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LIMIT SETS OF GEOMETRICALLY FINITE FREE KLEINIAN GROUPS

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Introduction. Let G be a geometrically finite Kleinian group. Let

$$S(z) = (az + b)/(cz + d)\ , \quad ad - bc = 1$$

be any element of G and let id be the identity transformation. The Poincaré dimension $P(G)$ and the Hausdorff dimension $d(\Lambda(G))$ of the limit set $\Lambda(G)$ for G are defined respectively as follows:

$$P(G) = \inf\left\{t \,\middle|\, \sum_{S \in G - \{\mathrm{id}\}} |c|^{-t} < +\infty\right\}$$

and

$$d(\Lambda(G)) = \inf\{\mu/2 \,|\, M_{\mu/2}(\Lambda(G)) = 0\}\ ,$$

where $M_{\mu/2}(\Lambda(G))$ denotes the $\mu/2$-dimensional Hausdorff measure of $\Lambda(G)$.

Suppose that G is a Schottky group. The former author proved the following relation ([1]):

$$(*) \qquad d(\Lambda(G)) = P(G)/2\ .$$

If G is a Fuchsian group of the first kind, the above $(*)$ is trivial. It was proved by Patterson ([6]) that $(*)$ holds for a Fuchsian group of the second kind without parabolic elements and for one with parabolic elements in the case $d(\Lambda(G)) \geqq 2/3$. He then posed the problem whether or not $(*)$ holds for $1/2 < d(\Lambda(G)) < 2/3$. Sullivan ([7]) solved this problem affirmatively by using the method of the space group and recently announced further in [8] that $(*)$ is true for a geometrically finite Kleinian group and that the proof will appear in [9].

In the previous paper [2] we proved that $(*)$ holds for a finitely generated Kleinian group with a fundamental domain bounded by a finite number of circles which are mutually disjoint or tangent externally to each other and posed the problem whether or not $(*)$ holds for more general geometrically finite free groups. The purpose of this paper is to show that $(*)$ is valid for such groups. Because our method is very

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different from Sullivan's, it is worthwhile to give another proof to (*) for a geometrically finite free Kleinian group, in spite of Sullivan's proof being valid for general geometrically finite Kleinian groups.

In §1, we shall state preliminaries and notations about a geometrically finite free Kleinian group G and give the relation between the Hausdorff measure and the measure defined by the special covering formed by the isometric circles for the limit set of G. We shall prove the main theorem giving the relation between the computing function and the Hausdorff measure of the limit set of G in §2. Finally, in §3 we shall give the relation (*) between the Poincaré dimension and the Hausdorff dimension of the limit set for G by using the main theorem.

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1. Preliminaries and notations. 1. Let G be a geometrically finite, free Kleinian group with basis $\{T_1, \cdots, T_p\}$ $(p\geqq 2)$. We denote by $\Omega(G)$ and $\Lambda(G)$ the region of discontinuity and the limit set of G, respectively. We put $\mathscr{G} = \{T_1, T_1^{-1}, \cdots, T_p, T_p^{-1}\}$. Then, for any $S\in G$, there exist $T_{\nu_1}, \cdots, T_{\nu_n}\in\mathscr{G}$ such that S can be represented uniquely in the normal form $S = T_{\nu_n}\circ\cdots\circ T_{\nu_1}$, where $T_{\nu_i}\neq T_{\nu_{i+1}}^{-1}$ $(i = 1, \cdots, n-1)$. So we shall call the number n the grade of S and use the notation $S(n)$ to clarify the grade n of S.

Throughout this paper we assume $\infty\in\Omega(G)$. So it can be easily seen that any element of G which fixes ∞ is the identity.

Consider two arbitrary elements $S, T\in G$ with $S\neq T^{-1}$. Denote by I_S, I_T and $I_{S\circ T}$ the isometric circles of S, T and $S\circ T$, respectively. Let R_S, R_T and $R_{S\circ T}$ be the radii of I_S, I_T and $I_{S\circ T}$, respectively. As to these values, the following equalities hold (see [5]):

$$R_{S\circ T} = R_S R_T/|T(\infty) - S^{-1}(\infty)| \tag{1.1}$$

$$\begin{aligned}|(S\circ T)^{-1}(\infty) - T^{-1}(\infty)| &= R_{S\circ T}R_T/R_S \\ &= R_T^2/|T(\infty) - S^{-1}(\infty)|\,.\end{aligned} \tag{1.2}$$

By using (1.1) and (1.2), we have the following proposition ([2]).

PROPOSITION 1. *Let $\{S(n)\}$ be a sequence of G satisfying $S(n) = T_{\nu_n}\circ\cdots\circ T_{\nu_1}$ $(T_{\nu_1}, \cdots, T_{\nu_n})\in\mathscr{G}$ and $S(n+1) = T_{\nu_{n+1}}\circ S(n)$ for all $n\in N$, the set of all natural numbers. Then there exist two positive constants $k_0 = k_0(G) < 1$ and $k_1 = k_1(G)$ depending only on G such that*

$$k_0 \leqq R_{S(n+1)}^2/R_{S(n)}^2 \leqq k_1 \tag{1.3}$$

for all $n\in N$.

2. It is well known that every limit point of a geometrically finite group is either a point of approximation or a cusped parabolic fixed point ([4]). Let us denote by $\Lambda_a(G)$ the set of all points of approximation of G. Note that the difference between $\Lambda_a(G)$ and $\Lambda(G)$ is only a countable set. Hence we can see that the Hausdorff measure of $\Lambda(G)$ is equal to that of $\Lambda_a(G)$. As to such a subset $\Lambda_a(G)$ of $\Lambda(G)$, the following proposition is important ([2]).

PROPOSITION 2. *For any $z \in \Lambda_a(G)$, there exist $\{S(n)\} \subset G$ and $K_G > 0$ depending only on G such that*

$$|z - S^{-1}(n)(\infty)| < K_G R^2_{S(n)} .$$

For any sufficiently small $\delta > 0$, we denote by $I(\delta)$ a family of closed discs $\{D_\lambda\}$ of radii $l_\lambda \leqq \delta$ such that every point of $\Lambda(G)$ is contained in some $\mathrm{Int}(D_\lambda)$. We shall call the quantity

$$M_{\mu/2}(\Lambda(G)) = \lim_{\delta \to 0}\left[\inf_{\{I(\delta)\}}\left\{\sum_{D_\lambda \in I(\delta)} (2l_\lambda)^{\mu/2}\right\}\right]$$

the $\mu/2$-dimensional Hausdorff measure of $\Lambda(G)$, where $\mu \in (0, 4]$. From now on we assume $\mu \in (0, 4]$. For any $S(n) \in G - \{\mathrm{id}\}$, we denote by

$$B_{S(n)} = \{z \mid |z - S^{-1}(n)(\infty)| \leqq K_G R^2_{S(n)}\} ,$$

where K_G is a positive constant depending only on G in Proposition 2. Putting

$$F(n_0, \delta/k_0) = \{B_{S(n)} \mid S(n) \in G,\ n \geqq n_0 \quad \text{and} \quad K_G R^2_{S(n)} \leqq \delta/k_0\}$$

for any $\delta > 0$ and any μ, we obtain the following ([2]).

