Tôhoku Math. Journ.
36 (1984), 293–314.

STABILITY OF CERTAIN MINIMAL SUBMANIFOLDS OF COMPACT HERMITIAN SYMMETRIC SPACES

MASARU TAKEUCHI

(Received June 18, 1983)

Introduction. In this paper we consider a compact totally real totally geodesic submanifold M of a Hermitian symmetric space $(\bar{M}, \bar{g})$ of compact type with $\dim M = \dim_C \bar{M}$, and study their classification and stability.

We shall show that *such a submanifold M is always a symmetric R-space* (cf. §1 for definition), *and these pairs $((\bar{M}, \bar{g}), M)$ correspond in one to one fashion to symmetric R-spaces.* Furthermore we shall prove that *M is stable in $(\bar{M}, \bar{g})$ as a minimal submanifold if and only if M is simply connected.*

Lawson-Simons [6] proved that a compact stable minimal submanifold of the complex projective n-space $P_n(C)$ endowed with the Kähler metric of constant holomorphic sectional curvature is always a complex submanifold. They showed also [6] that this is not true for a general Hermitian symmetric space of compact type, by giving an example of a compact stable minimal submanifold of $P_1(C) \times P_1(C)$ which is not a complex submanifold. The simply connected ones among our submanifolds include the example of Lawson-Simons and provide many examples with the same properties. For example, the quaternion Grassmann manifold $G_{p,q}(\boldsymbol{H})$ imbedded in the complex Grassmann manifold $G_{2p,2q}(C)$ is minimal and stable, but not a complex submanifold.

The author would like to express his hearty thanks to Professor Tadashi Nagano who gave him valuable advice during the preparation of this note.

1. Totally real totally geodesic submanifolds of compact Hermitian symmetric spaces.

In this section we shall classify compact totally real totally geodesic submanifolds M of a Hermitian symmetric space $(\bar{M}, \bar{g})$ of compact type with $\dim M = \dim_C \bar{M}$.

Let $(\bar{M}, \bar{g})$ be a Hermitian manifold. The inner product and the complex structure tensor on the tangent bundle $T\bar{M}$ are denoted by $\langle\ ,\ \rangle$ and J, respectively. A submanifold M of $\bar{M}$ is said to be *totally real* if $\langle JT_pM, T_pM\rangle = 0$ for each $p \in M$. A submanifold M is called a *real form* of $(\bar{M}, \bar{g})$ if there exists an involutive anti-holomorphic isometry σ of $(\bar{M}, \bar{g})$ such that

$$M = \{p \in \bar{M};\ \sigma(p) = p\} .$$

LEMMA 1.1. *Let $(\bar{M}, \bar{g})$ be a (complete) Hermitian manifold. Then any real form M of $(\bar{M}, \bar{g})$ is a (complete) totally real totally geodesic submanifold with* $\dim M = \dim_C \bar{M}$.

PROOF. Let σ be an involutive anti-holomorphic isometry of $(\bar{M}, \bar{g})$ which defines M. Then M coincides with the set of fixed points of the isometry σ of $(\bar{M}, \bar{g})$, and hence it is totally geodesic (cf. Kobayashi [4]).

Let $p \in M$ and σ_* denote the differential of σ at p. Then σ_* is an involutive linear isometry of $T_p\bar{M}$ with $\sigma_* J = -J\sigma_*$. Thus, denoting by $(T_p\bar{M})^{\pm}$ the (± 1)-eigenspace of σ_*, we have

$$T_p\bar{M} = (T_p\bar{M})^+ + (T_p\bar{M})^- \quad \text{(orthogonal sum)}$$

and $J(T_p\bar{M})^{\pm} = (T_p\bar{M})^{\mp}$. Since $(T_p\bar{M})^+ = T_pM$, we have that $\langle JT_pM, T_pM \rangle = 0$ and $\dim M = \dim_C \bar{M}$. q.e.d.

In the following we recall a construction of real forms, called symmetric R-spaces, of a Hermitian symmetric space of compact type (cf. Takeuchi [12]).

