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THE SECOND COHOMOLOGY GROUPS OF THE GROUP OF UNITS OF A Z_p-EXTENSION

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Let p be a prime number. We denote by K_0 a finite algebraic number field, by K_∞ a $\boldsymbol{Z}_p$-extension of K_0 and by K_n the cyclic subextension of degree p^n. Let E_∞ be the unit group of K_∞. Put $\Gamma_n = \mathrm{Gal}(K_n/K_0)$ and $\Gamma = \mathrm{Gal}(K_\infty/K_0)$. In connection with the Leopoldt conjecture and Greenberg conjecture, Iwasawa [5] posed the problem of studying the structure of $H^2(\Gamma, E_\infty)$. Let S_n be the set of prime ideals of K_n ramified in K_∞, and D_n be the p-Sylow subgroup of the ideal class group generated by the ideals $\prod \mathfrak{p}^\sigma$ for $\mathfrak{p} \in S_n$, where $\mathfrak{p}^\sigma$ runs through all different conjugates of $\mathfrak{p}$ over K_0. We consider the inductive limit D_∞ of D_n by means of the natural map. In this paper, we shall give a partial answer,

$$H^2(\Gamma, E_\infty) \cong (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{s_0 - r_p - 1}$$

where $r_p = \text{ess. rank}\, D_\infty$ and $s_0 = \sharp S_0$.

While preparing this paper, the author received the preprint by Iwasawa entitled "On cohomology groups of units for $\boldsymbol{Z}_p$-extensions" in which he also obtains a similar result. (The paper has since appeared in [6].)

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1. We define the essential rank for a $\boldsymbol{Z}_p$-module M, which is denoted by ess. rank M, as the dimension of $M \otimes_{Z_p} \boldsymbol{Q}_p$. For a p-primary torsion abelian group, we also define the essential rank as that of the Pontrjagin dual.

LEMMA 1. *Let $\{M_n\}_{n \geqq 0}$ be a family of finite abelian p-groups with bounded p-ranks. For $m > n \geqq 0$, let $\varphi_{m,n}: M_n \to M_m$* (resp. *$\psi_{m,n}: M_m \to M_n$*) *be homomorphisms giving rise to an inductive system $\{M_n, \varphi_{m,n}\}$* (resp. *projective system $\{M_n, \psi_{m,n}\}$*). *If the orders of* $\mathrm{Ker}(\varphi_{m,n})$ *and* $\mathrm{Coker}(\psi_{m,n})$ *are bounded with respect to m and n, then we have* $\text{ess. rank ind lim}\{M_n, \varphi_{m,n}\} = \text{ess. rank proj lim}\{M_n, \psi_{m,n}\}$.

PROOF. Let M_n^* be the dual abelian group of M_n and $\psi_{m,n}^*$ be the dual map induced by $\psi_{m,n}$. Then $\{M_n^*, \psi_{m,n}^*\}$ is an inductive system whose inductive limit is the dual of $\operatorname{proj\,lim} M_n$. Hence ess. rank $\operatorname{ind\,lim}_{\psi^*} M_n^* =$ $\operatorname{proj\,lim}_{\psi} M_n$. Put $A = \operatorname{ind\,lim}_{\varphi} M_n$ and $a =$ ess. rank A. Let $\varphi_n: M_n \to A$ be the canonical map and A_n be its image. We have $p^t A = \bigcup_{n=0}^{\infty} p^t A_n \cong (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^a$ for some t and $p^t M_n / p^t M_n \cap \operatorname{Ker}(\varphi_n) \cong p^t A_n$. Since $p^t M_n$ is an abelian group, it has a subgroup N_n such that $N_n \cong p^t A_n$. We have $[p^t M_n: N_n] = [p^t M_n: p^t M_n \cap \operatorname{Ker}(\varphi_n)] \cdot [p^t M_n \cap \operatorname{Ker}(\varphi_n): 0]/[N_n: 0] \leqq [\operatorname{Ker}(\varphi_n): 0]$. Since the orders of $\operatorname{Ker}(\varphi_{m\,n})$ are bounded, so are those of $\operatorname{Ker}(\varphi_n)$. Let $p^c = \max_{n \geqq 0}(\# \operatorname{Ker}(\varphi_{m,n}))$. We have $p^{t+c} M_n \subset N_n$. Put

$$b = \text{ess. rank ind lim}_{\psi^*} p^{t+c} M_n^* ,$$

which is equal to ess. rank $\operatorname{ind\,lim}_{\psi^*} M_n^*$. Since $p^{t+c} M_n^* \cong p^{t+c} M_n$, we have p-rank $(p^{t+c} M_n^*) = p$-rank $(p^{t+c} M_n) \leqq p$-rank $(N_n) \leqq a$. Hence we have $b \leqq a$. We also have $a \leqq b$. Thus we have $a = b$. q.e.d.