PROPOSITION 3. *For any μ, there exists a natural number N_0 depending only on G such that*

$$\begin{aligned}
&\lim_{\delta\to 0}\left[\inf_{\{F(n_0,\delta/k_0)\}}\left\{\sum_{B_{S(n)} \in F(n_0,\delta/k_0)} (2R^2_{S(n)})^{\mu/2}\right\}\right] \\
&\leqq N_0 (K_G k_0)^{-\mu/2} \lim_{\delta\to 0}\left[\inf_{\{I(\delta)\}}\left\{\sum_{D_\lambda \in I(\delta)} (2l_\lambda)^{\mu/2}\right\}\right] \\
&= N_0 (K_G k_0)^{-\mu/2} M_{\mu/2}(\Lambda(G)) .
\end{aligned} \tag{1.4}$$

2. Computing functions and Hausdorff measure of $\Lambda(G)$. 1. Since $\infty \in \Omega(G)$, the set $\{S^{-1}(\infty) \mid S \in G - \{\mathrm{id}\}\}$ is bounded. Hence, for any $T \in \mathscr{G}$ and any $S(n) = T \circ S(n-1) \in G$ $(n \in N)$, there exists a positive constant k_T depending only on T such that

$$S(n)(\infty) \in \{z \mid |z - T(\infty)| < k_T R_T\} .$$

Here we put $k_G = \max_{T\in\mathscr{G}}\{k_T\}$. It can be easily seen that k_G is positive and depends only on G. Let us denote by $D'_T = \{z \mid |z - T(\infty)| < k_G R_T\}$. Then, for any $T \in \mathscr{G}$ and any $S(n) = T\circ S(n-1) \in G$ $(n \in N)$, we have $D'_T \ni S(n)(\infty) = (T\circ S(n-1))(\infty)$.

First of all we shall prove the following.

LEMMA 1. *Assume that* $U(m+1), V(n+1) \in G$ *are of the form* $U(m+1) = T^*\circ U(m)$, $V(n+1) = \tilde{T}\circ V(n)$, *where* $m, n \in N$, $\tilde{T}, T^* \in \mathscr{G}$ *and* $U(m), V(n) \in G$. *If* $T^* \neq \tilde{T}$, *then there exists a positive constant* k^* *depending only on* G *such that*

$$|U(m+1)(\infty) - V(n+1)(\infty)| \geqq k^*$$

for all $m, n \in N$.

PROOF. Putting $T = T^*\circ U(m)$ and $S^{-1} = \tilde{T}\circ V(n)$ in (1.2), we have

$$(2.1) \quad \begin{aligned} &|(T^*\circ U(m))(\infty) - (\tilde{T}\circ V(n))(\infty)| \\ &= R^2_{T^*\circ U(m)}/|(U^{-1}(m)\circ T^{*^{-1}}\circ \tilde{T}\circ V(n))(\infty) - (U^{-1}(m)\circ T^{*^{-1}})(\infty)|\,. \end{aligned}$$

Since $U^{-1}(m+1)\circ V(n+1) = U^{-1}(m)\circ T^{*^{-1}}\circ \tilde{T}\circ V(n)$, the grade of $U^{-1}(m+1)\circ V(n+1)$ is $(m+1)+(n+1) = m+n+2$. Then it follows that $B_{U(m+1)} \supset B_{V^{-1}(n+1)\circ U(m+1)}$ for sufficiently large n's. Therefore we can take a sequence $\{S(n_k)\} \subset G$ such that

$$B_{V^{-1}(n+1)\circ U(m+1)} \supset B_{S(n_1)\circ V^{-1}(n+1)\circ U(m+1)} \supset \cdots \supset B_{S(n_k)\circ\cdots\circ S(n_1)\circ V^{-1}(n+1)\circ U(m+1)} \supset \cdots .$$

It is trivial that $\bigcap_{k=1}^{\infty} B_{S(n_k)\circ\cdots\circ S(n_1)\circ V^{-1}(n+1)\circ U(m+1)} \subset \Lambda_a(G)$. Then we can take $z \in \Lambda_a(G)$, so that $z \in B_{V^{-1}(n+1)\circ U(m+1)} \cap B_{U(m+1)}$. Hence we obtain from Proposition 2 the following:

$$(2.2) \quad \begin{aligned} &|(U^{-1}(m)\circ T^{*^{-1}}\circ \tilde{T}\circ V(n))(\infty) - (U^{-1}(m)\circ T^{*^{-1}})(\infty)| \\ &\leqq |(U^{-1}(m+1)\circ V(n+1))(\infty) - z| + |z - U^{-1}(m+1)(\infty)| \\ &\leqq K_G R^2_{V^{-1}(n+1)\circ U(m+1)} + K_G R^2_{U(m+1)} \\ &\leqq 2K_G R^2_{U(m+1)}\,. \end{aligned}$$

Applying (2.2) to (2.1), we have

$$|(T^*\circ U(m))(\infty) - (\tilde{T}\circ V(n))(\infty)| \geqq 1/2K_G\,. \qquad \text{q.e.d.}$$

For each $T \in \mathscr{G}$, we put $D_T = D'_T - \cup\{z \mid |z - S(n)(\infty)| < k^*/2\}$, where the union is taken over all $S(n) \in G$ with $S(n) = T'\circ S(n-1)$ for a $T' \in \mathscr{G} - \{T\}$. The set D_T is not empty by Lemma 1.

2. Let $S(n) = T_{\nu_n}\circ\cdots\circ T_{\nu_1} \in G - \{\mathrm{id}\}$ be of the form $S(n)(z) = (az+b)/(cz+d)$, $ad - bc = 1$. Taking the derivative of $S(n)$, we get

$$(2.3) \quad |dS(n)(z)/dz|^{\mu/2} = |cz+d|^{-\mu} = (R_{S(n)}/|S^{-1}(n)(\infty) - z|)^{\mu}\,.$$

Take any fixed element $T \in \mathscr{G}$. Forming the sum of $(2p-1)^n$ terms with respect to all $S(n)$ in (2.3) with $T_{\nu_1} \neq T^{-1}$, we have the following function

$$\sum_{S(n)} R^{\mu}_{S(n)}/|S^{-1}(n)(\infty) - z|^{\mu} = \sum_{S(n)} |dS(n)(z)/dz|^{\mu/2} . \tag{2.4}$$

We denote it by $\chi^{(\mu;T)}_n(z)$ and call it the μ-dimensional computing function of order n on T. The domain of definition of $\chi^{(\mu;T)}_n(z)$ is D_T.

Assume that $S(l) \in G$ is of the form $S(l) = T \circ S(l-1)$ $(T \in \mathscr{G})$. It can be easily seen that $S(l)(\infty) \in D_T$. Then we can obtain from (1.1) and (2.4)

$$\begin{aligned} \chi^{(\mu;T)}_n(S(l)(\infty)) &= \sum_{S(n)} R^{\mu}_{S(n)}/|S^{-1}(n)(\infty) - S(l)(\infty)|^{\mu} \\ &= \sum_{S(n)} R^{\mu}_{S(n)\circ S(l)}/R^{\mu}_{S(l)} , \quad \text{where} \quad S(n)\circ S(l) = S(n+l) . \end{aligned} \tag{2.5}$$

The relation between two computing functions on the different elements of $\mathscr{G}$ is given as follows ([2]).