Let $(\mathfrak{g}, \tau)$ be a *positive definite symmetric graded Lie algebra* (cf. Satake [10]), that is,

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 , \quad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} ,$$

is a real semi-simple graded Lie algebra such that $\mathfrak{g}_{-1} \neq 0$ and $\mathfrak{g}_0$ acts effectively on $\mathfrak{g}_{-1}$, and τ is a Cartan involution of $\mathfrak{g}$ with $\tau\mathfrak{g}_p = \mathfrak{g}_{-p}$ ($p = -1, 0, 1$). Then $\mathfrak{u} = \mathfrak{g}_0 + \mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}$. Let G be the connected Lie group with the trivial center such that $\operatorname{Lie} G$, the Lie algebra of G, is $\mathfrak{g}$. Put

$$U = \{a \in G;\ \operatorname{Ad}(a)\mathfrak{u} = \mathfrak{u}\} .$$

Then we have $\operatorname{Lie} U = \mathfrak{u}$. The homogeneous space $M = G/U$ is compact and called the *symmetric R-space* associated to $(\mathfrak{g}, \tau)$. The origin U of M will be denoted by o.

Let $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{u}}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{u}$, respectively and $\bar{G}$ the connected complex Lie group with the trivial center such that $\operatorname{Lie} \bar{G} = \bar{\mathfrak{g}}$. We regard G as a subgroup of $\bar{G}$. Put

$$\bar{U} = \{a \in \bar{G};\ \operatorname{Ad}(a)\bar{\mathfrak{u}} = \bar{\mathfrak{u}}\} .$$

Then $\bar{U}$ is a connected complex Lie subgroup of $\bar{G}$ with $\operatorname{Lie} \bar{U} = \bar{\mathfrak{u}}$ and

$\bar{U} \cap G = U$. The complex homogeneous space $\bar{M} = \bar{G}/\bar{U}$ is compact, and the identity component $\mathrm{Aut}^0(\bar{M})$ of the group of all holomorphic automorphisms of $\bar{M}$ is identified with $\bar{G}$ (cf. Takeuchi [14]). Moreover we obtain a natural G-equivariant imbedding $f: M \to \bar{M}$ by virtue of $\bar{U} \cap G = U$. It is called the *canonical imbedding* associated to $(\mathfrak{g}, \tau)$. In what follows we shall often regard M as a submanifold of $\bar{M}$ through the imbedding f.

Let σ be the complex conjugation of $\bar{\mathfrak{g}}$ with respect to $\mathfrak{g}$ and denote the extension of σ to $\bar{G}$ also by σ. Since $\bar{U}$ is connected we have $\sigma(\bar{U}) = \bar{U}$, and thus σ induces an involutive anti-holomorphic diffeomorphism σ of $\bar{M}$. Then $M \subset \bar{M}$ is given by

$$M = \{p \in \bar{M};\ \sigma(p) = p\} .$$

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated to τ. Then $\mathfrak{g}_u = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ is a compact real form of $\bar{\mathfrak{g}}$. Let τ denote the complex conjugation of $\bar{\mathfrak{g}}$ with respect to $\mathfrak{g}_u$. Then $\mathfrak{g}$ is stable under τ and τ coincides with the original τ on $\mathfrak{g}$. From the semi-simplicity of $\mathfrak{g}$, there exists uniquely an element $Z \in \mathfrak{g}_0$ such that

$$\mathfrak{g}_p = \{X \in \mathfrak{g};\ [Z, X] = pX\} \quad (p = -1, 0, 1) .$$

The condition $\tau\mathfrak{g}_p = \mathfrak{g}_{-p}$ $(p = -1, 0, 1)$ implies $\tau Z = -Z$, and hence $Z \in \mathfrak{p}$. Let K and G_u be the connected subgroups of $\bar{G}$ generated by $\mathfrak{k}$ and $\mathfrak{g}_u$, respectively, and put

$$K_0 = \{a \in K;\ \mathrm{Ad}(a)Z = Z\} , \quad \mathfrak{k}_0 = \mathrm{Lie}\, K_0 ,$$
$$K_u = \{a \in G_u;\ \mathrm{Ad}(a)Z = Z\} , \quad \mathfrak{k}_u = \mathrm{Lie}\, K_u .$$