Let C_n be the p-Sylow subgroup of the ideal class group of K_n. We define the natural homomorphism $i_{m,n}: C_n \to C_m$ by $i_{m,n}((\mathfrak{a})) = (\mathfrak{a})$ for the ideal $\mathfrak{a}$ of K_n, where we denote by $(\mathfrak{a})$ the ideal class determined by $\mathfrak{a}$. Let C_∞ (resp. C) be the inductive limit (resp. the projective limit) with respect to $i_{m,n}$ (resp. the norm map $N_{m,n}: C_m \to C_n$) for $m > n$. Let γ be a $\boldsymbol{Z}_p$-generator of Γ and γ_n be the generator of Γ_n which is the restriction of γ onto K_n. Let M be any Γ-module. Let $1-\gamma$ be the endomorphism on M such that $x^{1-\gamma} = x/x^\gamma$ for $x \in M$. We denote by $M^{1-\gamma}$ its image and by M^Γ its kernel. Similarly we define $1-\gamma_n$, $M_n^{1-\gamma_n}$ and $M_n^{\Gamma_n}$ on any Γ_n-module M_n.

LEMMA 2. ess. rank $C_\infty^\Gamma =$ ess. rank $(C/C^{1-\gamma})$.

PROOF. By the exact sequence $1 \to C_n^{\Gamma_n} \to C_n \xrightarrow{1-\gamma_n} C_n^{1-\gamma_n} \to 1$, we have

$$1 \to \operatorname{proj\,lim} C_n^{\Gamma_n} \to C \xrightarrow{1-\gamma} \operatorname{proj\,lim} C_n^{1-\gamma_n} \to 1$$

$$1 \to \operatorname{ind\,lim} C_n^{\Gamma_n} \to C_\infty \xrightarrow{1-\gamma} \operatorname{ind\,lim} C_n^{1-\gamma_n} \to 1 .$$

It is obvious that $\operatorname{proj\,lim} C_n^{1-\gamma_n} = C^{1-\gamma}$ and $\operatorname{ind\,lim} C_n^{1-\gamma_n} = C_\infty^{1-\gamma}$. Then we have $\operatorname{proj\,lim} C_n^{\Gamma_n} = C^\Gamma$ and $\operatorname{ind\,lim} C_n^{\Gamma_n} = C_\infty^\Gamma$. By the fundamental theorem of $\boldsymbol{Z}_p$-extensions, we see that the p-rank of $p^b C$ is bounded for a certain integer b. Since the orders of $\operatorname{Coker}(N_{m,n})$ are bounded and so are those of $\operatorname{Ker}(\varphi_{m,n})$ by the well known theorem of Iwasawa [4], we have ess. rank $p^b C =$ ess. rank $p^b C_\infty$ by Lemma 1. Hence we have ess. rank $C =$ ess. rank C_∞. Similarly we have ess. rank $C^{1-\gamma} =$ ess. rank $C_\infty^{1-\gamma}$. By the definition of ess. rank, we have ess. rank $C^\Gamma = \dim_{\boldsymbol{Q}_p} C^\Gamma \otimes \boldsymbol{Q}_p = \dim_{\boldsymbol{Q}_p} C \otimes \boldsymbol{Q}_p - \dim_{\boldsymbol{Q}_p} C^{1-\gamma} \otimes \boldsymbol{Q}_p$. Hence we have ess. rank $C^\Gamma =$ ess. rank $C -$ ess. rank $C^{1-\gamma}$. Since we also have ess. rank $C_\infty^\Gamma =$ ess. rank $C_\infty -$ ess. rank $C_\infty^{1-\gamma}$, we have ess. rank C_∞^Γ = ess. rank C − ess. rank $C^{1-\gamma}$ = ess. rank$(C/C^{1-\gamma})$. q.e.d.

2. In the following we denote by $H^m(A_n)$ (resp. $H^m(A_\infty)$) the cohomology group $H^m(\Gamma_n, A_n)$ (resp. $H^m(\Gamma, A_\infty)$) for a Γ_n-module A_n (resp. Γ-module A). Let U_n be the group of unit idéles of K_n and E_n be the group of global units of K_n. Let I_n be the ideal group of K_n.

