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THE FUNDAMENTAL THEOREM OF ANALYTIC SPACE CURVES AND APPARENT SINGULARITIES OF FUCHSIAN DIFFERENTIAL EQUATIONS

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We first recall the classical fundamental theorem of space curves, i.e., curves in Euclidean 3-space $\boldsymbol{E}^3$.

(i) Let $x^i = x^i(s)$ $(i = 1, 2, 3)$ be a curve of class C^3, where s is the arc-length parameter. Moreover we assume that its curvature $\kappa(s)$ *does not vanish anywhere*. Then there exists an orthonormal frame $\{\boldsymbol{e}_i(s)\}$ which satisfies the Frenet-Serret equation

$$(1)\qquad \begin{cases} d\boldsymbol{e}_1/ds = \kappa \boldsymbol{e}_2 \\ d\boldsymbol{e}_2/ds = -\kappa \boldsymbol{e}_1 + \tau \boldsymbol{e}_3 \\ d\boldsymbol{e}_3/ds = -\tau \boldsymbol{e}_2 \end{cases}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are the tangent, principal normal and binormal unit vectors, respectively, and $\tau(s)$ is the torsion.

(ii) Given a function $\kappa(s)$ of class C^1 and a continuous function $\tau(s)$, there exists a curve of class C^3 which admits an orthonormal frame $\{\boldsymbol{e}_i\}$ satisfying the equation (1) with given κ and τ as its curvature and torsion, respectively. Such a curve is uniquely determined up to a motion of $\boldsymbol{E}^3$.

In this paper we shall study the fundamental theorem of *analytic* space curves of which curvatures *have discrete zero points*. At zero points of the curvature, principal normal and binormal vectors are discontinuous in general (e.g., $y = x^3$, $z = 0$ in (x, y, z)-space; by the definition, $\boldsymbol{e}_2 = (-3x^3/(|x|(1 + 9x^4)^{1/2}), x/(|x|(1 + 9x^4)^{1/2}), 0)$, $\boldsymbol{e}_3 = (0, 0, 1)$ for $x > 0$ and $= (0, 0, -1)$ for $x < 0$) and the curvature is not always differentiable even if the curve is analytic. This fact shows that, for instance, when such a function κ and any analytic function τ were given, it is hard to see whether the curve determined by those is analytic or not (if κ is merely continuous, a solution of the equation (1) exists; in case $\kappa > 0$, see Hartman and Wintner [1]).

On this problem there are several investigations (e.g., [2], [5] and [6]). Especially, Nomizu [2] studied Frenet curves in detail and showed that an analytic curve is always a Frenet curve. Though his work extremely clarifies the ambiguities of the classical theorem stated above, there are still some indefiniteness on the choice of normal vectors, the sign of the curvature and the torsion (in [2], two invariants $k_1 = \pm\kappa$ and $k_2 = \pm\tau$ were taken instead of κ and τ, see also, Wong and Lai [6], p. 9).

In this paper we shall take a length-varying frame of which normal directions coincide with those of the classical frame at non-zero points of κ and shall assert that the essential invariants for analytic curves are $\kappa^2(s)$ and $\tau(s)$. A similar method is useful for investigating analytic curves with singularities [3]. Our main results are Theorem A in §1 and Theorem B in §3 which correspond to (i) and (ii) above, respectively. In §1, we shall define an orthogonal frame and shall obtain a formula which corresponds to the Frenet-Serret equation. In §2, we shall solve it by a method in the theory of regular singularities in the complex domain and shall show the existence of curves and frames. In the last section, we shall prove orthogonality of frames.

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1. Orthogonal Frame and Theorem A. Let x^i $(i = 1, 2, 3)$ be coordinates in E^3. Let C be an analytic curve defined by $x^i = x^i(t)$, where t runs through some interval and $x^i(t)$ be analytic in t. We assume that C is non-singular, i.e., $\sum_{i=1}^{3}(dx^i(t)/dt)^2$ is nowhere zero. Therefore we can parametrize C by its arc length s. From now on, we only consider C in the following form:

$$C\colon x = x(s) = (x^1(s), x^2(s), x^3(s)) \,, \qquad s \in L \,,$$

where $x(s)$ is analytic in s and L is a non-empty open interval. We assume that the curvature $\kappa(s)$ of C is not identically zero.