PROPOSITION 4. *For any two computing functions on the different elements of $\mathscr{G}$, there exists a positive constant $k(l, \mu)$ depending only on l and μ such that*

$$\chi^{(\mu;T)}_{n+l}(z) \geqq k(l, \mu)\chi^{(\mu;T')}_n(S(l)(z)) , \tag{2.6}$$

where $\lim_{l\to\infty} k(l, \mu) = 0$ *and* $S(l) = T' \circ S(l-1)$.

Next we shall look for the relation between two computing functions on the same T of different orders.

LEMMA 2. *Take any $T \in \mathscr{G}$ and any $z \in D_T$. Then for any positive integer n there exist two positive constants $k_1(n, \mu)$ and $k_2(n, \mu, z)$ depending only on n, μ and n, μ, z, respectively, such that*

$$k_1(n, \mu)\chi^{(\mu;T)}_l(z) \leqq \chi^{(\mu;T)}_{n+l}(z) \leqq k_2(n, \mu, z)\chi^{(\mu;T)}_l(z) \tag{2.7}$$

for all $l \in N$.

PROOF. For any fixed integer $n > 0$, we have

$$\chi^{(\mu;T)}_{n+l}(z) = \sum_{S(n+l)} R^{\mu}_{S(n)} R^{\mu}_{S(l)} |S^{-1}(n)(\infty) - S(l)(z)|^{-\mu} |S^{-1}(l)(\infty) - z|^{-\mu} , \tag{2.8}$$

where $S(n+l) = S(n)\circ S(l) = T_{\nu_{n+l}} \circ \cdots \circ T_{\nu_{1+l}} \circ T_{\nu_l} \circ \cdots \circ T_{\nu_1}$ $(T^{-1}_{\nu_1} \neq T)$. Noting $(S^{-1}(l) \circ S^{-1}(n))(\infty) = (T^{-1}_{\nu_1} \circ \cdots \circ T^{-1}_{\nu_{n+l}})(\infty) \notin D_T$, we see $|S^{-1}(n)(\infty) - S(l)(z)| \neq 0$. Since the natural number n is fixed, there exists $\delta = \delta(n, z) > 0$ such that $|S^{-1}(n)(\infty) - S(l)(z)| \geqq \delta$ for all $S(n) \in G$ and all $l \in N$. Furthermore there exists $r > 0$ such that $|S^{-1}(n)(\infty) - S(l)(z)| < r$, since $\infty \in \Omega(G)$. Hence we obtain

(2.9) $$\delta^\mu \leqq |S^{-1}(n)(\infty) - S(l)(z)|^\mu \leqq r^\mu .$$

Putting $\sigma_1 = \min_{S(n)\in G} R_{S(n)}$ and $\sigma_2 = \max_{S(n)\in G} R_{S(n)}$, we have the following from (2.9)

(2.10) $$\sigma_1^\mu/r^\mu \leqq R_{S(n)}^\mu/|S^{-1}(n)(\infty) - S(l)(z)|^\mu \leqq \sigma_2^\mu/\delta^\mu .$$

By combining (2.10) with (2.8), we obtain

$$\begin{aligned}&(2p-1)^n(\sigma_1/n)^\mu \chi_l^{(\mu;T)}(z)\\ &\leqq \sum_{S(n+l)} R_{S(n)}^\mu R_{S(l)}^\mu |S^{-1}(n)(\infty) - S(l)(z)|^{-\mu} |S^{-1}(l)(\infty) - z|^{-\mu}\\ &\leqq (2p-1)^n(\sigma_2/\delta)^\mu \chi_l^{(\mu;T)}(z) .\end{aligned}$$

Putting $(2p-1)^n(\sigma_1/r)^\mu = k_1(n, \mu)$ and $(2p-1)^n(\sigma_2/\delta)^\mu = k_2(n, \mu, z)$, we have (2.7). q.e.d.

3. Now let us give a lemma on a sequence of computing functions.

LEMMA 3. *Let* $\{\chi_n^{(\mu;T)}(z)\}$ *be a sequence of computing functions. Suppose that* $\lim_{n\to\infty} \chi_n^{(\mu;T^*)}(z_0) = \infty$ (resp. 0) *on some* $T^* \in \mathscr{G}$ *and some* $z_0 \in D_{T^*}$. *Then* $\lim_{n\to\infty} \chi_n^{(\mu;T^*)}(z) = \infty$ (resp. 0) *uniformly on* D_{T^*}.

PROOF. (I) The case of the limit ∞. For each $n \in N$, we put $S(n) = S(n-1)\circ T_{\nu_1}$ $(T_{\nu_1}^{-1} \neq T^*)$. Since $S^{-1}(n)(\infty) = (T_{\nu_1}^{-1}\circ S^{-1}(n-1))(\infty) \notin D_{T^*}$, we can easily see $|S^{-1}(n)(\infty) - z_0| \geqq k^*/2$ for all $S(n) = S(n-1)\circ T_{\nu_1} \in G$ $(T_{\nu_1}^{-1} \neq T^*)$. Here we choose a sufficiently large number $r > 0$ such that $\{z \mid |z - z_0| < r\} \supset \bigcup_{T\in\mathscr{G}} D_T$. Obviously we can take $k_0 > 0$ so that $r \leqq k_0 k^*/2$. Then we see

$$|S^{-1}(n)(\infty) - z| \leqq 2r \leqq k_0 k^* \leqq 2k_0 |S^{-1}(n)(\infty) - z_0|$$

for all $z \in D_{T^*}$. Hence we obtain from the above

(2.11) $$\chi_n^{(\mu;T^*)}(z) = \sum_{S(n)} R_{S(n)}^\mu/|S^{-1}(n)(\infty) - z|^\mu \geqq (2k_0)^{-\mu}\chi_n^{(\mu;T^*)}(z_0) .$$

Putting $K = (2k_0)^{-\mu}$, we have from (2.11) the following inequality

$$\chi_n^{(\mu;T^*)}(z) \geqq K\chi_n^{(\mu;T^*)}(z_0)$$

for all $z \in D_{T^*}$. This shows that $\lim_{n\to\infty} \chi_n^{(\mu;T^*)}(z) = \infty$ uniformly on D_{T^*}.

(II) The case of the limit 0. It can be easily seen that $|S^{-1}(n)(\infty) - z_0| \leqq r$. For any $S(n) = S(n-1)\circ T_{\nu_1} \in G$ $(T_{\nu_1}^{-1} \neq T^*)$, we see $|S^{-1}(n)(\infty) - z| \geqq k^*/2$ for all $z \in D_{T^*}$. Hence, in a way similar to the case of the limit ∞, we obtain

$$|S^{-1}(n)(\infty) - z_0| \leqq r \leqq k_0 k^*/2 \leqq k_0 |S^{-1}(n)(\infty) - z|$$

for all $z \in D_{T^*}$. Putting $K' = k_0^\mu$, we have

$$\chi_n^{(\mu;T^*)}(z) \leqq K'\chi_n^{(\mu;T^*)}(z_0)$$

for all $z \in D_{T^*}$. q.e.d.