LEMMA 3. (1) $H^1(U_n) \cong I_n^{\Gamma_n}/I_0$
(2) $H^1(U_n/E_n) \cong C_n^{\Gamma_n}/i_{n,0}(C_0)$

PROOF. Let J_n be the idéle group of K_n. We notice that $U_n \cdot K_n^\times/K_n^\times \cong U_n/E_n$ and that $(J_n/K_n^\times)^{\Gamma_n} = J_0 \cdot K_n^\times/K_n^\times$. Let C_0' (resp. C_n') be the ideal class group of K_0 (resp. $K_n^\times$) and $i\colon C_0' \to C_n'$ be the natural map. We have (1) and $H^1(U_n/E_n) = C_n'^{\Gamma_n}/i(C_0')$ by the cohomology long exact sequences

$$1 \to U_0 \to J_0 \to I_n^{\Gamma_n} \to H^1(U_n) \to H^1(J_n) = 1 \quad \text{and}$$
$$1 \to (U_n \cdot K_n^\times/K_n^\times)^{\Gamma_n} \to (J_n/K_n^\times)^{\Gamma_n} \to C_n'^{\Gamma_n} \to H^1(U_n \cdot K_n^\times/K_n^\times) \to H^1(J_n/K_n^\times) = 1$$

associated to the short exact sequences $1 \to U_n \to J_n \to I_n \to 1$ and $1 \to U_n \cdot K_n^\times/K_n^\times \to J_n/K_n^\times \to C_n' \to 1$, respectively. Since $C_n'^{\Gamma_n}/i(C_0')$ is a p-group, we have $C_n'^{\Gamma_n}/i(C_0') = C_n^{\Gamma_n}/i(C_0)$. q.e.d.

REMARK. The isomorphisms in Lemma 3 are compatible with the inflation maps from K_n to K_m for $m > n$ and natural maps $I_n^{\Gamma_n}/I_0 \to I_m^{\Gamma_m}/I_0$ and $C_n^{\Gamma_n}/i_{n,0}(C_0) \to C_m^{\Gamma_m}/i_{m,0}(C_0)$.

We have the exact sequence $H^1(E_n) \to I_n^{\Gamma_n}/I_0 \to C_n^{\Gamma_n}/i_{n,0}(C_0) \to H^2(E_n)$ by the cohomology long exact sequence associated to the short exact sequence $1 \to E_n \to U_n \to U_n/E_n \to 1$ and by Lemma 3. Let D_n' be the ideal group which is generated by $I_n^{\Gamma_n}$. We have $\mathrm{Image}(I_n^{\Gamma_n}/I_0 \to C_n^{\Gamma_n}/i_{n,0}(C_0)) \cong D_n' \cdot i(C_0')/i(C_0')$. Since this group is a p-group, we have $D_n' \cdot i(C_0')/i(C_0') \cong D_n \cdot i_{n,0}(C_0)/i_{n,0}(C_0)$. Hence we have the exact sequence

$$(1) \qquad 1 \to D_n \cdot i_{n,0}(C_0)/i_{n,0}(C_0) \to C_n^{\Gamma_n}/i_{n,0}(C_0) \to H^2(E_n) \to H^2(U_n) \to H^2(U_n/E_n) \to H^3(E_n)\,.$$

We take the inductive limit of this sequence with respect to the inflation maps and the natural maps induced by $i_{m,n}$. Let $E_\infty = \bigcup_{n=0}^\infty E_n$. Let $i_\infty\colon C_0 \to C_\infty$ be the canonical map. Since the cohomological dimension of Γ is 2, we have the exact sequence

$$(2) \qquad 1 \to D_\infty \cdot i_\infty(C_0)/i_\infty(C_0) \to C_\infty^\Gamma/i_\infty(C_0) \to H^2(E_\infty) \to \operatorname{ind\,lim} H^2(U_n) \to \operatorname{ind\,lim} H^2(U_n/E_n) \to 1\,.$$

3. We compute the inductive limit of $H^2(U_n)$ and $H^2(U_n/E_n)$. Let a be the smallest integer n such that every prime ideal of K_n ramified in K_∞ is totally ramified. Let $N_n: U_n \to U_0$ be the norm map.

LEMMA 4. $\operatorname{ind\,lim} H^2(U_n) \cong (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{s_0}$.

