) .$$

Now we proceed with the estimation of $M_p^{(22)}(x)$. Applying representation (4.14) and the Cauchy inequality:

$$[M_p^{(22)}(x)]^2 \leqq 2^p \sum_{m=2^p+1}^{2^{p+1}} [\varDelta_{10}\sigma_{m,2^p}(x)]^2 . \tag{5.3}$$

Setting

$$g_{10}^2(x) := \sum_{p=0}^{\infty} 2^p\lambda^2(2^p) \sum_{m=2^p+1}^{2^{p+1}} [\varDelta_{10}\sigma_{m,2^p}(x)]^2 ,$$

we have by (4.15)

$$\int g_{10}^2(x)d\mu(x) = \sum_{p=0}^{\infty} 2^p\lambda^2(2^p) \sum_{m=2^p+1}^{2^{p+1}} \sum_{i=2}^{m} \sum_{k=1}^{2^p} \left[\frac{i-1}{(m-1)m}\left(1 - \frac{k-1}{2^p}\right)\right]^2 a_{ik}^2$$

$$\leqq \sum_{p=0}^{\infty} 2^p \lambda^2(2^p) \sum_{m=2^p+1}^{2^{p+1}} \sum_{i=2}^{m} \sum_{k=1}^{2^p} \frac{i^2}{m^4} a_{ik}^2$$

$$\leqq \sum_{p=0}^{\infty} \lambda^2(2^p) 2^{-2p} \sum_{i=2}^{2^{p+1}} \sum_{k=1}^{2^p} i^2 a_{ik}^2$$

$$= \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 i^2 \sum_{p: 2^p \geqq \max(i/2, k)} \lambda^2(2^p) 2^{-2p}$$

$$\leqq C \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 \lambda^2(\max(i, k)) < \infty .$$

From here and (5.3) we get that

$$M_p^{(22)}(x) = o_x\{\lambda^{-1}(2^p)\} \quad \text{a.e.}$$

and

$$\lambda(2^p) M_p^{(22)}(x) \leqq g_{10}(x) \quad \text{a.e.} \quad (p = 0, 1, \cdots) .$$

Similar inequalities can be obtained for $M_p^{(23)}(x)$, too. □

6. **Proof of Theorem 3.** First we present two lemmas.

LEMMA 8. *If* $\{\lambda(m)\} \in \Lambda_2$, *then*

(6.1) $$m\lambda^{-1}(m) \to \infty \quad as \quad m \to \infty$$

and

(6.2) $$p^{-2} \sum_{m=1}^{p} m\lambda^{-2}(m) \leqq C\lambda^{-2}(p) \quad (p = 1, 2, \cdots) .$$

If $\{\lambda(m)\} \in \Lambda_{\sqrt{2}}$, *then*

(6.3) $$m\lambda^{-2}(m) \to \infty \quad as \quad m \to \infty$$

and

(6.4) $$p^{-1} \sum_{m=1}^{p} \lambda^{-2}(m) \leqq C\lambda^{-2}(p) \quad (p = 1, 2, \cdots) .$$

PROOF. As a matter of fact, the right inequality in (2.1) already implies (6.1)-(6.4). It is not so hard to show this and therefore it is omitted. □

LEMMA 9. (a) *Under condition* (2.5) *with such a* $\{\lambda(m)\}$ *that satisfies* (4.5), (6.1) *and* $\{m\lambda^{-1}(m)\}$ *is nondecreasing, for every* $\theta > 1$ *we have*

(6.5) $$m^{-2} \sum_{i=1}^{m} \sum_{k=\theta^{-1}i}^{\theta i} [s_{ik}(x) - \sigma_{ik}(x)]^2 = o_x\{\lambda^{-2}(m)\} \quad a.e. ,$$

and there exists $h(x) \in L^1$ *such that*

$$\lambda^2(m)m^{-2}\sum_{i=1}^{m}\sum_{k=\theta^{-1}i}^{\theta i}[s_{ik}(x)-\sigma_{ik}(x)]^2 \leqq h(x) \quad a.e. \quad (m=1,2,\cdots)\,.$$

(b) *Under condition* (2.5) *with such a* $\{\lambda(m)\}$ *that satisfies* (4.5), (6.3) *and* $\{m\lambda^{-2}(m)\}$ *is nondecreasing, for every* $\theta>1$ *we have*

$$m^{-1}\sum_{i=1}^{m} i^{-1}\sum_{i=\theta^{-1}i}^{\theta i}[s_{ik}(x)-\sigma_{ik}(x)]^2 = o_x\{\lambda^{-2}(m)\} \quad a.e.\,, \tag{6.6}$$

and there exists $h(x)\in L^1$ *such that*

$$\lambda^2(m)m^{-1}\sum_{i=1}^{m} i^{-1}\sum_{k=\theta^{-1}i}^{\theta i}[s_{ik}(x)-\sigma_{ik}(x)]^2 \leqq h(x) \quad a.e. \quad (m=1,2,\cdots)\,.$$

