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NONLINEAR SEMIGROUPS AND A CHARACTERIZATION OF THE VALUE PROCESS IN STOCHASTIC CONTROL

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

HIROAKI MORIMOTO AND MASATAKA IZUMISAWA

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1. Introduction. Let (Ω, F, P) be a complete probability space equipped with an increasing, right continuous family $\mathscr{F}$ of complete sub-σ-fields $(F_t)_{t\geqq 0}$ such that $F = \bigvee_{t\geqq 0} F_t$. Let us denote, by $\mathscr{O}(\mathscr{F})$, $\mathscr{T}(\mathscr{F})$ and $\mathscr{U}$, the $\mathscr{F}$-optional σ-field, the class of all $\mathscr{F}$-stopping times and the set of $\mathscr{F}$-progressively measurable processes $u = \{u(t); t \geqq 0\}$ with values in a σ-compact subset of $\boldsymbol{R}^d$, respectively. We define a stopped process u^S of u at time $S \in \mathscr{T}(\mathscr{F})$, $\mathscr{U}(u, S)$, a restriction S_A of S to an $\mathscr{F}_S$-measurable set A and a concatenation $u\cdot S\cdot v$ as follows:

$$u^S = \{u(t \wedge S); t \geqq 0\}\ , \qquad \mathscr{U}(u, S) = \{v \in \mathscr{U}; v^S = u^S\}\ ,$$

$$S_A = \begin{cases} S & \text{on } A \\ \infty & \text{on } A^c \end{cases} \quad \text{and} \quad u\cdot S\cdot v = uI_{[0,S]} + vI_{]S,\infty[} \text{ for } u, v \in \mathscr{U}\ .$$

Now we consider a subclass $\mathscr{D}$ of $\mathscr{U}$ as an admissible control. $\mathscr{D}$ is assumed to be stable for a stopping and a concatenation. Furthermore, we put the following assumptions on $\mathscr{D}$:

(A.1) For each admissible control u of $\mathscr{D}$, there exists a right continuous $(\mathscr{F}, P)$-local martingale N^u such that its jumps are strictly greater than -1 and $N^u(0) = 0$.

(A.2) $N^u(t \wedge S) = N^v(t \wedge S)$ if $v \in \mathscr{D}(u, S) = \mathscr{D} \cap \mathscr{U}(u, S)$ and
$N^{u\cdot S\cdot w}(t \vee S) - N^u(S) = N^{v\cdot S\cdot w}(t \vee S) - N^v(S)$ for
$u, v, w \in \mathscr{D}$, $S \in \mathscr{T}(\mathscr{F})$.

(A.3) Let z^u be an exponential local martingale with respect to N^u, i.e.,

$$z^u(t) = \mathscr{E}(N^u)(t) = \exp\{N^u(t) - (1/2) < (N^u)^c > (t)\} \prod_{s\leqq t} (1 + \Delta N^u(s))e^{-\Delta N^u(s)}$$

where $\langle (N^u)^c \rangle$ is the $\mathscr{F}$-predictable increasing process associated with the continuous part $(N^u)^c$ of N^u and $\Delta N^u(s) = N^u(s) - N^u(s-)$. Then there exists a constant $p > 1$ such that $\sup_{u\in\mathscr{D}} \|\sup_t |z^u(t)|\, \|_{L^p(P)} < \infty$.

From (A.3), z^u is a strictly positive and uniformly P-integrable martingale and we can define the probability measure P^u which has density z^u with respect to P and which is equivalent to P for each $u \in \mathscr{D}$. We define the cost associated with $u \in \mathscr{D}$. Let $c^u = \{c^u(t); t \geqq 0\}$ be a right continuous $\mathscr{F}$-adapted P^u-integrable increasing process. We consider $c^u(\infty)$ as the loss function, $c^u(t)$ as the evolution cost associated to the policy of control u. We suppose the following conditions on c^u:

(C.1) $c^u(t \wedge S) = c^v(t \wedge S)$ if $v \in \mathscr{D}(u, S)$ and

$$c^{u \cdot S \cdot w}(t \vee S) - c^u(S) = c^{v \cdot S \cdot w}(t \vee S) - c^v(S) \quad \text{for} \quad u, v, w \in \mathscr{D},\ S \in \mathscr{T}(\mathscr{F}) .$$

(C.2) $\sup_{u \in \mathscr{D}} E^u[c^u(s) - c^u(s_0)] \to 0$ as $s > s_0$ and $s \to s_0$ where $E^u[\ \ ]$ denotes the expectation with respect to P^u.