4. Now let us give the main theorem.

THEOREM 1. *The following three propositions are equivalent to each other*:

(i) $\lim_{n\to\infty} \chi_n^{(\mu;T^*)}(z_0) = \infty$ (resp. 0) *on some* $T^* \in \mathscr{G}$ *and some* $z_0 \in D_{T^*}$.

(ii) $\lim_{n\to\infty} \chi_n^{(\mu;T)}(z) = \infty$ (resp. 0) *uniformly on* D_T *for any* $T \in \mathscr{G}$.

(iii) $M_{\mu/2}(\Lambda(G)) = \infty$ (resp. 0).

As the proof of this theorem is fairly complicated, we divide it into five lemmas. First we shall show that (i) is equivalent to (ii). For this purpose, it suffices to show that (i) implies (ii).

LEMMA 4. *Suppose that* $\lim_{n\to\infty} \chi_n^{(\mu;T^*)}(z_0) = \infty$ (resp. 0) *on some* $T^* \in \mathscr{G}$ *and some* $z_0 \in D_{T^*}$. *Then* $\lim_{n\to\infty} \chi_n^{(\mu;T)}(z) = \infty$ (resp. 0) *uniformly on* D_T *for any* $T \in \mathscr{G}$.

PROOF. (I) The case of the limit ∞. From Lemma 3, there exists a constant $K > 0$ such that $\chi_n^{(\mu;T^*)}(z) > K\chi_n^{(\mu;T^*)}(z_0)$ for all $n \in N$ and all $z \in D_{T^*}$. For any large $M_0 > 0$, there exists $n_0(M_0, T^*) > 0$ depending only on M_0 and T^* so that $\chi_n^{(\mu;T^*)}(z_0) \geqq M_0/K$ for any $n \geqq n_0(M_0, T^*)$. Then we conclude

$$\chi_n^{(\mu;T^*)}(z) \geqq M_0 \tag{2.12}$$

for all $n \geqq n_0(M_0, T^*)$ and all $z \in D_{T^*}$.

For any fixed $T \in \mathscr{G}$, there exist $z_T \in D_T$ and $S(n_T) \in G$ ($n_T \in N$) such that $S(n_T)(z_T) \in D_{T^*}$. Then we have the following from Proposition 4:

$$\chi_{n+n_T}^{(\mu;T)}(z_T) \geqq k(n_T, \mu)\chi_n^{(\mu;T^*)}(S(n_T)(z_T)) \tag{2.13}$$

for any $n \geqq n_0(M_0, T^*)$. As $S(n_T)(z_T) \in D_{T^*}$, there holds

$$\lim_{n\to\infty} \chi_n^{(\mu;T^*)}(S(n_T)(z_T)) = \infty .$$

Hence from (2.12) and (2.13), there exists $n_0(M_0, T) \in N$ which depends only on M_0 and T so that $\chi_{n+n_T}^{(\mu;T)}(z_T) > M_0$ for any $n \geqq n_0(M_0, T)$. Here we put $n^*(M_0) = \max_{T\in\mathscr{G}}\{n_0(M_0, T) + n_T\}$. Then $\chi_n^{(\mu;T)}(z_T) > M_0$ for any $T \in \mathscr{G}$ and any $n \geqq n^*(M_0)$. Hence $\chi_n^{(\mu;T)}(z_T)$ and $\chi_n^{(\mu;T^*)}(z_0)$ diverge uniformly to ∞ for all $T \in \mathscr{G}$. From Lemma 3, we obtain $\lim_{n\to\infty} \chi_n^{(\mu;T)}(z) = \infty$ uniformly on D_T for all $T \in \mathscr{G}$.

(II) The case of the limit 0. From Lemma 3, there exists a constant $K' > 0$ such that $\chi_n^{(\mu;T^*)}(z) \leqq K'\chi_n^{(\mu;T^*)}(z_0)$ for any $z \in D_{T^*}$. For any small $\varepsilon > 0$, there exists $n_0(\varepsilon, T^*) \in N$ depending only on ε and T^* such that

$\chi_n^{(\mu;T^*)}(z_0) < \varepsilon/K'$ for any $n \geqq n_0(\varepsilon, T^*)$. Then we have

$$\chi_n^{(\mu;T^*)}(z) < \varepsilon \tag{2.14}$$

for all $n \geqq n_0(\varepsilon, T^*)$ and all $z \in D_{T^*}$.

For any fixed $T \in \mathscr{G}$, there exist $z_T^* \in D_{T^*}$ and $S(n_T) \in G$ $(n_T \in N)$ such that $S(n_T)(z_T^*) \in D_T$. Hence we have from Proposition 4

$$\chi_{n+n_T}^{(\mu;T^*)}(z_T^*) > k(n_T, \mu)\chi_n^{(\mu;T)}(S(n_T)(z_T^*)) \tag{2.15}$$

for any $n \geqq n_0(\varepsilon, T^*)$. As $z_T^* \in D_{T^*}$, we have $\lim_{n\to\infty} \chi_{n+n_T}^{(\mu;T^*)}(z_T^*) = 0$. Hence from (2.14) and (2.15), there exists $n_0(\varepsilon, T) \in N$ depending only on ε and T such that

$$\chi_n^{(\mu;T)}(S(n_T)(z_T^*)) < \varepsilon \quad \text{for any } n \geqq n_0(\varepsilon, T)\,.$$

Here we put $n_0(\varepsilon) = \max_{T\in\mathscr{G}}\{n_0(\varepsilon, T)\}$. Then we obtain $\chi_n^{(\mu,T)}(S(n_T)(z_T^*)) < \varepsilon$ for any $T \in \mathscr{G}$ and any $n \geqq n_0(\varepsilon)$. Hence we complete the proof of this lemma. q.e.d.

5. Next we shall show that (ii) implies (iii).

LEMMA 5. *Suppose that* $\lim_{n\to\infty} \chi_n^{(\mu;T)}(z) = \infty$ (resp. 0) *uniformly on* D_T *for any* $T \in \mathscr{G}$. *Then* $M_{\mu/2}(\Lambda(G)) = \infty$ (resp. 0).

PROOF. (I) The case of the limit ∞. From the assumption of this lemma, for any $T \in \mathscr{G}$ and any $M > 1$, there exists $n_0(M) \in N$ depending only on M such that

$$\chi_n^{(\mu;T)}(z) > M \tag{2.16}$$

for any $z \in D_T$ and any $n \geqq n_0(M)$. Let an integer n_1 $(n_1 > n_0)$ be fixed. Consider the $2p(2p-1)^{n_1-1}$ elements of grade n_1 of G. Take and fix an element $S(n_1) = S(n_1 - 1)\circ T^{-1}$ of grade n_1 among them. Let $F(\tilde{n}_0, \delta/k_0)$ be a covering of $\Lambda_a(G)$ defined in §1. We take a covering consisting of a finite number of closed discs $B_{S(m_1)}, \cdots, B_{S(m_Q)} \in F(\tilde{n}_0, \delta/k_0)$ of $\Lambda_a(G) \cap B_{S(n_1)}$, i.e., $\bigcup_{j=1}^{Q} B_{S(m_j)} \supset \Lambda_a(G) \cap B_{S(n_1)}$. Here we assume that $\delta > 0$ is a sufficiently small number such that $\tilde{n}_0 - n_1 > n_0$.