PROOF. Our first aim is to show that the function $h(x)$ defined by

$$h(x){:}=\sum_{m=1}^{\infty}\lambda^2(m)m^{-2}\sum_{n=\theta^{-1}m}^{\theta m}[s_{mn}(x)-\sigma_{mn}(x)]^2$$

is in L^1. To this end, we consider the termwise integrated series and use the representation corresponding to (4.8):

$$\begin{aligned}\int h(x)d\mu(x) &= \sum_{m=1}^{\infty}\lambda^2(m)m^{-2}\sum_{n=\theta^{-1}m}^{\theta m}\sum_{i=1}^{m}\sum_{k=1}^{n}\left[\frac{i-1}{m}\left(1-\frac{k-1}{n}\right)+\frac{k-1}{n}\right]^2 a_{ik}^2\\ &\leqq 2(I_5+I_6)\,.\end{aligned}$$

By (2.5) and (4.5),

$$\begin{aligned}I_5{:} &= \sum_{m=1}^{\infty}\lambda^2(m)m^{-2}\sum_{n=\theta^{-1}m}^{\theta m}\sum_{i=1}^{m}\sum_{k=1}^{n}\left[\frac{i-1}{m}\left(1-\frac{k-1}{n}\right)\right]^2 a_{ik}^2\\ &\leqq (\theta-\theta^{-1}+1)\sum_{m=1}^{\infty}\lambda^2(m)m^{-3}\sum_{i=2}^{m}\sum_{k=1}^{\theta m}(i-1)^2a_{ik}^2\\ &= (\theta-\theta^{-1}+1)\sum_{i=2}^{\infty}\sum_{k=1}^{\infty}a_{ik}^2(i-1)^2\sum_{m:\max(i,\theta^{-1}k)\leqq m}\lambda^2(m)m^{-3}<\infty\,,\end{aligned}$$

where the last sum can be treated similarly to (5.2). In the same way, we get that

$$\begin{aligned}I_6{:} &= \sum_{m=1}^{\infty}\lambda^2(m)m^{-2}\sum_{n=\theta^{-1}m}^{\theta m}\sum_{i=1}^{m}\sum_{k=1}^{n}(k-1)^2n^{-2}a_{ik}^2\\ &\leqq \theta^2(\theta-\theta^{-1}+1)\sum_{i=1}^{\infty}\sum_{k=2}^{\infty}a_{ik}^2(k-1)^2\sum_{m:\max(i,\theta^{-1}k)\leqq m}\lambda^2(m)m^{-3}<\infty\,.\end{aligned}$$

Now it remains to apply the well-known Kronecker lemma (see, e.g. [2, p. 72]), while taking into account (6.1) and (6.3), respectively. □

After these prerequisitities we can complete the proof of Theorem 3 as follows. By Theorem 2, in case $\{\lambda(m)\}\in\Lambda_2$ and $\theta>1$, the differences $\sigma_{ik}(x)-f(x)$ are of the order of magnitude $o_x\{\lambda^{-1}(i)\}$ a.e. and $\lambda(i)|\sigma_{ik}(x)-$

$f(x)|$ is majorized by some $g(x) \in L^2$ a.e., provided $\theta^{-1} \leqq k/i \leqq \theta$. Consequently,

$$i^{-1} \sum_{k=\theta^{-1}i}^{\theta i} [\sigma_{ik}(x) - f(x)]^2 = o_x\{\lambda^{-2}(i)\} \quad \text{a.e.} \tag{6.7}$$

Forming again the first arithmetic mean, this time with respect to i, we find that

$$\begin{aligned} m^{-1} \sum_{i=1}^{m} i^{-1} \sum_{k=\theta^{-1}i}^{\theta i} [\sigma_{ik}(x) - f(x)]^2 \\ = m^{-1} \sum_{i=1}^{m} o_x\{\lambda^{-2}(i)\} = o_x\{\lambda^{-2}(m)\} \quad \text{a.e.}\ , \end{aligned} \tag{6.8}$$

but here we have to assume the fulfilment of (6.4), i.e. that $\{\lambda(m)\} \in \Lambda_{\sqrt{2}}$.

In the case when only $\{\lambda(m)\} \in \Lambda_2$, by (6.2) and (6.7),

$$\begin{aligned} m^{-2} \sum_{i=1}^{m} \sum_{k=\theta^{-1}i}^{\theta i} [\sigma_{ik}(x) - f(x)]^2 \\ = m^{-2} \sum_{i=1}^{m} o_x\{i\lambda^{-2}(i)\} = o_x\{\lambda^{-2}(m)\} \quad \text{a.e.} \end{aligned} \tag{6.9}$$

Now putting (6.5) and (6.9) together, we find (2.8); and putting (6.6) and (6.8) together, we find (2.9).