(C.3) The difference of c^u is exponentially uniformly bounded in $S \in \mathscr{T}(\mathscr{F})$ and in $u \in \mathscr{D}$, i.e., there exist constants $a > 0$ and $C(a)$, which depends on a, such that $0 \leqq c^u(\infty) - c^u(S) \leqq C(a)e^{-aS}$.

The cost $\Gamma(u)$, the conditional cost $\Gamma(u, S)$ and the conditional minimal cost $J(u, S)$ are then defined as follows:

$$\Gamma(u) = E^u[c^u(\infty)],\ \Gamma(u, S) = E^u[c^u(\infty) \mid F_S] \quad \text{and}$$
$$J(u, S) = \operatorname{ess\,inf}\{\Gamma(v, S); v \in \mathscr{D}(u, S)\} .$$

Under some suitable assumptions on N^u and c^u, Benes [1], Brémaud [3] and Duncan-Varaiya [5] showed that there exists an optimal control $u^* \in \mathscr{D}$, i.e., $\Gamma(u^*) \leqq \Gamma(u)$ for all $u \in \mathscr{D}$, or equivalently, $J(u^*, S) = \Gamma(u^*, S)$ for all $S \in \mathscr{T}(\mathscr{F})$. It is known (see El-Karoui [6; Chap. 3]) that there exists a right continuous $\mathscr{F}$-optional process W, called the value process, such that $W(S) = \operatorname{ess\,inf}\{E^u[c^u(\infty) - c^u(S) \mid F_S]; u \in \mathscr{D}\}$ P-a.s. on $\{S < \infty\}$

(i) $c^u(S) + W(S) = J(u, S)$ P-a.s. for all $S \in \mathscr{T}(\mathscr{F})$.

(ii) $E^u[W(0)] = \inf\{J(u, 0); u \in \mathscr{D}\}$.

(iii) $c^u + W$ is a positive P^u-submartingale.

(iv) u^* is an optimal control if and only if $c^{u^*} + W$ is a positive P^{u^*}-martingale.

In this paper, our aim is to characterize the value process by the method of nonlinear semigroups of conditioned shifts in non-Markovian case. We base ourselves on martingale theory, Bellman's principle and Nisio's results [7] (see also Bensoussan [2]) of nonlinear semigroups in the control of Markov processes.

Let us denote by $\mathscr{X}$ the Banach space of all essentially bounded right continuous $\mathscr{F}$-adapted process $x = \{x(t); t \geqq 0\}$ with its norm $\|x\| = \|\sup_t |x(t)|\|_{L^\infty(P)} < \infty$ and with the usual order. Let Φ be a subclass of

$\mathscr{X}$ such that $\{c^u(t) + e^{-at}x(t)\}$ is a right continuous P^u-submartingale for each $u \in \mathscr{D}$. We seek a semigroup $\{G_h\}$ of operators acting on Φ whose fixed point is equal to the value process. Such a semigroup will be obtained as the envelope of the semigroups $\{K_h^u\}$ of conditioned shifts on $\mathscr{X}$ where $K_h^u x$ is defined as the $\mathscr{F}$-optional P^u-projection of the process $\{e^{at}(c^u(t+h) - c^u(t)) + e^{-ah}x(t+h)\}$. In fact we prove the following theorems in § 4.

THEOREM 1. *There exists a nonlinear semigroup* $\{G_h; h \geqq 0\}$ *on* Φ *satisfing the following conditions*:

(i) *semigroup property*; $G_0 x = x$, $G_{h+k}x = G_h G_k x = G_k G_h x$.

(ii) *monotone*; $G_h x \leqq G_h y$ *whenever* $x \leqq y$.

(iii) *contractive*; $\|G_h x - G_h y\| \leqq e^{-ah}\|x - y\|$.

(iv) $K_h^u x \geqq G_h x$ *for all* $u \in \mathscr{D}$.

