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ABSOLUTE RIESZ SUMMABILITY FACTORS OF FOURIER SERIES

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

KÔSI KANNO

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1. Let $\lambda(\omega)$ be a monotone increasing and differentiable function on (C, ∞) for some finite positive number C tending to infinity as $\omega \to \infty$. A given infinite series $\sum u_n$ is said to be absolutely summable by Riesz's method of order r and type $\lambda(\omega)$ and denoted $\sum u_n \in |R, \lambda(\omega), r|$, $r > 0$, if

$$\int_A^\infty \frac{\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \Big|\sum_{n<\omega} \{\lambda(\omega) - \lambda(n)\}^{r-1}\lambda(n)u_n\Big| d\omega < \infty ,$$

where A is a finite positive number (Mohanty [6], Obrechkoff [7]).

Let $f(t) \in L(-\pi, \pi)$ be a 2π-periodic function. Without any loss of generality we may assume that the constant term of the Fourier series of $f(t)$ is zero, so that

$$f(t) \sim \sum_{n=1}^\infty (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^\infty A_n(t) .$$

Throughout this paper we shall use the following notations:

$$\varphi(t) = (f(x+t) + f(x-t))/2 ,$$

$$\Phi_\beta(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1}\varphi(u)du , \qquad \beta > 0 ,$$

$$\varphi_\beta(t) = \Gamma(\beta+1)t^{-\beta}\Phi_\beta(t) ,$$

$$e(\omega) = \exp \omega^\alpha , \qquad 0 < \alpha < 1 ,$$

$$Q(n, \omega, s) = [e(\omega) - e(n)]^{r-1}e(n)n^s , \qquad 0 < r < 1 ,$$

$$E(\omega, t, s) = \sum_{n<\omega} Q(n, \omega, s) \exp int ,$$

where s is a real number,

$$g(\omega, u, s) = \int_u^\pi (t-u)^{-\beta} \operatorname{Re} E(\omega, t, s)dt ,$$

$$G(\omega, u, s) = \int_0^u v^{\delta+\beta}\frac{\partial}{\partial v}g(\omega, v, s)dv , \qquad \delta \geqq 0 ,$$

$$H(\omega, u, s) = \int_u^\pi v^{\delta+\beta} \frac{\partial}{\partial v} g(\omega, v, s) dv \ .$$

$C_1, C_2, \cdots$ denote absolute constants, possibly different at different occurrences.

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2. Absolute Riesz summability for the Fourier series of a function of bounded variation are investigated by many authors, for example, P. Chandra, G. D. Dikshit, K. Matsumoto, R. Mohanty and others. In this paper we prove the following new theorem:

THEOREM. *Let* $0 \leqq \beta < 1$, $\delta \geqq 0$, $0 < s + \beta < \delta + \beta < 1$ *and* $\alpha = (\delta - s)/(\delta + \beta)$, *then* $t^{-\delta}\varphi_\beta(t) \in BV(0, \pi)$ *implies that* $\sum_{n=1}^\infty n^s A_n(x)$ *is summable* $|R, e(\omega), r|$, with $r > \delta + \beta$.

We shall use the following Lemmas to prove our Theorem.

LEMMA 1. *Let* $0 < r < 1$, *and* $1 < A \leqq \omega$ *for a suitable positive constant* A. *Then*

(i) $E(\omega, t, s) = O\{t^{-r}\omega^{(\alpha-1)(r-1)+s}(e(\omega))^r + Q(m, \omega, s)\}$, *for* $\omega^{-a} < t < \pi$ $(0 < a < 1)$,

(ii) $E(\omega, t, s) = O\{\omega^{s+1-\alpha}(e(\omega))^r + Q(m, \omega, s)\}$, *for all* $t > 0$.

PROOF. $E(\omega, t, s) = \sum_{n<\omega} Q(n, \omega, s) \exp int = (\sum_{n=1}^{\omega_1} + \sum_{n=\omega_1+1}^{m}) Q(n, \omega, s) \times \exp int = S_1 + S_2$, say, where $\omega_1 = [\omega - 1/t]$ and $m = [\omega]$.