It is quite obvious that

$$m^{-1} \sum_{i=1}^{m} i^{-1} \sum_{k=\theta^{-1}i}^{\theta i} [\sigma_{ik}(x) - f(x)]^2 \leqq (\theta - \theta^{-1} + 1) g^2(x) \quad \text{a.e.} \quad (m = 1, 2, \cdots)\ .$$

7. Proofs of Theorems 5–7.

We set

$$A_r = \left\{ \sum_{(i,k) \in Q_r \backslash Q_{r-1}} a_{ik}^2 \right\}^{1/2}$$

and

$$\Phi_r(x) = \begin{cases} A_r^{-1} \sum_{(i,k) \in Q_r \backslash Q_{r-1}} a_{ik} \varphi_{ik}(x) & \text{if} \quad A_r \neq 0\ , \\ |Q_r \backslash Q_{r-1}|^{-1/2} \sum_{(i,k) \in Q_r \backslash Q_{r-1}} \varphi_{ik}(x) & \text{if} \quad A_r = 0\ , \end{cases}$$

where by $|Q_r \backslash Q_{r-1}|$ we denote the number of the lattice points $(i, k) \in N^2$ contained in $Q_r \backslash Q_{r-1}$ $(r = 1, 2, \cdots)$.

It is obvious that $\{\Phi_r(x): r = 1, 2, \cdots\}$ is an (ordinary) single orthonormal system and by (3.1)

$$\sum_{r=1}^{\infty} A_r^2 \lambda^2(r) < \infty\ .$$

Thus, we can apply the relevant generalizations of the results of [6] and [8] in order to conclude Theorems 5 and 6.

If merely condition (1.2) is satisfied, then

$$\sum_{r=1}^{\infty} A_r^2 < \infty$$

and by applying the result of [4] we obtain Theorem 7.

8. **Extension to multiparameter case.** Let N^d be the set of d-tuples $k = (k_1, \cdots, k_d)$ with positive integers for coordinates, where d is a fixed positive integer. Let $\{\varphi_k(x): k \in N^d\}$ be an orthonormal system on the measure space $(X, \mathscr{F}, \mu)$. We consider the d-multiple orthogonal series

$$\sum_{k \in N^d} a_k \varphi_k(x) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} a_{k_1,\cdots,k_d} \varphi_{k_1,\cdots,k_d}(x) , \tag{1.1'}$$

where $\{a_k: k \in N^d\}$ is a d-multiple sequence of real numbers, for which

$$\sum_{k \in N^d} a_k^2 < \infty . \tag{1.2'}$$

By the Riesz-Fischer theorem there exists a function $f(x) \in L^2(X, \mathscr{F}, \mu)$ such that the rectangular partial sums of (1.1′) defined by

$$s_n(x) = \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} a_k \varphi_k(x) \quad (n \in N^d)$$

converge to $f(x)$ in L^2-metric:

$$\int [s_n(x) - f(x)]^2 d\mu(x) \to 0 \quad \text{as} \quad \min_{1 \leq j \leq d} n_j \to \infty .$$

Denote by $\sigma_n(x)$ the first arithmetic means of the rectangular partial sums:

$$\begin{aligned}\sigma_n(x) &= \left(\prod_{j=1}^{d} n_j^{-1}\right) \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} s_k(x) \\ &= \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} \left[\prod_{j=1}^{d} \left(1 - \frac{k_j - 1}{n_j}\right)\right] a_k \varphi_k(x) \quad (n \in N^d) .\end{aligned}$$

Given two d-multiple sequence $\{\kappa(n): n \in N^d\}$ and $\{\lambda(n): n \in N^d\}$ of real numbers, the relation

$$\kappa(n) = o\{\lambda(n)\}$$

is defined by the requirements that

$$\kappa(n)/\lambda(n) \to 0 \quad \text{as} \quad \min_{1 \leq j \leq d} n_j \to \infty$$

(including the assumption that $\lambda(n) \neq 0$ if each n_j is large enough) and

$$|\kappa(n)| \leqq C |\lambda(n)| \quad (n \in N^d) .$$

Now, the extensions of Theorems 1-4 read as follows.