We put $m^* = \min_{1\leqq j\leqq Q}\{m_j\}$. We amend these closed discs $B_{S(m_1)}, \cdots, B_{S(m_Q)}$ as follows: (i) if $m_j - m^* = n_0 r$ $(r \in \boldsymbol{Z}, r \geqq 0)$, then we put $m_j = m_j'$, and (ii) if $m_j - m^* = n_0 r + s$ $(r, s \in \boldsymbol{Z}, r \geqq 0, 1 \leqq s \leqq n_0 - 1)$, then we replace the closed disc $B_{S(m_j)}$ by $(2p-1)^{n_0-s}$ discs $B_{S_k(m_j')}$, $k = 1, 2, \cdots, (2p-1)^{n_0-s}$, of grade $m_j' = m^* + n_0(r+1) = m_j + (n_0 - s)$. By this procedure, we get a new covering of $\Lambda_a(G) \cap B_{S(n_1)}$ consisting of $B_{S(m_1')}, \cdots, B_{S(m_R')}$, $(Q \leqq R)$. Then there exists from (1.3) a constant $K(n_0, \mu) > 0$ depending only on n_0 and μ such that

$$(2.17)\qquad \sum_{j=1}^{Q} R^{\mu}_{S(m_j)} \geqq K(n_0, \mu) \sum_{j=1}^{R} R^{\mu}_{S(m'_j)} .$$

We again amend these closed discs $B_{S(m'_1)}, \cdots, B_{S(m'_R)}$ in the following manner.

In the set of closed discs $B_{S(m'_1)}, \cdots, B_{S(m'_R)}$, there exist a finite number of systems $W_{m^*_k}$ $(1 \leqq k \leqq n)$ with the following properties: (i) each $W_{m^*_k}$ has $(2p-1)^{n_0}$ closed discs of grade m^*_k and (ii) the grades of closed discs in different systems are not necessarily equal. Here we put $W_{m^*_k} = \{B_{S_j(n_0)\circ S(m^*_k - n_0)} \mid j = 1, 2, \cdots, (2p-1)^{n_0}\}$. We replace these $(2p-1)^{n_0}$ closed discs in each system $W_{m^*_k}$ by closed discs whose grades are $m^*_k - 1$. Repeat such procedure n_0 times for each $W_{m^*_k}$ $(1 \leqq k \leqq n)$. Then we see from (2.5) and (2.16)

$$(2.18)\qquad \sum_{S(n_0)} R^{\mu}_{S(m^*_k - n_0)\circ S(n_0)} > R^{\mu}_{S(m^*_k - n_0)} .$$

After such replacement we reach a new covering of $\Lambda_a(G) \cap B_{S(n_1)}$ consisting of closed discs $B_{S(m''_1)}, \cdots, B_{S(m''_U)} (U < R)$.

Repeating the above procedure to these closed discs and continuing $(r-1)$ times, we obtain the following:

$$(2.19)\qquad \sum_{j=1}^{R} R^{\mu}_{S(m'_j)} \geqq \sum_{S(m^* - n_1)} R^{\mu}_{S(m^*)} ,$$

where $S(m^*) = S(n_1)\circ S(m^* - n_1)$ and the summation on the right hand side is taken over all transformations in G of the form $S(m^*) = S(n_1)\circ S(m^* - n_1)$. Then we have from (2.5) and (2.16)

$$(2.20)\qquad \begin{aligned} \sum_{S(m^*-n_1)} R^{\mu}_{S(m^*)} &= \sum_{S^{-1}(m^*-n_1)} (R^{\mu}_{S^{-1}(m^*-n_1)\circ S^{-1}(n_1)} / R^{\mu}_{S^{-1}(n_1)}) \times R^{\mu}_{S^{-1}(n_1)} \\ &= \chi^{(\mu;T)}_{m^*-n_1}(S^{-1}(n_1)(\infty)) \times R^{\mu}_{S^{-1}(n_1)} \geqq M R^{\mu}_{S^{-1}(n_1)} , \end{aligned}$$

where $S(n_1) = S(n_1 - 1)\circ T^{-1}$ and the summation in (2.20) is taken over all the transformations of the form $S(n_1) = S(n_1 - 1)\circ T^{-1}$. Hence we obtain from (2.18), (2.19) and (2.20)

$$(2.21)\qquad \begin{aligned} \sum_{j=1}^{Q} R^{\mu}_{S(m_j)} &\geqq K(n_0, \mu) \sum_{j=1}^{R} R^{\mu}_{S(m'_j)} \\ &\geqq K(n_0, \mu) \sum_{S(m^*-n_1)} R^{\mu}_{S(m^*)} \geqq K(n_0, \mu)\cdot M \cdot R^{\mu}_{S(n_1)} . \end{aligned}$$

Noting that (2.21) holds for any closed discs $B_{S(n_1)}$, we obtain from (1.4) and (2.21) the following:

$$(2.22)\qquad \begin{aligned} &N_0(2K_G k_0)^{-\mu/2} M_{\mu/2}(\Lambda(G) \cap D_T) \\ &\geqq K(n_0, \mu)\Big(\sum_{S(n_1)} R^{\mu}_{S^{-1}(n_1)} / R^{\mu}_{T^{-1}}\Big) \times R^{\mu}_{T^{-1}} \times M . \end{aligned}$$

Since M is any positive number and n_1 is any fixed integer greater than n_0, we obtain from (2.22) that $M_{\mu/2}(\Lambda(G)) = \infty$ by letting n_1 to go to infinity.

(II) The case of the limit 0. For any $T \in \mathscr{G}$ and any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in N$ depending only on ε such that

$$\chi_{n_0}^{(\mu;T)}(z) < \varepsilon \tag{2.23}$$

for any $z \in D_T$. For any $x \in N$, we put $[x] = 2p(2p-1)^{x-1}$. Take any sufficiently large integer l ($l > n_0$) and let it be fixed. Then there exist $S_j(l) \in G$ ($j = 1, 2, \cdots, [l] = 2p(2p-1)^{l-1}$) such that

$$\bigcup_{j=1}^{[l]} B_{S_j(l)} \supset \Lambda_a(G) .$$

Note that $S(k)(\infty) \in D_T$, if $S(k) = T \circ S(k-1)$ for any $k \in N$. So we get from (2.5) and (2.23)

$$\chi_{n_0}^{(\mu;T)}(S(k)(\infty)) = \sum_{S(n_0)} R^{\mu}_{S(n_0)\circ S(k)}/R^{\mu}_{S(k)} < \varepsilon .$$

Hence we have

$$\sum_{S(n_0)} R^{\mu}_{S(n_0)\circ S(k)} < \varepsilon R^{\mu}_{S(k)} . \tag{2.24}$$

Let us put $l = rn_0 + s$ ($r, s \in N$, $s \leqq n_0 - 1$). Since $S(l) = S(n_0)\circ S(l-n_0) = S(n_0)\circ S((r-1)n_0 + s)$, we can see from (2.24)

$$\sum_{S(n_0)} R^{\mu}_{S(l)} < \varepsilon R^{\mu}_{S(l-n_0)} . \tag{2.25}$$

Taking the summation on both sides of (2.25) over all transformations of grade $l - n_0 = (r-1)n_0 + s$, we obtain

$$\sum_{j=1}^{[l]} R^{\mu}_{S_j(l)} < \varepsilon \sum_{j=1}^{[l-n_0]} R^{\mu}_{S_j(l-n_0)} .$$