(v) *maximal*; *Let* $\{H_h; h \geqq 0\}$ *be a semigroup on* Φ *which satisfies* (i)-(iv). *Then* $H_h x \leqq G_h x$.

THEOREM 2. *There exists a unique solution* $x^* \in \Phi$ *such that* $G_h x^* = x^*$ *for all* $h > 0$ *and that* x^* *is identical with the process* $\{a^{at}W(t); t \geqq 0\}$. *Furthermore* x^* *is a maximal element of* Φ.

2. **Preliminaries.** (1) The optional projection. Throughout this paper, we identify, as usual, two indistinguishable processes. So we have the following lemma concerning the optional projection of processes.

LEMMA 1. (Dellacherie-Meyer [4; VI-43, 47]). *For a measurable bounded process* x, *there exists a unique optional process* y *such that*

$$E[x(T)\,|\,F_T] = y(T) \quad P\text{-a.s. } on \quad \{T < \infty\} \quad for\ all \quad T \in \mathscr{T}(\mathscr{F})\,.$$

Furthermore, if x *is right continuous, then so is* y.

This process y is called the optional projection of x.

(2) The formula of Bayes' type. An easy calculation shows the following formula of Bayes' type:

$$E^u[X\,|\,F_T] = E[X(z^u(\infty)/z^u(T))\,|\,F_T] \quad P\text{-a.s.} \tag{2.1}$$

for every essentially bounded random variable X. For simplicity we omit P-a.s. in inequalities or in equalities from now on, if no confusion occurs. Using (2.1), we can calculate the expectation with respect to the probability law induced by a concatenation control. From (A.2), we get $N^{u\cdot S\cdot v}(t) = N^{u\cdot S\cdot v}(t \wedge S) + N^{u\cdot S\cdot v}(t \vee S) - N^{u\cdot S\cdot v}(S) = N^u(t \wedge S) + N^v(t \vee S) - N^v(S) = (I_{[0,S]}\circ N^u)(t) + (I_{]S,\infty[}\circ N^v)(t)$ for $u, v \in \mathscr{D}$ where $H\circ N$ denotes the stochastic integral of a predictable process H relative to a local martingale N. Since $I_{[0,S]}\circ N^u$ is orthogonal to $I_{]S,\infty[}\circ N^v$, we have $z^{u\cdot S\cdot v}(t)=\mathscr{E}(N^{u\cdot S\cdot v})(t)=\mathscr{E}(I_{[0,S]}\circ N^u)(t)\ \mathscr{E}(I_{]S,\infty[}\circ N^v)(t)=z^u(t\wedge S)z^v(t\vee S)/z^v(S)$. Hence we get

$$z^{u\cdot S\cdot v}(T)=\begin{cases} z^u(S)z^v(T)/z^v(S) & \text{if } T\geqq S \\ z^u(T) & \text{if } T<S\,, \end{cases} \tag{2.2}$$

especially, $z^{u\cdot S\cdot v}(S)=z^u(S)$ and $z^{u\cdot S\cdot v}(\infty)=z^u(S)z^v(\infty)/z^v(S)$. Let X be an essentially bounded random variable. Then we have

$$E^{u\cdot S\cdot v}[X\,|\,F_T]=\begin{cases} E^v[X\,|\,F_T] & \text{if } T\geqq S \\ E^u[E^v[X\,|\,F_S]\,|\,F_T] & \text{if } T<S \end{cases} \tag{2.3}$$

and $E^{u\cdot S\cdot v}[X]=E^u[E^v[X\,|\,F_S]]$. Indeed, if $T\geqq S$, we have

$$\begin{aligned} E^{u\cdot S\cdot v}[X\,|\,F_T] &= E[Xz^{u\cdot S\cdot v}(\infty)/z^{u\cdot S\cdot v}(T)\,|\,F_T] \\ &= E[Xz^v(\infty)/z^v(T)\,|\,F_T]=E^v[X\,|\,F_T] \end{aligned}$$

by (2.1) and (2.2). Similarly, if $T<S$, we get

$$\begin{aligned} E^{u\cdot S\cdot v}[X\,|\,F_T] &= E[Xz^v(\infty)z^u(S)/z^v(S)z^u(T)\,|\,F_T] \\ &= E[E[Xz^v(\infty)/z^v(S)\,|\,F_S]z^u(S)/z^u(T)\,|\,F_T]=E^u[E^v[X\,|\,F_S]\,|\,F_T]\,. \end{aligned}$$