Since $Q(n, \omega, s)$ and $e(n)n^s$ are ultimately increasing in n, we have

$$\begin{aligned} |S_1| &\leqq Q(\omega_1, \omega, s) \max_{1 \leqq p \leqq q \leqq \omega_1} |\sum_{n=p}^q \exp int| \\ &= O[\{(\omega - \omega_1)\omega_*^{\alpha-1} e(\omega_*)\}^{r-1} e(\omega_1)\omega_1^s t^{-1}] \\ &= O\{t^{-r}\omega^{(\alpha-1)(r-1)+s}(e(\omega))^r\} \end{aligned}$$

for $0 < t < \pi$ $(\omega_1 < \omega_* < \omega)$. And

$$\begin{aligned} |S_2| &\leqq \int_{\omega_1}^{\omega} Q(x, \omega, s) dx + Q(m, \omega, s) \\ &= O\{(e(\omega) - e(\omega_1))^r \omega_1^{s-\alpha+1}\} \\ &\quad + O\{(e(\omega) - e(\omega_1))^r\} \left| \int_{\omega 1}^{\omega_1'} x^{s-\alpha} dx \right| + Q(m, \omega, s) \ , \quad \omega_1 < \omega_1' < w \\ &= O\{t^{-r}\omega^{(\alpha-1)r}(e(\omega))^r \omega_1^{s-\alpha+1}\} \\ &\quad + O\{t^{-r}\omega^{(\alpha-1)r}(e(\omega))^r \max(\omega^{s-\alpha+1}, \omega_1^{s-\alpha+1})\} \\ &\quad + Q(m, \omega, s) \ . \end{aligned}$$

Since it is easy to see that $(w - 1/t)^l \sim \omega^l$ for any l if $t^{-1} < \omega^a$ $(0 < a < 1)$, we obtain (i) by $|S_1|$ and $|S_2|$.

Let $\omega_2^\alpha = \omega^\alpha - 1$, then

$$|E(\omega, t, s)| \leqq \sum_{n<\omega} Q(n, \omega, s) \leqq \int_A^\omega Q(x, \omega, s)dx + Q(m, \omega, s)$$
$$= \left\{\int_A^{\omega_2} + \int_{\omega_2}^\omega\right\} Q(x, \omega, s)dx + Q(m, \omega, s)$$
$$= T_1 + T_2 + Q(m, \omega, s)\ , \quad \text{say.}$$

Since $(e(x))^{1/2}x^{s-\alpha+1}$ is increasing for $x > A$, we have:

$$T_1 \leqq (e(\omega) - e(\omega_2))^{r-1} \int_A^{\omega_2} \frac{d}{dx}(e(x))^{1/2}\frac{2}{\alpha}x^{s-\alpha+1}(e(x))^{1/2}dx$$
$$= O\{(e(\omega))^{r-1}(e(\omega_2))^{1/2}\omega_2^{s-\alpha+1}(e(\omega_2))^{1/2}\}$$
$$= O\{(e(\omega))^r\omega^{s-\alpha+1}\}\ .$$
$$T_2 = \int_{\omega_2}^\omega Q(x, \omega, s)dx = O\{\omega^{s-\alpha+1}[(e(\omega) - e(x))^r]_{\omega_2}^\omega\}$$
$$= O\{(e(\omega))^r\omega^{s-\alpha+1}\}\ .$$

From T_1, T_2 we obtain the estimate (ii).

LEMMA 2. *If* $0 < a < 1$, $0 \leqq \beta < 1$ *and* $0 < \delta + \beta < r$

$$\text{(i)} \quad g(\omega, u, s) = \begin{cases} O\{u^{-r}\omega^{(\alpha-1)(r-1)+s+\beta-1}(e(w))^r + Q(m, \omega, s+\beta-1)\}\ , \\ \qquad\qquad \textit{for} \quad \omega^{-a} < u < \pi\ , \\ O\{\omega^{s+\beta-\alpha}(e(\omega))^r + Q(m, \omega, s+\beta-1)\}\ , \\ \qquad\qquad \textit{for all} \quad u > 0\ , \end{cases}$$

$$\text{(ii)} \quad G(\omega, u, s) = O\{u^{\delta+\beta}\omega^{s+\beta-\alpha}(e(\omega))^r\} + u^{\delta+\beta}Q(m, \omega, s+\beta-1)\}\ , \quad \textit{for all} \quad u > 0\ ,$$

$$\text{(iii)} \quad H(\omega, u, s) = O\{u^{\delta+\beta-r}\omega^{(\alpha-1)(r-1)+s+\beta-1}(e(\omega))^r + Q(m, \omega, s+\beta-1)\}\ , \quad \textit{for} \quad \omega^{-a} < u < \pi\ .$$