Now we define an orthogonal frame $\{E_i(s)\}$ as follows:

$$E_1 = dx/ds \,, \quad E_2 = dE_1/ds \,, \quad E_3 = E_1 \times E_2 \,,$$

where $E_1 \times E_2$ is the vector product of E_1 and E_2. The relations between those and the classical Frenet frame $\{e_i(s)\}$ at non-zero points of κ are

$$E_1 = e_1 \,, \quad E_2 = \kappa e_2 \,, \quad E_3 = \kappa e_3 \,. \tag{1.1}$$

Thus $E_2(s_0) = E_3(s_0) = 0$ when $\kappa(s_0) = 0$ and squares of the length of E_2 and E_3 vary analytically in s. By the definition of E_i or (1.1), a simple calculation shows that

$$(d/ds)\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & (\log\kappa^2)'/2 & \tau \\ 0 & -\tau & (\log\kappa^2)'/2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} , \tag{1.2}$$

where a dash denotes the differentiation with respect to the arc length s and $\tau = \tau(s) = \det(dx/ds, d^2x/ds^2, d^3x/ds^3)/\kappa^2$ is the torsion of C. From the Frenet-Serret equation, we know that any zero point of κ^2 is a removable singularity of τ. The equation (1.2) corresponds to the Frenet-Serret equation in the classical case. Moreover, $\{E_i\}$ satisfies:

$$\begin{cases} (E_1, E_1) = 1\,, & (E_2, E_2) = (E_3, E_3) = \kappa^2\,, \\ (E_i, E_j) = 0 & \text{for} \quad i \neq j\,, \end{cases} \tag{1.3}$$

where (,) denotes the inner product of E^3. We note that the essential quantities in (1.2) and (1.3) are $\kappa^2(s)$ and $\tau(s)$ which are analytic in s. Thus we conclude:

THEOREM A. *Let $x(s)$ be an analytic space curve of which curvature does not vanish identically. Then there exists an orthogonal frame $\{E_i(s)\}$ which satisfies the equation* (1.2) *and* (1.3).

REMARK 1.1. For the next section we note the following fact: Let s_0 be a zero point of κ. Let us represent E_2 by the Taylor series at $s = s_0$. Since $E_2(s_0) = 0$, it can be written as $E_2 = \sum_{k=n}^{\infty} \boldsymbol{a}_k (s - s_0)^k$, $\boldsymbol{a}_n \neq 0$ $(n > 0)$. Therefore, by the formula $\kappa^2 = (E_2, E_2)$, we obtain

$$\kappa^2 = (\boldsymbol{a}_n, \boldsymbol{a}_n)(s - s_0)^{2n} + \cdots\,. \tag{1.4}$$

This shows that the first non-vanishing term of the power series expansion of κ^2 at $s = s_0$ has an even power of $s - s_0$.

Next, by a rotation of E^3, we may assume that

$$E_1(s_0) = (1, 0, 0)\,. \tag{1.5}$$

Thus, by the definition of $\{E_i\}$, the following must be satisfied;

$$\boldsymbol{a}_n = (0, a_{n2}, a_{n3}) \tag{1.6}$$

$$E_3 = (0, -a_{n3}, a_{n2})(s - s_0)^n + \cdots\,. \tag{1.7}$$

2. Existence of Curves and Regular Singularity. Let L be a non-empty open interval. For simplicity we may assume that L contains the origin $s = 0$ at which a given analytic function K vanishes. Though a negative arc-length is curious, no absurdity occurs in mathematical meaning.

Now we take two real analytic functions $K(s) \geqq 0$ and $\tau(s)$ on L, and consider natural extensions of those and the equation (2.1) below into the complex domain.

$$(2.1)\qquad dE/ds = \begin{bmatrix} 0 & 1 & 0 \\ -K & (\log K)'/2 & \tau \\ 0 & -\tau & (\log K)'/2 \end{bmatrix} E ,$$

where $E = {}^t(E_1, E_2, E_3)$. Let D be a sufficiently small disk with center $s = 0$ on which K and τ are holomorphic and $s = 0$ is the only zero point of K. The power series expansions of K and τ on D can be taken as follows (cf., Remark 1.1):

$$(2.2)\qquad \begin{cases} K = \sum_{k=2n}^{\infty} \kappa_k s^k , & \kappa_{2n} > 0 , \quad n \geqq 1 , \\ \tau = \sum_{k=0}^{\infty} \tau_k s^k , & \kappa_k, \tau_k \in \boldsymbol{R} . \end{cases}$$