If we repeat this procedure $(r-1)$ times, we obtain

$$\sum_{j=1}^{[l]} R^{\mu}_{S_j(l)} < \varepsilon^r \sum_{j=1}^{[s]} R^{\mu}_{S_j(s)} \leqq \varepsilon^r \max_{1\leqq m \leqq s} \left\{ \sum_{j=1}^{[m]} R^{\mu}_{S_j(m)} \right\} . \tag{2.26}$$

Since the right hand side of (2.26) tends to zero as r tends to the infinity, we have

$$\lim_{l\to\infty} \sum_{j=1}^{[l]} (R^2_{S_j(l)})^{\mu/2} = 0 .$$

Hence we conclude $M_{\mu/2}(\Lambda_a(G)) = M_{\mu/2}(\Lambda(G)) = 0$. q.e.d.

6. Now we shall show (iii) implies (i). First we shall show this in the case of the limit ∞ as follows.

LEMMA 6. *If* $M_{\mu/2}(\Lambda(G))=\infty$, *then* $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z_0)=\infty$ *for some* $T^*\in\mathscr{G}$ *and some* $z_0\in D_{T^*}$.

PROOF. Let n_0 be a fixed natural number. From the assumption, we have $S(n_0)\in G$ such that

$$M_{\mu/2}(\Lambda_a(G)\cap B_{S(n_0)})=\infty\ . \tag{2.27}$$

Put $S(n_0)=T^*\circ S(n_0-1)$. We can easily see $S(n_0)(\infty)\in D_{T^*}$. Then for any $n_1\in N$ $(n_1>n_0)$ and any $z\in\Lambda_a(G)\cap B_{S(n_0)}$, there exists $S(n_1)\in G$ such that $z\in B_{S(n_1)}$, where $S_{(n_1)}=S(n_1-n_0)\circ S(n_0)$. Hence we find that there exist $S_j(n_1)\in G$ $(j=1,2,\cdots,N_0=(2p-1)^{n_1-n_0})$ such that

$$\bigcup_{j=1}^{N_0}B_{S_j(n_1)}\supset\Lambda_a(G)\cap B_{S(n_0)}\ .$$

From the definition of Hausdorff measure we have

$$M_{\mu/2}(\Lambda_a(G)\cap B_{S(n_0)})\leqq\sum_{j=1}^{N_0}(2K_GR^2_{S_j(n_1)})^{\mu/2}\ . \tag{2.28}$$

From (2.5) it follows that

$$\begin{aligned}\sum_{j=1}^{N_0}R^\mu_{S_j(n_1)}&=\sum_{S(n_1-n_0)}R^\mu_{S(n_1-n_0)\circ S(n_0)}/R^\mu_{S(n_0)}\times R^\mu_{S(n_0)}\\&=\chi^{(\mu;T^*)}_{n_1-n_0}(S(n_0)(\infty))\times R^\mu_{S(n_0)}\ .\end{aligned} \tag{2.29}$$

Hence we obtain the following from (2.28) and (2.29)

$$M_{\mu/2}(\Lambda_a(G)\cap B_{S(n_0)})\leqq(2K_GR^2_{S(n_0)})^{\mu/2}\lim_{n_1\to\infty}\chi^{(\mu;T^*)}_{n_1-n_0}(S(n_0)(\infty))\ .$$

Putting $S(n_0)(\infty)=z_0$, we see $z_0\in D_{T^*}$. Thus from (2.27) we have $\lim_{n_1\to\infty}\chi^{(\mu;T^*)}_{n_1-n_0}(z_0)=\infty$. q.e.d.

7. In order to prove that (iii) implies (i) in the case of the limit zero, we have to prove the following.

LEMMA 7. *Suppose that there exists a subsequence* $\{\chi^{(\mu;T^*)}_{n_i}(z)\}$ *of* $\{\chi^{(\mu;T^*)}_n(z)\}$ *with respect to some* $T^*\in\mathscr{G}$ *such that* $\lim_{i\to\infty}\chi^{(\mu;T^*)}_{n_i}(z_0)=\infty$ (resp. 0) *for some* $z_0\in D_{T^*}$. *Then* $\lim_{n\to\infty}\chi^{(\mu;T^*)}_n(z_0)=\infty$ (resp. 0).

PROOF. (I) The case of the limit ∞. Replacing $\{\chi^{(\mu;T^*)}_n(z_0)\}$ by $\{\chi^{(\mu;T^*)}_{n_i}(z_0)\}$ in Lemma 3, we have $\lim_{i\to\infty}\chi^{(\mu;T^*)}_{n_i}(z)=\infty$ uniformly on D_{T^*}. For any large $M'>0$, there exists $n'_0=n'_0(M')\in N$ depending only on M' such that $\chi^{(\mu;T^*)}_{n'_0}(z)>M'$ for any $z\in D_{T^*}$. Here we put $\mathscr{G}=\{T_1,T_2,\cdots,T_p,T_{p+1}=T_1^{-1},T_{p+2}=T_2^{-1},\cdots,T_{2p}=T_p^{-1}\}$. Then, for any $T_j\in\mathscr{G}$, there exist $z_j\in D_{T_j}$ and $S_j(\tilde{n})\in G$ $(\tilde{n}\in N)$ such that $S_j(\tilde{n})(z_j)\in D_{T^*}$, where $S_j(\tilde{n})$ depends only on $T_j\in\mathscr{G}$ $(j=1,\cdots,2p)$. Hence we have from (2.6).

$$(2.30)\qquad \chi^{(\mu;T_j)}_{n_0'+\tilde{n}}(z_j) \geqq k(\tilde{n},\mu)\chi^{(\mu;T^*)}_{n_0'}(S_j(\tilde{n})(z_j)) \geqq k(\tilde{n},\mu)M' .$$

Since $\lim_{i\to\infty}\chi^{(\mu;T^*)}_{n_i}(S_j(\tilde{n})(z_j)) = \infty$ for any $T_j \in \mathscr{G}$, we can see

$$\lim_{i\to\infty}\chi^{(\mu;T_j)}_{n_i+\tilde{n}}(z_j) = \infty$$

for any $T_j \in \mathscr{G}$. From the proof of Lemma 3, for any $T_j \in \mathscr{G}$, there exists $K_j > 0$ depending only on $T_j \in \mathscr{G}$ such that

$$(2.31)\qquad \chi^{(\mu;T_j)}_{n_i+\tilde{n}}(z_j) \leqq K_j\chi^{(\mu;T_j)}_{n_i+\tilde{n}}(z)$$

for any n_i and any $z \in D_{T_j}$. Here we put $K_0 = \max_{1\leq j\leq 2p}\{K_j\}$. Hence we obtain the following inequality from (2.30) and (2.31)

$$(2.32)\qquad \chi^{(\mu;T_j)}_{n_0'+\tilde{n}}(z) \geqq K_0^{-1}\chi^{(\mu;T_j)}_{n_0'+\tilde{n}}(z_j) \geqq K_0^{-1}k(\tilde{n},\mu)M'$$

for any $z \in D_{T_j}$ and any $T_j \in \mathscr{G}$. Note that $k(\tilde{n},\mu)$ depends only on $\tilde{n}$ and μ. Take a sufficiently large number $M' > 0$ such that $K_0^{-1}k(\tilde{n},\mu)M' = M > 1$. Then there exists $n_0''(M) \in N$ such that $\chi^{(\mu;T_j)}_{n_0''+\tilde{n}}(z) \geqq M > 1$ for any $z \in D_{T_j}$ and $T_j \in \mathscr{G}$. Here let us put $n_0 = n_0''(M) + \tilde{n}$. Then we can easily see

$$(2.33)\qquad \chi^{(\mu;T_j)}_{n_0}(z) \geqq M > 1$$

for any $z \in D_{T_j}$ and any $T_j \in \mathscr{G}$.