PROOF. Using the first and second mean value theorems

$$\int_u^\pi (t-u)^{-\beta}\cos nt dt$$
$$= \left(\int_u^{u+\pi/n} + \int_{u+\pi/n}^\pi\right)(t-u)^{-\beta}\cos nt dt$$
$$= \cos nu_1 \int_u^{u+\pi/n}(t-u)^{-\beta}dt + \frac{n^\beta}{\pi^\beta}\int_{u+\pi/n}^{u_2}\cos nt dt\ ,$$
$$u \leqq u_1 \leqq u + \pi/n \leqq u_2 \leqq \pi$$
$$= \frac{\pi^{1-\beta}}{1-\beta}n^{\beta-1}\cos nu_1 + \frac{n^\beta}{\pi^\beta}\left(\frac{\sin nu_2}{n} - \frac{\sin(nu+\pi)}{n}\right)\ .$$

Thus

$$\begin{aligned} g(\omega, u, s) &= \int_u^\pi (t-u)^{-\beta} \operatorname{Re} E(\omega, t, s)dt \\ &= \sum_{n<\omega} Q(n, \omega, s) \int_u^\pi (t-u)^{-\beta} \cos ntdt \\ &= C_1 \operatorname{Re} E(\omega, u_1, s+\beta-1) + C_2\{\operatorname{Im} E(\omega, u_2, s+\beta-1) \\ &\quad + \operatorname{Im} E(\omega, u, s+\beta-1)\} . \end{aligned}$$

Hence, we get (i) by the estimates in Lemma 1. And

$$\begin{aligned} G(\omega, u, s) &= \int_0^u v^{\delta+\beta} \frac{\partial}{\partial u} g(\omega, v, s)dv \\ &= u^{\delta+\beta} g(\omega, u, s) - (\delta+\beta) \int_0^u v^{\delta+\beta-1} g(\omega, v, s)dv \\ &= O\{u^{\delta+\beta} \omega^{s+\beta-\alpha}(e(\omega))^r\} + O\{u^{\delta+\beta} Q(m, \omega, s+\beta-1)\} \end{aligned}$$

by (i) of this Lemma. Finally, by the estimate of $g(\omega, u, s)$ for $\omega^{-\alpha} < u < \pi$ and $r > \delta + \beta$, we get

$$\begin{aligned} H(\omega, u, s) &= \int_u^\pi v^{\delta+\beta} \frac{\partial}{\partial v} g(\omega, v, s)dv \\ &= [v^{\delta+\beta} g(\omega, v, s)]_u^\pi - (\delta+\beta) \int_u^\pi v^{\delta+\beta-1} g(\omega, v, s)dv \\ &= O\{u^{\delta+\beta-r} \omega^{(\alpha-1)(r-1)+s+\beta-1}(e(\omega))^r + Q(m, \omega, s+\beta-1)\} . \end{aligned}$$

3. Proof of Theorem. We have to show that

$$I = \int_A^\infty \left| \frac{e'(\omega)}{(e(\omega))^{r+1}} \sum_{n<\omega} Q(n, \omega, s) A_n(x) \right| d\omega < \infty$$

for some constant $A > 0$. As the method $|R, e(\omega), r|$ is absolutely regular (Obrechkoff [7]), it is sufficient to consider the case $0 < r < 1$. Since

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \varphi(t) \cos ntdt ,$$

we get

$$\begin{aligned} \sum_{n<\omega} Q(n, \omega, s) A_n(x) &= \frac{2}{\pi} \int_0^\pi [\sum_{n<\omega} Q(n, \omega, s) \cos nt] \varphi(t)dt \\ &= C_1 \int_0^\pi \operatorname{Re} E(\omega, t, s) \int_0^t (t-u)^{-\beta} d\Phi_\beta(u)dt \\ &= C_1 \int_0^\pi g(\omega, u, s) d\Phi_\beta(u) \end{aligned}$$

$$= C_1[\Phi_\beta(u)g(\omega, u, s)]_0^\pi - C_1\int_0^\pi \frac{\partial}{\partial u}g(\omega, u, s)\Phi_\beta(u)du$$

$$= C_2\int_0^\pi u^{\delta+\beta}\frac{\partial}{\partial u}g(\omega, u, s)u^{-\delta}\varphi_\beta(u)du$$

$$= C_2[u^{-\delta}\varphi_\beta(u)G(\omega, u, s)]_0^\pi - C_2\int_0^\pi G(\omega, u, s)d\{u^{-\delta}\varphi_\beta(u)\} .$$