Furthermore, we get

$$E^{u\cdot S_A\cdot v}[X\,|\,F_S]=E^v[X\,|\,F_S]I_A+E^u[X\,|\,F_S]I_{A^c} \quad \text{for} \quad A\in F_S\,. \tag{2.4}$$

In fact, since $A\cap\{S_A\leqq t\}=A\cap\{S\leqq t\}$ and $A^c\cap\{S_A\leqq t\}=\varnothing$, we get $B\cap A^c\in F_{S_A}$ and $C\cap A\in F_S$ for $B\in F$ and $C\in F_{S_A}$. So we have

$$\begin{aligned} E^v[X\,|\,F_{S_A}] &= E^v[XI_A\,|\,F_{S_A}]+XI_{A^c}=E^v[X\,|\,F_{S_A}]I_A+XI_{A^c} \\ &= E^v[E^v[X|F_{S_A}]I_A\,|\,F_S]+XI_{A^c}=E^v[X\,|\,F_S]I_A+XI_{A^c} \end{aligned}$$

and

$$\begin{aligned} E^{u\cdot S_A\cdot v}[X\,|\,F_S] &= E^u[E^v[X\,|\,F_{S_A}]\,|\,F_S]=E^u[E^v[X\,|\,F_S]I_A+XI_{A^c}\,|\,F_S] \\ &= E^v[X\,|\,F_S]I_A+E^u[X\,|\,F_S]I_{A^c}\,. \end{aligned}$$

(3) The essential infimum.

LEMMA 2. (El-Karoui [6; Appendix]) (i) *For each family* $\{Y^i;\,i\in I\}$ *of random variables, there exists a random variable* Y *such that*

(a) $Y^i\geqq Y$ P-a.s. *for all* $i\in I$.

(b) *If* Z *is a random variable such that* $Z\leqq Y^i$ P-a.s. *for all* i, *then* $Z\leqq Y$ P-a.s.

This Y, *which is the greatest lower bounded of the family* $\{Y^i;\,i\in I\}$ *in the sense of* P-a.s. *inequality, is denoted by* $P\text{-ess inf}\,Y^i$.

Further there exists at least one countable sequence $\{Y^n.\ n\in N\}$ *taken* *from* $\{Y^i\}$ *such that* P-ess inf $Y^i = \inf_N Y^n$ P-a.s.

(ii) *If the family* $\{Y^i; i \in I\}$ *is directed downwards, the sequence* $\{Y^n; n \in N\}$ *can be chosen to be decreasing* P-a.s., P-ess inf $Y^i = \lim_n \downarrow Y^n$ P-a.s. *and* $E[\text{ess inf } Y^i | G] = P\text{-ess inf } E[Y^i | G]$ *for every sub-*σ*-field* G *of* F.

3. Semigroups of conditioned shifts and its envelope. (1) First, we consider the semigroups $\{K_h^u; h \geqq 0\}$ of conditioned shifts. By the definition of $K_h^u x$, we have the following relation:

$$K_h^u x(T) = E^u[e^{aT}\{c^u(T+h) - c^u(T)\} + e^{-ah}x(T+h) | F_T] \tag{3.1}$$

P-a.s. on $\{T < \infty\}$ for $T \in \mathscr{T}(\mathscr{F})$. Also we get the following proposition.

PROPOSITION. *Let* $h, k \in [0, \infty)$.

(i) K_h^u *is an operator on* $\mathscr{X}$.

(ii) *semigroup property*; $K_0^u x = x$, $K_{h+k}^u x = K_h^u K_k^u x = K_k^u K_h^u x$.

(iii) *monotone*; $K_h^u x \leqq K_h^u y$ *whenever* $x \leqq y$.

(iv) *contractive*; $\| K_h^u x - K_h^u y \| \leqq e^{-ah} \|x - y\|$.