Then we have smooth identifications

$$M = K/K_0 , \quad \bar{M} = G_u/K_u .$$

We define an involutive automorphism θ of $\bar{G}$ by

$$\theta(a) = \exp(\pi\sqrt{-1}Z)a(\exp(\pi\sqrt{-1}Z))^{-1} \quad \text{for} \quad a \in \bar{G} .$$

Then $\theta(K) = K$, $\theta(G_u) = G_u$ and

$$(K_\theta)^0 \subset K_0 \subset K_\theta , \quad K_u = (G_u)_\theta ,$$

where K_θ (resp. $(G_u)_\theta$) denotes the subgroup of all fixed points of θ in K (resp. in G_u) and $(K_\theta)^0$ the identity component of K_θ. Thus both (K, K_0) and (G_u, K_u) are compact symmetric pairs. If we define

$$\mathfrak{m} = \{X \in \mathfrak{k};\ \theta X = -X\} ,$$
$$\mathfrak{m}_u = \{X \in \mathfrak{g}_u;\ \theta X = -X\} ,$$

denoting also by θ the differential of θ, we have direct sum decompositions

$$\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{m} , \quad \mathfrak{g}_u = \mathfrak{k}_u + \mathfrak{m}_u$$

as vector spaces. Thus $\mathfrak{m}$ and $\mathfrak{m}_u$ are identified with T_oM and $T_o\bar{M}$, respectively. Then $H_0 = -\sqrt{-1}Z$ is the unique element of the center of $\mathfrak{k}_u$ such that $\mathrm{ad}(H_0)|\mathfrak{m}_u$ gives the complex structure tensor J_o of $\bar{M}$ at o. Denote by $(\ ,\ )$ the Killing form of $\bar{\mathfrak{g}}$, and define a $\mathfrak{g}_u$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}_u$ by

$$\langle X, Y\rangle = -(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}_u .$$

The K-invariant (resp. G_u-invariant) Riemannian metric on M (resp. on $\bar{M}$) which extends $\langle\ ,\ \rangle|\mathfrak{m}\times\mathfrak{m}$ (resp. $\langle\ ,\ \rangle|\mathfrak{m}_u\times\mathfrak{m}_u$) is denoted by g (resp. by $\bar{g}$), and called the *canonical Riemannian metric* on M (resp. on $\bar{M}$). Then

(i) (M, g) (resp. $(\bar{M}, \bar{g})$) is a compact symmetric space (resp. a Hermitian symmetric space of compact type) such that the identity component $I^0(M, g)$ (resp. $I^0(\bar{M}, \bar{g})$) of the group of all isometries of (M, g) (resp. of $(\bar{M}, \bar{g})$) is identified with K (resp. with G_u), and the canonical imbedding $f: (M, g) \to (\bar{M}, \bar{g})$ is isometric.

Moreover σ is an isometry of $(\bar{M}, \bar{g})$, and hence M is a real form of $(\bar{M}, \bar{g})$. Thus, by Lemma 1.1.

(ii) M is a totally real totally geodesic submanifold of $(\bar{M}, \bar{g})$ with $\dim M = \dim_C \bar{M}$.

REMARK 1. If $\mathfrak{g}$ is simple, the Riemannian metrics g and $\bar{g}$ satisfying (i) and (ii) are unique up to homothety. In this case, the symmetric R-space M or (M, g) is said to be *irreducible*.

REMARK 2. Let $\bar{M}^*$ be the symmetric bounded domain dual to $\bar{M}$ which is imbedded into $\bar{M}$ as an open submanifold of $\bar{M}$ by means of Harish-Chandra imbedding. It can be shown (Takeuchi [12]) that then $M^* = \bar{M}^* \cap M$ is a non-compact symmetric space dual to M and it is a real form of $\bar{M}^*$.