Now let us consider the computing function $\chi^{(\mu;T^*)}_{qn_0}(z)$ at z_0 for $q \in N$. For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\chi^{(\mu;T^*)}_{qn_0}(z_0) > \chi^{(\mu;T^*)}_{qn_0}(z) - \varepsilon$$

for any $z \in D_{T^*} \cap \{z \mid |z - z_0| < \delta(\varepsilon)\}$. Take a sufficiently large $l \in N$. Then there exist $\varepsilon > 0$ and $S(l) \in G$ such that $S(l)(\infty) \in D_{T^*} \cap \{z \mid |z - z_0| < \delta(\varepsilon)\}$ and so

$$(2.34)\qquad \chi^{(\mu;T^*)}_{qn_0}(z_0) > \chi^{(\mu;T^*)}_{qn_0}(S(l)(\infty)) - \varepsilon .$$

Now we have from (2.5)

$$(2.35)\qquad \chi^{(\mu;T^*)}_{qn_0}(S(l)(\infty)) = \sum_{S(qn_0)} R^{\mu}_{S(qn_0)\circ S(l)}/R^{\mu}_{S(l)} .$$

Modifying the right hand side of (2.35), we obtain

$$(2.36)\qquad R^{-\mu}_{S(l)}\sum_{S(qn_0)} R^{\mu}_{S(qn_0)\circ S(l)} = \prod_{j=1}^{q}\left[\sum_{S(jn_0)} R^{\mu}_{S(jn_0)\circ S(l)} \Big/ \sum_{S((j-1)n_0)} R^{\mu}_{S((j-1)n_0)\circ S(l)}\right],$$

where $S(0) = \text{id}$. Since

$$R^{-\mu}_{S((j-1)n_0)\circ S(l)} \sum_{S(n_0)} R^{\mu}_{S(jn_0)\circ S(l)} = \chi^{(\mu;T_i)}_{n_0}(S((j-1)n_0)\circ S(l)(\infty))$$

$(j \geqq 1)$, we have from (2.33)

$$\chi^{(\mu;T_i)}_{n_0}(S((j-1)n_0)\circ S(l)(\infty)) \geqq M\,, \quad (j \geqq 1) \tag{2.37}$$

where $S((j-1)n_0)\circ S(l) = T_i \circ S((j-1)n_0 + l - 1)$, $T_i \in \mathscr{G}$. If we apply (2.35), (2.36) and (2.37) to (2.34), then we obtain

$$\chi^{(\mu;T^*)}_{qn_0}(z_0) > M^q - \varepsilon\,.$$

Hence we conclude

$$\lim_{q\to\infty} \chi^{(\mu;T^*)}_{qn_0}(z_0) = \infty\,. \tag{2.38}$$

For any positive integer $m = qn_0 + r$ ($q, r \in \boldsymbol{Z}$, $q, r \geqq 0$, $r \leqq n_0 - 1$), let us put $n = r$ and $l = qn_0$ in Lemma 2. Then we have from (2.7)

$$k_1(n_0, \mu)\chi^{(\mu;T^*)}_{qn_0}(z_0) \leqq \chi^{(\mu;T^*)}_{m}(z_0) \leqq k_2(n_0, \mu, z_0)\chi^{(\mu;T^*)}_{qn_0}(z_0)\,. \tag{2.39}$$

Therefore from (2.38) and (2.39) we conclude $\lim_{n\to\infty} \chi^{(\mu;T^*)}_{n}(z_0) = \infty$.

(II) The case of the limit 0. For any $T_j \in \mathscr{G}$ there exists $S_j(\tilde{n}) \in G$ $(\tilde{n} \in \boldsymbol{N})$ depending only on T_j such that $S_j(\tilde{n})(z_0) \in D_{T_j}$. Put $n_i - \tilde{n} = n'_i$. Then we have

$$\chi^{(\mu;T^*)}_{n_i}(z_0) > k(\tilde{n}, \mu)\chi^{(\mu;T_j)}_{n'_i}(S_j(\tilde{n})(z_0))$$

for any $T_j \in \mathscr{G}$. Since $\lim_{i\to\infty} \chi^{(\mu;T^*)}_{n_i}(z_0) = 0$, we have $\lim_{i\to\infty} \chi^{(\mu;T_j)}_{n'_i}(S_j(\tilde{n})(z_0)) = 0$ for any $T_j \in \mathscr{G}$. Hence, from Lemma 3, for any small $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \boldsymbol{N}$ such that

$$\chi^{(\mu;T_j)}_{n_0}(z) < \varepsilon \tag{2.40}$$

for any $z \in D_{T_j}$ and any $T_j \in \mathscr{G}$.

In a way analogous to the case of the limit ∞, we conclude from (2.40) $\lim_{n\to\infty} \chi^{(\mu;T^*)}_{n}(z_0) = 0$. q.e.d.

8. Now we can show that (iii) implies (i) in the case of the limit zero.

LEMMA 8. *If* $M_{\mu/2}(\Lambda(G)) = 0$, *then* $\lim_{n\to\infty} \chi^{(\mu;T^*)}_{n}(z_0) = 0$ *for some* $T^* \in \mathscr{G}$ *and some* $z_0 \in D_{T^*}$.

PROOF. Assume the contrary. Since (i) and (ii) of Theorem 1 are equivalent to each other from Lemma 4, there exists a subsequence $\{\chi^{(\mu;T)}_{n_i}(z)\}$ of $\{\chi^{(\mu;T)}_{n}(z)\}$ and $0 < d \leqq \infty$ such that

$$\lim_{i\to\infty} \chi^{(\mu;T^*)}_{n_i}(z_0) = d$$

for some $T^* \in \mathscr{G}$ and some $z_0 \in D_{T^*}$.

If $d=\infty$, then from Lemma 7 we can see $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z_0)=\infty$. Hence from Lemma 5 we have $M_{\mu/2}(\Lambda(G))=\infty$, a contradiction. So we may assume $0<d<\infty$. Then

$$0<\liminf_{n\to\infty}\chi_n^{(\mu;T)}(z)\leqq\limsup_{n\to\infty}\chi_n^{(\mu;T)}(z)<\infty \tag{2.41}$$

for any $T\in\mathscr{G}$ and any $z\in D_T$.