PROOF. (i) From Lemma 1, $K_h^u x$ is right continuous. Since z^u is a P-uniformly integrable martingale, we get

$$\begin{aligned}
&\|\sup_t |K_h^u x(t)|\|_{L^\infty(P)} \\
&\quad = \|\sup_t |E^u[e^{at}\{c^u(t+h) - c^u(t)\} + e^{-ah}x(t+h) | F_t]\|_{L^\infty(P)} \\
&\quad \leqq \|\sup_t E[\{C(a) + \|x\|\}\{z^u(\infty)/z^u(t)\} | F_t]\|_{L^\infty(P)} = C(a) + \|x\| < \infty
\end{aligned}$$

by (C.3) and the formula of Bayes' type.

(ii) It is trivial that K_0^u is the identity operator. For $T \in \mathscr{T}(\mathscr{F})$, we have

$$\begin{aligned}
e^{-aT}K_{h+k}^u x(T) &= E^u[c^u(T+h+k) - c^u(T) + e^{-a(T+h+k)}x(T+h+k) | F_T] \\
&= E^u[c^u(T+h) - c^u(T) + e^{-a(T+h)}E^u[e^{a(T+h)}\{c^u(T+h+k)\} - c^u(T+h)\} \\
&\qquad\qquad + e^{-ak}x(T+h+k) | F_{T+h}] | F_T] \\
&= e^{-aT}E^u[e^{aT}\{c^u(T+h) - c^u(T)\} + e^{-ah}K_k^u x(T+h) | F_T] = e^{-aT}K_h^u K_k^u x(T) .
\end{aligned}$$

(iii) is immediate from (3.1). For (iv), we have

$$\begin{aligned}
\|\sup_t |K_h^u x(t) - K_h^u y(t)|\|_{L^\infty(P)} &\leqq \|\sup_t E^u[e^{-ah}|x(t+h) - y(t+h)| | F_t]\|_{L^\infty(P)} \\
&= \|\sup_t E[e^{-ah}|x(t+h) - y(t+h)|(z^u(t+h)/z^u(t)) | F_t]\|_{L^\infty(P)} \\
&\leqq e^{-ah}\|x - y\| .
\end{aligned}$$

(2) Before we define the envelope of $\{K_h^u\}$, we show that Φ is closed in $\mathscr{X}$. Indeed, let $x_n \in \Phi$ and $x_n \to x$ in $\mathscr{X}$. Then we have

$\sup_n |x_n(t)| \leqq \sup_n \|x_n\| < \infty$ P-a.s. and $\lim_{n\to\infty} x_n(t) = x(t)$ P-a.s. for each t. By the submartingale property, we get $E^u[c^u(t) + e^{-at}x_n(t) | F_s] \geqq c^u(s) + e^{-as}x_n(s)$ for $t \geqq s$. Letting $n \to \infty$, we obtain $E^u[c^u(t) + e^{-at}x(t) | F_s] \geqq c^u(s) + e^{-as}x(s)$ by the boundedness of $\{x_n\}$. Therefore $x \in \Phi$. We remark that non-positive constant processes $\{x(t) = b; t \geqq 0\}$ with $b \leqq 0$ belong to Φ, for the process c^u is increasing.

(3) Now we define the envelope G_h of $\{K_h^u\}$ as follows:

$$G_h x(T) = P\text{-ess}\inf_{u \in \mathscr{D}} K_h^u x(T) \quad \text{for} \quad x \in \mathscr{X},\ T \in \mathscr{T}(\mathscr{F}) .$$

Clearly, $\{G_h x(t); t \geqq 0\}$ is an $\mathscr{F}$-adapted process.

LEMMA 3. (The Bellman principle) *Let* $x \in \mathscr{X}$, $T \in \mathscr{T}(\mathscr{F})$ *and* $h, k \in [0, \infty)$. *Then*

$$G_{h+k} x(T) = P\text{-ess}\inf_{u \in \mathscr{D}} E^u[e^{aT}\{c^u(T+k) - c^u(T)\} + e^{-ak} G_h x(T+k) | F_T]$$

on $\{T < \infty\}$.