Therefore, by $e'(\omega) = \alpha\omega^{\alpha-1}e(\omega)$

$$I = \int_A^\infty \left| \frac{\alpha}{\omega^{1-\alpha}(e(\omega))^r} \sum_{n<\omega} Q(n, \omega, s)A_n(x) \right| d\omega$$

$$\leqq C_3\int_A^\infty \frac{|G(\omega, \pi, s)|}{\omega^{1-\alpha}(e(\omega))^r}d\omega$$

$$+ C_3\int_0^\pi \left[\int_A^\infty \frac{|G(\omega, u, s)|}{\omega^{1-\alpha}(e(\omega))^r}d\omega\right] |d\{u^{-\delta}\varphi_\beta(u)\}| .$$

Thus, it suffices for our purpose to prove that

$$J = \int_A^\infty \frac{|G(\omega, \pi, s)|}{\omega^{1-\alpha}(e(\omega))^r}d\omega < \infty$$

and

$$K = \int_A^\infty \frac{|G(\omega, u, s)|}{\omega^{1-\alpha}(e(\omega))^r}d\omega = O(1) \qquad \text{for} \quad 0 < u < \pi ,$$

since $u^{-\delta}\varphi_\beta(u)$ is of bounded variation in $(0, \pi)$. Now,

$$G(w, \pi, s) = \int_0^\pi u^{\delta+\beta}\frac{\partial}{\partial u}g(\omega, u, s)du$$

$$= \pi^{\delta+\beta}g(w, \pi, s) - (\delta + \beta)\int_0^\pi u^{\delta+\beta-1}g(\omega, u, s)du$$

$$= -(\delta + \beta)\int_0^\pi u^{\delta+\beta-1}\int_u^\pi (t - u)^{-\beta}\,\mathrm{Re}\, E(\omega, t, s)dtdu$$

$$= -(\delta + \beta)\int_0^\pi \mathrm{Re}\, E(\omega, t, s)\int_0^t u^{\delta+\beta-1}(t - u)^{-\beta}dudt$$

$$= -(\delta + \beta)B(\delta + \beta, 1 - \beta)\sum_{n<\omega} Q(n, \omega, s)\int_0^\pi t^\delta \cos ntdt$$

$$= O(1)\left\{\sum_{n<\omega} Q(n, \omega, s - \delta - 1)\right\}$$

by

$$\int_0^\pi t^\delta \cos ntdt = O(n^{-1-\delta}) , \qquad 0 \leqq \delta \leqq 1 .$$

Since $\sum n^{-(\delta+1-s)} < \infty$ $(\delta > s)$, we obviously have $J < \infty$ by the absolute regularity of the method $|R, e(\omega), r|$.

Next we consider about K. As $G(\omega, u, s) = G(\omega, \pi, s) - H(\omega, u, s)$, we have

$$K = \int_A^\infty \frac{|G(\omega, u, s)|}{\omega^{1-\alpha}(e(\omega))^r} d\omega$$
$$\leqq \int_A^\zeta \frac{|G(\omega, u, s)|}{\omega^{1-\alpha}(e(\omega))^r} d\omega + \int_\zeta^\infty \frac{|G(\omega, \pi, s)|}{\omega^{1-\alpha}(e(\omega))^r} d\omega$$
$$+ \int_\zeta^\infty \frac{|H(\omega, u, s)|}{\omega^{1+\alpha}(e(\omega))^r} d\omega = K_1 + K_2 + K_3 , \qquad \text{say} ,$$

where $\zeta = u^{-1/a}$ and $a = (s + \beta)/(\delta + \beta)$. Since $K_2 = O(J) = O(1)$, we estimate K_1 first. Using (ii) of Lemma 2, we have

$$K_1 = O(u^{\delta+\beta}) \int_A^\zeta \frac{\omega^{s+\beta-\alpha}}{\omega^{1-\alpha}} d\omega + O(u^{\delta+\beta}) \int_A^\zeta \frac{Q(m, \omega, s + \beta - 1)}{\omega^{1-\alpha}(e(\omega))^r} d\omega$$
$$= O(u^{\delta+\beta}\zeta^{s+\beta}) + O(K_1') = O(1) + O(K_1') ,$$

because $0 < s + \beta$ and $\delta + \beta - (s + \beta)/a = 0$, where

$$(1) \quad K_1' = \int_A^\zeta \frac{Q(m, \omega, s + \beta - 1)}{\omega^{1-\alpha}(e(\omega))^r} d\omega .$$