Two positive definite symmetric graded Lie algebras $(\mathfrak{g}, \tau)$ and $(\mathfrak{g}', \tau')$ are said to be *isomorphic* if there exists a Lie isomorphism $\phi: \mathfrak{g} \to \mathfrak{g}'$ such that $\phi\mathfrak{g}_p = \mathfrak{g}'_p$ $(p = -1, 0, 1)$ and $\phi\circ\tau = \tau'\circ\phi$. Let $\mathscr{S}$ denote the set of all isomorphism classes of positive definite symmetric graded Lie algebras. The set $\mathscr{S}$ was completely determined (Kobayashi-Nagano [5], Takeuchi [12]). Next we consider a pair $((\bar{M}, \bar{g}), M)$ of a connected Hermitian symmetric space $(\bar{M}, \bar{g})$ of compact type and a compact connected totally real totally geodesic submanifold M of $(\bar{M}, \bar{g})$ with $\dim M = \dim_C \bar{M}$. Such a pair is called a TRG-*pair*. For a finite number of TRG-pairs $((\bar{M}_i, \bar{g}_i), M_i)$, $1 \leqq i \leqq s$, the *direct product* $((\bar{M}, \bar{g}), M) = ((\bar{M}_1, \bar{g}_1), M_1) \times \cdots \times ((\bar{M}_s, \bar{g}_s), M_s)$, which is also a TRG-pair, is defined by $\bar{M} = \bar{M}_1 \times \cdots \times \bar{M}_s$, $\bar{g} = \bar{g}_1 \times \cdots \times \bar{g}_s$ and $M = M_1 \times \cdots \times M_s$. Two TRG-pairs $((\bar{M}, \bar{g}), M)$ and $((\bar{M}', \bar{g}'), M')$ are said to be *equivalent* if there exist direct product decompositions $((\bar{M}, \bar{g}), M) = ((\bar{M}_1, \bar{g}_1), M_1) \times \cdots \times ((\bar{M}_s, \bar{g}_s), M_s)$ and $((\bar{M}', \bar{g}'), M') =$

$((\bar{M}'_1, \bar{g}'_1), M'_1) \times \cdots \times ((\bar{M}'_{s'}, \bar{g}'_{s'}), M'_{s'})$ with $s = s'$ and homothetic biholomorphic maps $\phi_i: (\bar{M}_i, \bar{g}_i) \to (\bar{M}'_i, \bar{g}'_i)$, $1 \leqq i \leqq s$, such that the product map $\phi = \phi_1 \times \cdots \times \phi_s: \bar{M} \to \bar{M}'$ satisfies $\phi(M) = M'$. Let $\mathscr{T}$ denote the set of all equivalence classes of TRG-pairs.

THEOREM 1.2. *Our correspondence* $(\mathfrak{g}, \tau) \mapsto ((\bar{M}, \bar{g}), M)$ *induces a bijection* $\Phi: \mathscr{S} \to \mathscr{T}$.

PROOF. It follows from definition that our correspondence induces a map $\Phi: \mathscr{S} \to \mathscr{T}$. Conversely, for any TRG-pair $((\bar{M}, \bar{g}), M)$ we shall associate canonically a positive definite symmetric graded Lie algebra $(\mathfrak{g}, \tau)$. Let $\bar{G} = \mathrm{Aut}^0(\bar{M})$ which is a connected complex semi-simple Lie group with the trivial center, and let $G_u = I^0(\bar{M}, \bar{g})$ which is a subgroup of $\bar{G}$ because $(\bar{M}, \bar{g})$ is a compact Kähler manifold (cf. Kobayashi [4]). Let J denote the complex structure tensor of $\bar{M}$. We identify $\bar{\mathfrak{g}} = \mathrm{Lie}\,\bar{G}$ (resp. $\mathfrak{g}_u = \mathrm{Lie}\,G_u$) with the Lie algebra of all smooth vector fields X on $\bar{M}$ such that the Lie derivative of J with respect to X vanishes (resp. of all Killing vector fields on $(\bar{M}, \bar{g})$) with Lie product $[X, Y] = YX - XY$. Then by Matsushima's theorem on compact Kähler Einstein manifolds we have

$$\bar{\mathfrak{g}} = \mathfrak{g}_u + J\mathfrak{g}_u\,, \quad \mathfrak{g}_u \cap J\mathfrak{g}_u = 0\,. \tag{1.1}$$