PROOF. Let us fix $T \in \mathscr{T}(\mathscr{F})$. From the semigroup property of $\{K_h^u\}$, we have $K_{h+k}^u x(T) = K_k^u K_h^u x(T) \geqq E^u[e^{aT}\{c^u(T+k) - c^u(T)\} + e^{-ak} G_h x(T+k) | F_T]$ and so $G_{h+k} x(T) \geqq P\text{-ess}\inf E^u[e^{aT}\{c^u(T+k) - c^u(T)\} + e^{-ak} G_h x(T+k) | F_T]$. To prove the reverse inequality, we first show that $\{K_h^u x(S); u \in \mathscr{D}\}$ is directed downwards for $S \in \mathscr{T}(\mathscr{F})$. Put $A = \{K_h^u x(S) \geqq K_h^v x(S)\}$ and $w = u \cdot S_A \cdot v$. From (C.1), we see that $c^w(S+h) - c^w(S) = c^w(S_A + h) - c^w(S_A) = c^v(S_A + h) - c^v(S_A) = c^v(S+h) - c^v(S)$ on A and $c^w(S+h) - c^w(S) = c^w((S+h) \wedge S_A) - c^w(S \wedge S_A) = c^u(S+h) - c^u(S)$ on A^c. Using (2.4) and the above relations, we have

$$\begin{aligned} K_h^w x(S) &= E^w[e^{aS}\{c^w(S+h) - c^w(S)\} + e^{-ah} x(S+h) | F_S] \\ &= E^v[e^{aS}\{c^v(S+h) - c^v(S)\} + e^{-ah} x(S+h) | F_S] I_A \\ &\quad + E^u[e^{aS}\{c^u(S+h) - c^u(S)\} + e^{-ah} x(S+h) | F_S] I_{A^c} \\ &= K_h^v x(S) I_A + K_h^u x(S) I_{A^c} = K_h^v x(S) \wedge K_h^u x(S) . \end{aligned}$$

By virtue of Lemma 2-(ii), we can choose a countable sequence $\{u_n\}$ of $\mathscr{D}$ such that $G_h x(T+k) = \lim_{n\to\infty} K_h^{u_n} x(T+k)$. Hence we get

$$\begin{aligned} &G_{h+k} x(T) \\ &\quad \leqq E^{v\cdot(T+k)\cdot u_n}[e^{aT}\{c^{v\cdot(T+k)\cdot u_n}(T+h+k) - c^{v\cdot(T+k)\cdot u_n}(T)\} + e^{-a(h+k)} x(T+h+k) | F_T] \\ &\quad = E^{v\cdot(T+k)\cdot u_n}[e^{aT}\{c^v(T+k) - c^v(T)\} + e^{-ak} E^{v\cdot(T+k)\cdot u_n}[e^{a(T+k)}\{c^{u_n}(T+h+k) \\ &\qquad\qquad - c^{u_n}(T+k)\} + e^{-ah} x(T+h+k) | F_{T+k}] | F_T] \\ &\quad = E^v[e^{aT}\{c^v(T+k) - c^v(T)\} + e^{-ak} E^{u_n}[e^{a(T+k)}\{c^{u_n}(T+k+h) - c^{u_n}(T+k)\} \\ &\qquad\qquad + e^{-ah} x(T+k+h) | F_{T+k}] | F_T] . \end{aligned}$$

Letting $n\to\infty$, we have $G_{h+k}x(T)\leqq E^v[e^{aT}\{c^v(T+k)-c^v(T)\}+e^{-ak}G_hx(T+k)|F_T]$ which completes the proof.

4. Proof of theorems. (1) Proof of Theorem 1. Let x be an element of Φ. We first note that the mapping $t\to\|G_hx(t)\|_{L^\infty(P)}$ is bounded for each $h\in[0,\infty)$. We next show that $G_hx(T)\leqq G_{h+k}x(T)$ on $\{T<\infty\}$ for each $x\in\Phi$, $T\in\mathscr{T}(\mathscr{F})$. Since $\{c^u(t)+e^{-at}x(t)\}$ is an essentially bounded P^u-submartingale, we get $E^u[c^u(T+h+k)+e^{-a(T+h+k)}x(T+h+k)|F_{T+k}]\geqq c^u(T+h)+e^{-a(T+h)}x(T+h)$, hence

$$\begin{aligned}&E^u[e^{aT}\{c^u(T+h+k)-c^u(T)\}+e^{-a(h+k)}x(T+h+k)|F_T]\\&\quad\geqq E^u[e^{aT}\{c^u(T+h)-c^u(T)\}+e^{-ah}x(T+h)|F_T]\,.\end{aligned}$$