By (iii) of Lemma 2, we get

$$K_3 = O(u^{\delta+\beta-r}) \int_\zeta^\infty \frac{\omega^{(\alpha-1)(r-1)+s+\beta-1}}{\omega^{1-\alpha}} d\omega$$
$$+ O(1) \int_\zeta^\infty \frac{Q(m, \omega, s + \beta - 1)}{\omega^{1-\alpha}(e(\omega))^r} d\omega$$
$$= O(u^{\delta+\beta-r}\zeta^{s+\beta+(\alpha-1)r}) + O(K_3')$$
$$= O(1) + O(K_3') ,$$

because

$$s + \beta + (\alpha - 1)r = s + \beta - r(s + \beta)/(\delta + \beta)$$
$$= (s + \beta)(\delta + \beta - r)/(\delta + \beta) < 0$$

and $\delta + \beta - r - \{s + \beta + (\alpha - 1)r\}/a = 0$, where

$$(2) \quad K_3' = \int_\zeta^\infty \frac{Q(m, \omega, s + \beta - 1)}{\omega^{1-\alpha}(e(\omega))^r} d\omega .$$

Now, we have to estimate $K_1' + K_3'$. Since $Q(m, \omega, s + \beta - 1) = (e(\omega) - e(m))^{r-1}e(m)m^{s+\beta-1}$ and

$$\int_m^{m+1} \frac{(e(\omega) - e(m))^{r-1}}{\omega^{1-\alpha}(e(\omega))^r} d\omega$$

$$= \frac{1}{\alpha r}\int_m^{m+1} \frac{1}{(e(\omega))^{r+1}} \frac{d}{d\omega}(e(\omega) - e(m))^r d\omega$$
$$\leqq \frac{1}{\alpha r}(e(m))^{-r-1}(e(m+1) - e(m))^r = O\{(e(m))^{-1}m^{(\alpha-1)r}\} ,$$

we obtain, by (1) and (2)

$$K_1' + K_3' \leqq \sum_{m=1}^{\infty}\int_m^{m+1} \frac{Q(m, \omega, s+\beta-1)}{\omega^{1-\alpha}(e(\omega))^r} d\omega$$
$$= O(1)\sum_{m=1}^{\infty} m^{s+\beta-1+r(\alpha-1)} = O(1) ,$$

because $s + \beta + r(\alpha - 1) < 0$.

Summing up the above estimates, we get $J + K = O(1)$, which is the required.

4. Corollaries.

In this section we consider some applications of our theorem.

If we put $\delta + \beta = \gamma$, $s = 0$ or $\delta = 0$ in our theorem respectively, we obtain the following Corollaries 1 or 2.

COROLLARY 1 (Matsumoto [4], cf. [5]). *Let* $0 < \beta < \gamma < 1$. *Then* $t^{-\gamma}\Phi_\beta(t) \in BV(0, \pi)$ *implies that* $\sum_{n=1}^{\infty} A_n(x)$ *is summable* $|R, e(\omega), r|$, *where* $r > \gamma$ *and* $\alpha = 1 - \beta/\gamma$.

COROLLARY 2 (cf. Dikshit [1], [2]). *Let* $0 < \beta < 1$ *and* $\varphi_\beta(t)$ *is of bounded variation in* $(0, \pi)$. *Then* $\sum_{n=1}^{\infty} n^{-\alpha\beta}A_n(x)$ *is summable* $|R, e(\omega), r|$, *where* $0 < \alpha < 1$ *and* $r > \beta$.

COROLLARY 3 (cf. Dikshit [3], Matsumoto [5]). *Let* $0 < s < \delta < 1$ *and* $t^{-\delta}\varphi(t) \in BV(0, \pi)$. *Then* $\sum n^s A_n(x)$ *is summable* $|R, e(\omega), r|$, *where* $\alpha = 1 - s/\delta$ *and* $r > \delta$.

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Department of Mathematics
Yamagata University
Yamagata, 990
Japan