Now take a compact set K in D_{T^*} such that $\mathrm{Int}(K)\cap\Lambda_a(G)\neq\varnothing$ and let it be fixed. Then there exist $c_1, c_2>0$ such that

$$0<c_1\leqq\liminf_{n\to\infty}\chi_n^{(\mu;T^*)}(z)\leqq\limsup_{n\to\infty}\chi_n^{(\mu;T^*)}(z)\leqq c_2<+\infty$$

for any $z\in K$. Taking a sufficiently small $\varepsilon>0$ ($\varepsilon<c_1$), we can easily see that there exists $n_0=n_0(\varepsilon, K)\in N$ depending on ε and K such that

$$0<c_1-\varepsilon\leqq\chi_n^{(\mu;T^*)}(z)\leqq c_2+\varepsilon<+\infty \tag{2.42}$$

for any $z\in K$ and any $n\geqq n_0$. For any sufficiently large $n_1\in N(n_1>n_0)$ we can take and fix $S(n_1)=T^*\circ S(n_1-1)\in G$ such that $B_{S^{-1}(n_1)}\subset\mathrm{Int}(K)$. Then for small $\delta>0$, there exists $n'=n'(\delta)\in N$ depending only on δ and closed disc $B_{S(m_1)},\cdots,B_{S(m_Q)}\in F(n',\delta/k_0)$ such that $m_j>n_1$ ($j=1,\cdots,Q$) and $\mathrm{Int}(K)\supset\bigcup_{j=1}^{Q}B_{S(m_j)}\supset\Lambda_a(G)\cap B_{S(n_1)}$. Here we can take a natural number n^* so large that $n^*-m_j\geqq n_0$ for $j=1,\cdots,Q$. Then we get from (2.42)

$$c_1-\varepsilon\leqq\sum_{S(n^*-m_j)}R^{\mu}_{S(n^*-m_j)\circ S(m_j)}/R^{\mu}_{S(m_j)}\leqq c_2+\varepsilon\,. \tag{2.43}$$

It holds from (2.43) that

$$\begin{aligned}R^{\mu}_{S(m_j)}&\geqq(c_2+\varepsilon)^{-1}\sum_{S(n^*-m_j)}R^{\mu}_{S(n^*-m_j)\circ S(m_j)}\\&=(c_2+\varepsilon)^{-1}\sum_{S(n^*-m_j)}R^{\mu}_{S(n^*)}\end{aligned} \tag{2.44}$$

for all $j=1,\cdots,Q$. Hence we have from (2.44)

$$\sum_{j=1}^{Q}R^{\mu}_{S(m_j)}>(c_2+\varepsilon)^{-1}\sum_{S(n^*-n_1)}R^{\mu}_{S(n^*)}\,. \tag{2.45}$$

Since

$$\sum_{S(n^*-n_1)}R^{\mu}_{S(n^*)}=R^{\mu}_{S(n_1)}\sum_{S(n^*-n_1)}R^{\mu}_{S(n^*-n_1)\circ S(n_1)}/R^{\mu}_{S(n_1)}=R^{\mu}_{S(n_1)}\times\chi^{(\mu;T^*)}_{n^*-n_1}(S(n_1)(\infty))\,,$$

we have from (2.45)

$$\sum_{j=1}^{Q}R^{\mu}_{S(m_j)}>(c_2+\varepsilon)^{-1}\chi^{(\mu;T^*)}_{n^*-n_1}(S(n_1)(\infty))\times R^{\mu}_{S(n_1)}\,. \tag{2.46}$$

Noting $n^*-n_1\geqq n_0$, we have the following from (2.42) and (2.46)

$$\sum_{j=1}^{Q}R^{\mu}_{S(m_j)}>(c_1-\varepsilon)(c_2+\varepsilon)^{-1}R^{\mu}_{S(n_1)}\,. \tag{2.47}$$

Hence we obtain from Proposition 3 and (2.47) the following relation:

$$N_0(K_G k_0)^{-\mu/2} M_{\mu/2}(\Lambda(G) \cap B_{S(n_1)}) \geqq \lim_{\delta \to 0} \left[\inf_{\{F(n', \delta/k_0)\}} \{\sum (2R^2_{S(m_j)})^{\mu/2}\} \right],$$

where the summation is taken over all $B_{S(m_j)} \in F(n', \delta/k_0)$. Clearly the right hand side of this inequality is greater than a positive number $2^{\mu/2}(c_1 - \varepsilon)(c_2 + \varepsilon)^{-1} R^{\mu}_{S(n_1)}$. This contradicts the assumption $M_{\mu/2}(\Lambda(G)) = 0$. q.e.d.

Therefore we complete the proof of Theorem 1.

3. Hausdorff dimension of $\Lambda(G)$ and Poincaré dimension of G.

1. In this section we shall consider the relation between the Hausdorff dimension of $\Lambda(G)$ and the Poincaré dimension of G. First of all we give the definitions.

Let Γ be a Kleinian group with $\infty \in \Omega(\Gamma)$ and $\{S \in \Gamma \mid S(\infty) = \infty\} = \{\mathrm{id}\}$. The Hausdorff dimension $d(\Lambda(\Gamma))$ of $\Lambda(\Gamma)$ is defined as

$$d(\Lambda(\Gamma)) = \inf \{\mu/2 \mid M_{\mu/2}(\Lambda(\Gamma)) = 0\}. \tag{3.1}$$

The Poincaré dimension of Γ is

$$P(\Gamma) = \inf \left\{ \mu \,\middle|\, \sum_{S \in \Gamma - \{\mathrm{id}\}} R^{\mu}_S < +\infty \right\}. \tag{3.2}$$

As to these values, the following two propositions are essential. They are direct consequences of Theorem 1 (see [2] for the proofs).

PROPOSITION 5. *Put* $d(\Lambda(G)) = \mu^*/2$. *Then*

$$0 < M_{\mu^*/2}(\Lambda(G)) < +\infty .$$

PROPOSITION 6. *If* $M_{\mu/2}(\Lambda(G)) = 0$, *then*

$$\sum_{S \in G - \{\mathrm{id}\}} R^{\mu}_S < +\infty .$$

2. The following two results are well known.

PROPOSITION 7 ([4]). *If* Γ *is a geometrically finite Kleinian group, then*

$$d(\Lambda(\Gamma)) \leqq P(\Gamma)/2 \leqq 2 .$$

PROPOSITION 8 ([3]). *Let* Γ *be a discrete subgroup of* PSL$(2, \boldsymbol{C})$. *If* $\Omega(\Gamma) \neq \emptyset$, *then*

$$\sum_{S \in \Gamma - \{\mathrm{id}\}} R^{4}_S < +\infty .$$

Now we can prove the following theorem.

THEOREM 2. $d(\Lambda(G)) = P(G)/2 < 2$.

Proof. From Propositions 6 and 7, it can be easily seen that $d(\Lambda(G)) = P(G)/2 \leqq 2$. If $d(\Lambda(G)) = 2$, then Proposition 5 yields $0 < M_2(\Lambda(G)) < +\infty$. Since $\sum_{S\in G-\{\mathrm{id}\}} R_S^4 < +\infty$ from Proposition 8, we conclude $M_2(\Lambda(G)) = 0$ from Proposition 7, a contradiction. Hence we obtain $d(\Lambda(G)) < 2$. q.e.d.

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