Thus $G_{h+k}x(T)\geqq G_hx(T)$. Combining the above inequality with Lemma 3, we get

$$\begin{aligned}c^u(t)+e^{-at}G_hx(t)&\leqq c^u(t)+e^{-at}G_{h+k}x(t)\\&\leqq c^u(t)+e^{-at}E^u[e^{at}\{c^u(t+k)-c^u(t)\}+e^{-ak}G_hx(t+k)|F_t]\\&=E^u[c^u(t+k)+e^{-a(t+k)}G_hx(t+k)|F_t]\,.\end{aligned}$$

Therefore $\{c^u(t)+e^{-at}G_hx(t);t\geqq0\}$ is a P^u-submartingale. To prove the right continuity of $\{c^u(t)+e^{-at}G_hx(t)\}$, it suffices to show that the mapping $t\to E^u[e^{-at}G_hx(t)]$ is right continuous for each $u\in\mathscr{D}$. For $s\geqq t$, let us denote

$$\begin{aligned}X_s^{u\cdot t\cdot v}&=c^{u\cdot t\cdot v}(s+h)-c^{u\cdot t\cdot v}(s)+e^{-a(s+h)}x(s+h)\\&=c^v(s+h)-c^v(s)+e^{-a(s+h)}x(s+h)\,,\end{aligned}$$

which follows from (C.1). From the right continuity of $z^u(u\in\mathscr{D})$, (A.3) and (2.2) it follows that

$$E[|z^{u\cdot(t+\varepsilon)\cdot v}(\infty)-z^{u\cdot t\cdot v}(\infty)|]\to0$$

and then, by (C.3),

$$|E^{u\cdot(t+\varepsilon)\cdot v}[X_t^{u\cdot t\cdot v}]-E^{u\cdot t\cdot v}[X_t^{u\cdot t\cdot v}]|\to0\quad\text{as}\quad\varepsilon\to0,\varepsilon>0\,.$$

For each $\delta>0$ there exists $v_\delta\in\mathscr{D}$ such that

$$E^{u\cdot t\cdot v_\delta}[X_t^{u\cdot t\cdot v_\delta}]<\inf_{v\in\mathscr{D}}E^{u\cdot t\cdot v}[X_t^{u\cdot t\cdot v}]+\delta$$

and thus

$$E^{u\cdot(t+\varepsilon)\cdot v_\delta}[X_t^{u\cdot t\cdot v_\delta}]\leqq\inf_{v\in\mathscr{D}}E^{u\cdot t\cdot v}[X_t^{u\cdot t\cdot v}]+\delta$$

for ε sufficiently small. Hence it is not difficult to see that for any $\delta>0$ there exists $\varepsilon_\delta>0$ such that for $\varepsilon_\delta>\varepsilon>0$,

$$\begin{aligned}&|\inf_{v\in\mathscr{D}} E^{u\cdot(t+\varepsilon)\cdot v}[X^{u\cdot t\cdot v}_{t+\varepsilon}] - \inf_{v\in\mathscr{D}} E^{u\cdot t\cdot v}[X^{u\cdot t\cdot v}_{t}]|\\ &\qquad \leqq \sup_{w\in\mathscr{D}} |E^{w}[X^{u\cdot t\cdot w}_{t+\varepsilon}] - E^{w}[X^{u\cdot t\cdot w}_{t}]| + \delta\,.\end{aligned}$$

On the other hand, we get by Lemma 2

$$E^{u}[e^{-at}G_h x(t)] = \inf_{v\in\mathscr{D}} E^{u\cdot t\cdot v}[X^{u\cdot t\cdot v}_{t}]\,.$$