Let $\mathfrak{g}(M)$ be the real subalgebra of $\bar{\mathfrak{g}}$ consisting of all $X \in \bar{\mathfrak{g}}$ such that the restriction $X|M$ is tangent to M, and $\mathfrak{k}(M)$ the Lie algebra of all Killing vector fields on M with respect to the Riemannian metric g induced from $\bar{g}$. We put

$$\mathfrak{k} = \mathfrak{g}(M) \cap \mathfrak{g}_u\,, \quad \mathfrak{p} = \mathfrak{g}(M) \cap J\mathfrak{g}_u\,,$$

and

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}\,. \tag{1.2}$$

Then $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, and hence $\mathfrak{g}$ is a real subalgebra of $\bar{\mathfrak{g}}$. We need here the following:

LEMMA 1.3. (1) *The map* $\mathfrak{k} \to \mathfrak{k}(M)$ *defined by* $X \mapsto X|M$ $(X \in \mathfrak{k})$ *is a Lie isomorphism.*

(2) *We have*

$$\mathfrak{g}_u = \mathfrak{k} + J\mathfrak{p}\,, \quad \mathfrak{k} \cap J\mathfrak{p} = 0\,. \tag{1.3}$$

Now, it follows from (1.1), (1.2) and (1.3) that $\mathfrak{g}$ is a real form of $\bar{\mathfrak{g}}$. Let σ and τ denote the complex conjugation of $\bar{\mathfrak{g}}$ with respect to $\mathfrak{g}$ and $\mathfrak{g}_u$, respectively. Then

$$\sigma JX = -J\sigma X \quad \text{for} \quad X \in \bar{\mathfrak{g}}\,, \tag{1.4}$$

$$\sigma\mathfrak{g}_u = \mathfrak{g}_u\,. \tag{1.5}$$

We fix a point $o \in M$ and put

$$K_u = \{a \in G_u;\ a(o) = o\}\ ,$$

which is known to be connected. (See Helgason [2] for fundamental results on symmetric spaces.) Then $\bar{M} = G_u/K_u$ as smooth manifold. Let $\mathfrak{k}_u = \text{Lie}\ K_u$ and $\mathfrak{g}_u = \mathfrak{k}_u + \mathfrak{m}_u$ be the associated Cartan decomposition. Let H_0 be the unique element of the center of $\mathfrak{k}_u$ such that $J_o = \text{ad}(H_0)|\mathfrak{m}_u$. Putting $Z = JH_0 \in \bar{\mathfrak{g}}$, we define

$$\begin{aligned} &\bar{\mathfrak{g}}_p = \{X \in \bar{\mathfrak{g}};\ [Z, X] = pX\} \quad (p = -1, 0, 1)\ , \\ &\bar{\mathfrak{u}} = \bar{\mathfrak{g}}_0 + \bar{\mathfrak{g}}_1\ , \\ &\bar{U} = \{a \in \bar{G};\ \text{Ad}(a)\bar{\mathfrak{u}} = \bar{\mathfrak{u}}\}\ . \end{aligned}$$

Then $\text{Lie}\ \bar{U} = \bar{\mathfrak{u}}$, $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_{-1} + \bar{\mathfrak{g}}_0 + \bar{\mathfrak{g}}_1$ and $\bar{M} = \bar{G}/\bar{U}$ as complex manifold. Note here that $\bar{\mathfrak{g}}_0$ acts on $\bar{\mathfrak{g}}_{-1}$ effectively. We define an involutive automorphism θ of $\bar{G}$ by

$$\theta(a) = \exp(\pi JZ)a(\exp(\pi JZ))^{-1} \quad \text{for} \quad a \in \bar{G}\ .$$