THEOREM 1′. *If*

$$\sum_{k\in N^d} a_k^2 \prod_{j=1}^{d} \lambda_j^2(k_j) < \infty ,$$

where each $\{\lambda_j(k_j): k_j = 1, 2, \cdots\}$ *belongs to* Λ_2, $1 \leqq j \leqq d$, *then*

$$\sigma_n(x) - f(x) = o_x\left\{\max_{1\leqq j\leqq d} \lambda_j^{-1}(n_j)\right\} \quad a.e. ,$$

and there exists a function $g(x) \in L^2(X, \mathscr{F}, \mu)$ *such that*

$$\min_{1\leqq j\leqq d}\{\lambda_j(n_j)\}|\sigma_n(x) - f(x)| \leqq g(x) \quad a.e. \quad (n \in N^d) .$$

THEOREM 2′. *If*

$$\sum_{k\in N^d} a_k^2 \lambda^2\left(\max_{1\leqq j\leqq d} k_j\right) < \infty , \tag{2.5′}$$

where $\{\lambda(n_1): n_1 = 1, 2, \cdots\} \in \Lambda_2$, *then for every* $\theta > 1$ *we have*

$$\max_{n_2:\theta^{-1}\leqq n_2/n_1\leqq\theta} \cdots \max_{n_d:\theta^{-1}\leqq n_d/n_1\leqq\theta} |\sigma_n(x) - f(x)| = o_x\{\lambda^{-1}(n_1)\} \quad a.e. , \tag{2.6′}$$

and there exists a function $g(x) \in L^2(X, \mathscr{F}, \mu)$ *such that the left-hand side of* (2.6′) *multiplied by* $\lambda(n_1)$ *does not exceed* $g(x)$ *a.e.* $(n_1 = 1, 2, \cdots)$.

THEOREM 3′. *If condition* (2.5′) *is satisfied with* $\{\lambda(n_1)\} \in \Lambda_2$ *and* $\{n_1\lambda^{-1}(n_1)\}$ *is nondecreasing, then for every* $\theta > 1$ *we have*

$$n_1^{-d} \sum_{k_1=1}^{n_1} \sum_{k_2=\theta^{-1}k_1}^{\theta k_1} \cdots \sum_{k_d=\theta^{-1}k_1}^{\theta k_1} [s_k(x) - f(x)]^2 = o_x\{\lambda^{-2}(n_1)\} \quad a.e. \tag{2.8′}$$

If (2.5′) *is satisfied with* $\{\lambda(n_1)\} \in \Lambda_{\sqrt{2}}$ *and* $\{n_1\lambda^{-2}(n_1)\}$ *is nondecreasing, then*

$$n_1^{-1} \sum_{k_1=1}^{n_1} k_1^{-(d-1)} \sum_{k_2=\theta^{-1}k_1}^{\theta k_1} \cdots \sum_{k_d=\theta^{-1}k_1}^{\theta k_1} [s_k(x) - f(x)]^2 = o_x\{\lambda^{-2}(n_1)\} \quad a.e. \tag{2.9′}$$

Furthermore, there exists a function $h(x) \in L^1(X, \mathscr{F}, \mu)$ *such that the left-hand sides of* (2.8′) *and* (2.9′) *both multiplied by* $\lambda^2(n_1)$ *do not exceed* $h(x)$ *a.e.* $(n_1 = 1, 2, \cdots)$.

THEOREM 4′. *If condition* (1.2′) *is satisfied and* $\sigma_n(x)$ *converges to* $f(x)$ *a.e. as* $\min_{1\leqq j\leqq d} n_j \to \infty$, *then for every* $\theta > 1$ *the left-hand side of* (2.9′) *is* $o_x\{1\}$ *a.e. and does not exceed a function* $h(x) \in L^1(X, \mathscr{F}, \mu)$ *a.e.* $(n_1 = 1, 2, \cdots)$.

The proofs can be carried out in a similar manner to those of Theorems 1-4, only the technical details will become somewhat more complicated.

Quite analogously, one can extend Theorems 5–7 from $d = 2$ to general d, too. Let $Q = \{Q_r\colon r = 1, 2, \cdots\}$ be a nondecreasing sequence of finite sets in N^d, whose union is N^d. If we write

$$s_r(Q; x) = \sum_{k \in Q_r} a_k \varphi_k(x) ,$$

$$\sigma_r(Q; x) = r^{-1} \sum_{\rho=1}^{r} s_\rho(Q; x)$$

$$= \sum_{\rho=1}^{r} \left(1 - \frac{\rho - 1}{r}\right) \sum_{k \in Q_\rho \setminus Q_{\rho-1}} a_k \varphi_k(x) \quad (r = 1, 2, \cdots; Q_0 = \emptyset) ,$$

and

$$\sum_{r=1}^{\infty} \left(\sum_{k \in Q_r \setminus Q_{r-1}} a_k^2 \right) \lambda^2(r) < \infty , \tag{3.1'}$$

then Theorems 5–7 in the form as they are stated in Section 3 remain valid for arbitrary $d \geqq 1$.

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