PROOF. Since Γ_n is cyclic, we have $H^2(U_n) \cong U_0/N_nU_n$. We have $H^2(U_a) \cong \prod_{i=1}^{s_0} \boldsymbol{Z}/p^{d_i}\boldsymbol{Z}$ by the semi-local theory. Hence we have $H^2(U_n) \cong \prod_{i=1}^{s_0} \boldsymbol{Z}/p^{n-a+d_i}\boldsymbol{Z}$ for $n \geqq a$. Let $\varphi_{m,n}: U_0/N_nU_n \to U_0/N_mU_m$ be the inflation map for $m > n$. Let $\{u\}_n$ be an element of U_0/N_nU_n which is the class of $u \in U_0$. Then $\varphi_{m,n}(\{u\}_n) = \{u^{p^{m-n}}\}_m$. We have $\operatorname{Image}(\varphi_{m,n}) \cong U_0^{p^{m-n}} \cdot N_mU_m/N_mU_m$. Hence we have $\operatorname{Image}(\varphi_{m,n}) \cong \prod_{i=1}^{s_0} \boldsymbol{Z}/p^{n-a+d_i}\boldsymbol{Z}$. This shows that $\varphi_{m,n}$ is injective for $m > n \geqq a$. Hence we have $\operatorname{ind\,lim} H^2(U_n) \cong (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{s_0}$. q.e.d.

Let L_n be the p-Hilbert class field of K_n. We denote by L_n^* its genus of the Galois extension K_n/K_0. Put $L_\infty^* = \bigcup_{n=0}^{\infty} L_n^*$. Let $E_{n,p}$ be the completion of E_n in $\prod_{\mathfrak{p}/p} U_{n,\mathfrak{p}}$ where $U_{n,\mathfrak{p}}$ is the unit group of the completion of K_n at $\mathfrak{p}$. Let $N_\infty U_\infty = \bigcap_{n=1}^{\infty} N_nU_n$. Let $V_n = U_0/N_\infty U_\infty$ and $W = E_{0,p} \cdot N_\infty U_\infty/N_\infty U_\infty$. We have the projective system $\{V_0/V_n \cdot W\}_{n \geqq 0}$ with respect to the canonical maps $V_0/V_n \cdot W \to V_0/V_n \cdot W$. Let V be its projective limit.

LEMMA 5.
$$\begin{aligned}\text{ess. rank}\, V &= \text{ess. rank}\, \operatorname{Gal}(L_\infty^*/L_0) \\ &= \text{ess. rank}\, (C/C^{1-\gamma}) + 1\,.\end{aligned}$$

PROOF. Since $\operatorname{Gal}(L_\infty^*/K_\infty) \cong C/C^{1-\gamma}$, we have the last equality. Let H_n be the full Hilbert class field of K_n and H_n^* be the genus field in H_n of the Galois extension K_n/K_0. We have $\operatorname{Gal}(H_n^*/K_0) \cong J_0/K_0^\times \cdot N_nU_n$. Hence $\operatorname{Gal}(H_n^*/H_0) \cong K_0^\times \cdot U_0/K_0^\times \cdot N_nU_n \cong U_0/E_0 \cdot N_nU_n \cong V_0/V_n \cdot W$ since $E_{0,p} \cdot N_uU_n = E_0 \cdot N_nU_n$. Since $U_0^{p^n} \subset N_nU_n$, it is a p-group. $\operatorname{Gal}(L_n^*/L_0)$ is canonically isomorphic to the p-Sylow subgroup of $\operatorname{Gal}(H_n^*/H_0)$. Hence we have $\operatorname{Gal}(L_n^*/L_0) \cong V_0/V_n \cdot W$. We have the following commutative diagram for $m > n$, with respect to the restriction maps of the Galois group and the canonical maps $V_0/V_m \cdot W \to V_0/V_n \cdot W$.

$$\begin{array}{ccc}\operatorname{Gal}(L_m^*/L_0) & \cong & V_0/V_m \cdot W \\ \downarrow & & \downarrow \\ \operatorname{Gal}(L_n^*/L_0) & \cong & V_0/V_n \cdot W\end{array}$$

Taking the projective limit, we have $\operatorname{Gal}(L_\infty^*/L_0) \cong V$. q.e.d.

LEMMA 7. $\text{ess. rank ind lim}\, H^2(U_n/E_n) = \text{ess. rank}\, C_\infty^\Gamma + 1$.