Let us find a solution of the equation (2.1). We may assume $E_1(0) = (1, 0, 0)$ which is obtained by a motion (especially, a rotation) of $\boldsymbol{E}^3$. Consequently, our problem is to obtain a real solution of (2.1) which, from (1.5), (1.6) and (1.7), satisfies:

$$(2.3)\qquad E(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$(d^n E(0)/ds^n)/n! = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \xi & \pm(\kappa_{2n} - \xi^2)^{1/2} \\ 0 & \mp(\kappa_{2n} - \xi^2)^{1/2} & \xi \end{bmatrix}$$

where

$$(2.4)\qquad |\xi| \leqq \kappa_{2n}^{1/2} , \qquad \xi \in \boldsymbol{R} .$$

Obviously, the linearity of the equation (2.1) and the condition (2.3) show that, by a rotation of the $(x^2 \cdot x^3)$-plane, one solution is obtained from another of which sign of $(\kappa_{2n} - \xi^2)^{1/2}$ in (2.3) is opposite to the former. There is still an indefiniteness depending on ξ, but, for two solutions E_{ξ_i} $(i = 1, 2)$ $(\xi_1 \neq \xi_2, |\xi_i| \leqq \kappa_{2n}^{1/2})$ corresponding to ξ_i, there also exists a rotation of the $(x^2 \cdot x^3)$-plane

$$(2.5)\qquad E_{\xi_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix} E_{\xi_2} ,$$

where

$$a = [\xi_1 \xi_2 + (\kappa_{2n} - \xi_1^2)^{1/2} (\kappa_{2n} - \xi_2^2)^{1/2}] / \kappa_{2n} ,$$
$$b = [\xi_1 (\kappa_{2n} - \xi_2^2)^{1/2} - \xi_2 (\kappa_{2n} - \xi_1^2)^{1/2}] / \kappa_{2n} .$$

LEMMA 2.1. *The equation* (2.1) *has a regular singularity at* $s = 0$ *and has characteristic exponents* $(0, n, n)$.

PROOF. From (2.2), $(\log K)'/2$ has a pole of order 1 with residue n at $s = 0$ and the other components of the coefficient matrix of (2.1) are holomorphic. Thus $s = 0$ is a regular singular point of (2.1) (Sauvage [4]). Let us rewrite (2.1) in the vector form:

$$s(de/ds) = Ae , \tag{2.6}$$

where e is a 3-dimensional vector ${}^t(e_1, e_2, e_3)$. Then, $A = \sum_{k=0}^{\infty} A_k s^k$ satisfies:

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{*}} \\ 0 & & \end{bmatrix}, \quad A_2, \cdots, A_{2n} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\boxed{*}} \\ 0 & & \end{bmatrix}. \tag{2.7}$$

By definition, the characteristic exponents of (2.1) are the roots ρ of $\det(A_0 - \rho I) = 0$, where I is the identity matrix. Lemma follows from (2.7). q.e.d.

Now we determine a solution of (2.1), namely, three independent solutions of (2.6). Let e_0, e_n and $\tilde{e}_n$ be solutions corresponding to the characteristic exponents 0, n and one more n, respectively (Lemma 2.1). First, we determine e_n and $\tilde{e}_n$.

Let $e_n = \sum_{k=n}^{\infty} B_k s^k$ and $\tilde{e}_n = \sum_{k=n}^{\infty} \tilde{B}_k s^k$. From (2.6), we obtain

$$[(n + k)I - A_0]B_{n+k} = \sum_{\substack{i+j=n+k \\ i \neq 0}} A_i B_j . \tag{2.8}$$

Since

$$nI - A_0 = \begin{bmatrix} n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

we can determine B_n and $\tilde{B}_n$ as follows (see (2.3)):

$$\begin{cases} B_n = {}^t(0, \xi, -(\kappa_{2n} - \xi^2)^{1/2}) \\ \tilde{B}_n = {}^t(0, (\kappa_{2n} - \xi^2)^{1/2}, \xi) , \end{cases} \tag{2.9}$$

where ξ satisfies (2.4). For any integer $k \geqq 1$, the coefficient matrix on the left hand side of (2.8) is non-singular. Thus B_k and $\tilde{B}_k$ $(k > n)$ are determined uniquely by (2.9) and, from the theory of regular singularity, e_n and $\tilde{e}_n$ represent holomorphic functions on D. Moreover, they take real values for real s, since the coefficient matrices of (2.8) and (2.9) are all real.

Next, we determine $e_0 = \sum_{k=0}^{\infty} C_k s^k$. C_k also must satisfy

$$(kI - A_0)C_k = \sum_{\substack{i+j=k \\ i\neq 0}} A_i C_j \ . \tag{2.10}$$

From (2.4), we take

$$C_0 = {}^t(1, 0, 0) \ . \tag{2.11}$$

It satisfies (2.10), for $A_0 C_0 = 0$ from (2.7). For any positive integer k ($<n$), $kI - A_0$ is non-singular and the right hand side of (2.10) is zero by (2.7). Thus $C_k = 0$ ($1 \leqq k < n$). In case $k = n$, (2.10) can be written as follows (see also (2.7)):

$$\begin{bmatrix} n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} C_n = 0 \ .$$