Thus we deduce

$$\begin{aligned}&|E^{u}[e^{-a(t+\varepsilon)}G_h x(t+\varepsilon)] - E^{u}[e^{-at}G_h x(t)]|\\ &\quad\leqq \sup_{w\in\mathscr{D}} E^{w}[\{c^{w}(t+h+\varepsilon) - c^{w}(t+h)\} + \{c^{w}(t+\varepsilon) - c^{w}(t)\}]\\ &\quad + \sup_{w\in\mathscr{D}} e^{-a(t+h+\varepsilon)}E[|x(t+h+\varepsilon) - x(t+h)|z^{w}(\infty)]\\ &\quad + \sup_{w\in\mathscr{D}} e^{-a(t+h)}E[|x(t+h)|z^{w}(\infty)](1-e^{-a\varepsilon}) + \delta\,.\end{aligned}$$

The first term of the last expression converges to zero as $\varepsilon\to 0$ by (C.2). The second term is dominated by

$$\sup_{w\in\mathscr{D}} e^{-a(t+h+\varepsilon)}\|x(t+h+\varepsilon) - x(t+h)\|_{L^{p/(p-1)}(P)}\|z^{w}(\infty)\|_{L^{p}(P)}\,.$$

By Lebesgue's dominated convergence theorem, $\lim_{\varepsilon\to 0}\|x(t+h+\varepsilon) - x(t+h)\|_{L^{p/(p-1)}(P)} = 0$. The third term converges also to zero as $\varepsilon\to 0$, for this term is dominated by $e^{-a(t+h)}\|x\|\sup_w\|z^{w}(\infty)\|_{L^{p}(P)}(1-e^{-a\varepsilon})$. Therefore, letting $\delta\to 0$, we obtain $G_h x\in\Phi$ for $x\in\Phi$. The properties (i) (ii) (iv) of G_h are obvious by Lemma 2 and the definition of G_h. Since

$$\begin{aligned}|G_h x(t) - G_h y(t)| &\leqq \operatorname*{ess\,sup}_{u\in\mathscr{D}} E^{u}[e^{-ah}|x(t+h) - y(t+h)||F_t]\\ &= e^{-ah}\operatorname*{ess\,sup}_{u\in\mathscr{D}} E[|x(t+h) - y(t+h)|(z^{u}(t+h)/z^{u}(t))|F_t]\\ &\leqq e^{-ah}\|x-y\|\,,\end{aligned}$$

we get $\|G_h x - G_h y\| \leqq e^{-ah}\|x-y\|$. The property (v) is immediate by the definition of essential infimum.

(2) The proof of Theorem 2. We first show that the value process is a maximal element of Φ. Let us denote by V the process $\{e^{at}W(t); t\geqq 0\}$. Note that $c^{u}(t) + e^{-at}x(t) \leqq E^{u}[c^{u}(\infty)|F_t]$ for all $x\in\Phi$. It follows from the property (iii) of W that V belongs to Φ. For all $x\in\Phi$, we have $G_\infty x(T) = e^{aT}W(T)$ and $c^{u}(T) + e^{-aT}x(T) \leqq E^{u}[c^{u}(\infty)|F_T]$. Thus $x(T)\leqq P\text{-ess}\inf_{u\in\mathscr{D}} e^{aT}E^{u}[c^{u}(\infty) - c^{u}(T)|F_T] = V(T)$ which implies the maximality of V. From the contraction mapping theorem, there exists a fixed point $x_h\in\Phi$ of G_h for each $h>0$. To prove the latter part of Theorem 2, it is sufficient to show that $x_h(T)\geqq e^{aT}W(T)$ on $\{T<\infty\}$ for all $h>0$. Since $\{c^{u}(t) + W(t)\}$ is a P^{u}-submartingale for each $u\in\mathscr{D}$, we have

$$E^u[e^{aT}\{c^u(T+h)-c^u(T)\}+e^{-ah}\{e^{a(T+h)}W(T+h)\}|F_T]\geqq e^{aT}W(T)$$

and hence $G_h V(T) \geqq V(T)$. Applying the operator G_h to the above inequality, we inductively get $V(T) \leqq (G_h)^n V(T)$, while, $(G_h)^n V(T)$ converges to the fixed point $x_h(T)$ as $n \to \infty$. So $x_h = V$ and x^* is independent of $h > 0$. Thus the proof is complete.

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Department of Mathematics
Faculty of General Education
Ehime University
Matsuyama, 790
Japan

and

Mathematical Institute
Tôhoku University
Sendai, 980
Japan