PROOF. We have $\operatorname{Image}(H^2(U_n) \to H^2(U_n/E_n)) \cong U_0/N_nU_n \cdot E_0$. Let

$\varphi_{m,n}$ be the restriction onto $U_0/N_nU_n\cdot E_0$ of the inflation map from $H^2(U_n/E_n)$ to $H^2(U_m/E_m)$ for $m>n$. By (2) we have $\mathrm{ind\,lim}_\varphi U_0/N_nU_n\cdot E_0\cong \mathrm{ind\,lim}\, H^2(U_n/E_n)$. Since $V_0/V_n\cdot W\cong U_0/N_nU_n\cdot E_0$, we also denote by $\varphi_{m,n}$ the induced map of $\varphi_{m,n}$ on $V_0/V_n\cdot W$. Then we have ess. rank $\mathrm{ind\,lim}_\varphi$ $U_0/N_nU_n\cdot E_0=$ ess. rank $\mathrm{ind\,lim}\,V_0/V_n\cdot W$. Let $\{v\}_n$ be the element of $V_0/V_n\cdot W$ generated by the element $v\in V_0$. We have $\mathrm{Ker}\,\varphi_{m,n}=\{\{v\}_n|v\in V_0, v^{p^{m-n}}\in V_m\cdot W\}$. Since $V_0/V_n\cong U_0/N_nU_n\cong\prod_{i=1}^{s_0}Z/p^{n-a+d_i}Z$ for $n\geqq a$, we have $V_a/V_n\cong(Z/p^{n-a}Z)^{s_0}$. Since $V_a^{p^{n-a}}\subset V_n$ and $V_a/V_a^{p^{n-a}}\cong(Z/p^{n-a}Z)^{s_0}$, we have $V_a^{p^{n-a}}=V_n$. Let $\{v\}_n\in\mathrm{Ker}(\varphi_{m,n})$. Then we have $v^{p^{m-n}}=u^{p^{m-n}}\cdot w$ for $u\in V_a$ and $w\in W$. Hence we have $(v\cdot u^{-p^{n-a}})^{p^{m-n}}=w$. Since $\{v\}_n=\{v\cdot u^{-p^{n-a}}\}_n$, we have $\mathrm{Ker}(\varphi_{m,n})=\{\{v\}_n|v\in V_0, v^{p^{m-n}}\in W\}$ for $n\geqq a$. This is the homomorphic image of the torsion subgroup of V_0/W. Hence the orders of $\mathrm{Ker}(\varphi_{m,n})$ are bounded. Let $V_\infty=\mathrm{ind\,lim}_\varphi V_0/V_n\cdot W$. We have by Lemma 1 ess. rank $V_\infty=$ ess. rank $V=$ ess. rank $(C/C^{1-\gamma})+1$. By Lemma 2 we have ess. rank $V_\infty=$ ess. rank $C_\infty^\Gamma+1$. q.e.d.

4. Theorem. *Let* $r_p=$ ess. rank D_∞ *and* s_0 *be the number of the prime ideals of* K_0 *which are ramified in* K_∞. *Then we have*

$$H^2(\Gamma, E_\infty)\cong(Q_p/Z_p)^{s_0-r_p-1}\ .$$

Proof. By (2), we have

$$\text{ess. rank } H^2(E_\infty)=-\text{ess. rank } D_\infty\cdot i_{n,0}(C_0)/i_{n,0}(C_0)+\text{ess. rank } C_\infty^\Gamma/i_{n,0}(C_0)$$
$$+s_0-\text{ess. rank } C_\infty^\Gamma-1=s_0-r_p-1\ .$$

Let $\varphi_n: H^2(E_n)\to H^2(E_\infty)$ be the canonical map. Since $H^2(E_n)\cong E_0/N_nE_n$, we denote by $\{x\}_n$ the element of E_0/N_nE_n which is the class of $x\in E_0$. Let $\inf_{n+1,n}: E_0/N_nE_n\to E_0/N_{n+1}E_{n+1}$ be the inflation map. Then we have $\varphi_n(\{x\}_n)=\varphi_{n+1}\circ\inf_{n+1,n}(\{x\}_n)=\varphi_{n+1}(\{x^p\}_n)$. Hence $\varphi_{n+1}(\{x\}_{n+1})^p=\varphi_n(\{x\}_n)$. This shows that $H^2(E_\infty)$ is p-divisible. Thus we have $H^2(E_\infty)\cong(Q_p/Z_p)^{s_0-r_p-1}$. q.e.d.

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