Thus $C_n = 0$ satisfies (2.10). Since $kI - A_0$ is non-singular for k ($>n$), C_k is determined uniquely by (2.11) and $C_n = 0$. The solution e_0 determined as above becomes also a holomorphic function on D and takes real value for real s. Obviously, from (2.9) and (2.11), the Wronskian W of the solutions has the following power series expansion:

$$W = \det(e_0, e_n, \tilde{e}_n) = \kappa_{2n} s^{2n} + \cdots \ .$$

Since W is not identically zero, they constitute a system of linearly independent solutions of (2.6). Namely, ${}^t(E_1, E_2, E_3) = (e_0, e_n, \tilde{e}_n)$ is a solution of (2.1) which satisfies (2.3). Since the solution was found in a class of holomorphic functions on D, $s = 0$ is an *apparent singularity* of (2.1). Restricting it to real s, we obtain a solution for which we have originally searched.

Integrating E_1 with respect to s, we obtain:

LEMMA 2.2. *Given two analytic functions* $K \geqq 0$ *and* τ, *there exists an analytic curve which admits three vectors* E_i ($i = 1, 2, 3$) *satisfying the equation* (2.1). *It is uniquely determined up to a motion of* E^3.

3. Orthogonality and Theorem B. The purpose of this section is to show that the solution $\{E_i\}$ of (2.1) satisfies (see (1.3)):

$$(E_1, E_1) = 1 \ , \quad (E_2, E_2) = (E_3, E_3) = K \ , \quad (E_i, E_j) = 0 \quad \text{for} \quad i \neq j \ , \tag{3.1}$$

i.e., $\{E_i\}$ is an orthogonal frame.

Let $F_{ij} = (E_i, E_j)$ ($i, j = 1, 2, 3$). Then, by (2.1),

$$dF_{11}/ds = 2F_{12} \ ,$$
$$dF_{12}/ds = -KF_{11} + (\log K)' F_{12}/2 + \tau F_{13} + F_{22} \ ,$$

$$
\begin{aligned}
dF_{13}/ds &= -\tau F_{12} + (\log K)' F_{13}/2 + F_{23} \, , \\
dF_{22}/ds &= -2KF_{12} + (\log K)' F_{22} + 2\tau F_{23} \, , \\
dF_{23}/ds &= -KF_{13} - \tau F_{22} + (\log K)' F_{23} + \tau F_{33} \, , \\
dF_{33}/ds &= -2\tau F_{23} + (\log K)' F_{33} \, .
\end{aligned} \tag{3.2}
$$

We consider the natural extension of the equations (3.2) into the complex plane. We rewrite (3.2) as follows.

$$
s(dF/ds) = GF \, , \qquad F = {}^t(F_{11}, F_{12}, F_{13}, F_{22}, F_{23}, F_{33}) \, . \tag{3.3}
$$

Then $G = \sum_{k=0}^{\infty} G_k s^k$ has the property that G_0 is a diagonal matrix diag $(0, n, n, 2n, 2n, 2n)$. We can easily show the following (cf., Lemma 2.1):

LEMMA 3.1. *The equation* (3.3) *has a regular singularity at* $s = 0$ *with characteristic exponents* $(0, n, n, 2n, 2n, 2n)$.

From (1.3), we have to solve (3.3) under conditions (cf., (2.3));

$$
\begin{aligned}
&F(0) = {}^t(1, 0, \cdots, 0) \, , \qquad (d^n F(0)/ds^n) = 0 \, , \\
&(d^{2n} F(0)/ds^{2n})/(2n)! = {}^t(0, 0, 0, \kappa_{2n}, 0, \kappa_{2n}) \, .
\end{aligned} \tag{3.4}
$$

Thus the solution in question corresponds to the characteristic exponent 0 and must be determined uniquely by (3.4).

On the other hand, there exists a trivial solution $F = {}^t(1, 0, 0, K, 0, K)$ of (3.3) which also satisfies (3.4). Thus they must coincide with each other.

The above argument shows that $\{E_i\}$ is an orthogonal frame of the curve obtained in Lemma 2.2. Then, by (2.1) and (3.1), $K^{1/2}$ and τ become its curvature and torsion, respectively. Therefore we obtain:

THEOREM B. *When two analytic functions* $K \geqq 0$ *and* τ *are given, there exists an analytic curve which admits an orthogonal frame* $\{E_i\}$ *satisfying* (2.1) *and* (3.1) *with given* $K^{1/2}$ *and* τ *as its curvature and torsion, respectively. Such a curve is uniquely determined up to a motion of* $\boldsymbol{E}^3$.

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