Then $\theta(G_u) = G_u$ and hence the differential of θ, denoted also by θ, satisfies $\theta\mathfrak{g}_u = \mathfrak{g}_u$. Morever we have

$$\mathfrak{k}_u = \{X \in \mathfrak{g}_u;\ \theta X = X\}\ , \tag{1.6}$$

$$\mathfrak{m}_u = \{X \in \mathfrak{g}_u;\ \theta X = -X\}\ . \tag{1.7}$$

A diffeomorphism θ of $\bar{M} = G_u/K_u$ is defined by the correspondence $a \cdot o \mapsto \theta(a) \cdot o (a \in G_u)$ because K_u is connected. It is the symmetry of $(\bar{M}, \bar{g})$ at o. Since M is totally geodesic in $(\bar{M}, \bar{g})$ we have $\theta(M) = M$, and hence $\theta\mathfrak{g}(M) = \mathfrak{g}(M)$. Therefore we have $\theta\mathfrak{k} = \mathfrak{k}$ and $\theta\mathfrak{p} = \mathfrak{p}$, and hence $\theta\mathfrak{g} = \mathfrak{g}$. Thus (1.5), (1.6) and (1.7) imply

$$\sigma\mathfrak{k}_u = \mathfrak{k}_u\ , \tag{1.8}$$

$$\sigma\mathfrak{m}_u = \mathfrak{m}_u\ . \tag{1.9}$$

Now it follows from (1.4) and (1.9) that $\sigma J_o = -J_o\sigma$ on $\mathfrak{m}_u = T_o(\bar{M})$, and thus $[\sigma H_o, \sigma X] = -J_o\sigma X$ for each $X \in \mathfrak{m}_u$, where σH_0 is an element of the center of $\mathfrak{k}_u$ by (1.8). Therefore the uniqueness of H_0 implies that $\sigma H_0 = -H_0$, and so $\sigma Z = Z$, that is, $Z \in \mathfrak{g}$. Thus, putting $\mathfrak{g}_p = \bar{\mathfrak{g}}_p \cap \mathfrak{g}$ $(p = -1, 0, 1)$ we get $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. Moreover τ restricted to $\mathfrak{g}$ is a Cartan involution with $\tau Z = -Z$, and thus $\tau\mathfrak{g}_p = \mathfrak{g}_{-p}$ $(p = -1, 0, 1)$. The effectiveness of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ follows from that of $\bar{\mathfrak{g}}_0$ on $\bar{\mathfrak{g}}_{-1}$. Therefore $(\mathfrak{g}, \tau)$ is a positive definite symmetric graded Lie algebra.

Next we shall show that our correspondence $((\bar{M}, \bar{g}), M) \mapsto (\mathfrak{g}, \tau)$ induces a map $\Psi: \mathscr{T} \to \mathscr{S}$. Let $((\bar{M}, \bar{g}), M)$ and $((\bar{M}', \bar{g}'), M')$ be equivalent.

Various objects for $((\bar{M}', \bar{g}'), M')$ will be denoted by the same notation as $((\bar{M}, \bar{g}), M)$ but with primes. Let $\phi: \bar{M} \to \bar{M}'$ be an equivalence. Then, since both $\phi(o)$ and o' are on M', by Lemma 1.3, (1) there exists $\phi' \in I^0(\bar{M}', \bar{g}')$ such that $\phi'(M') = M'$ and $\phi'(\phi(o)) = o'$. Therefore we may assume that $\phi(o) = o'$. Then the correspondence $a \mapsto \phi \circ a \circ \phi^{-1} (a \in \bar{G})$ defines an isomorphism $\phi: \bar{G} \to \bar{G}'$ such that the differential $\phi: \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}'$ is a Lie isomorphism with $\phi \circ J = J' \circ \phi$, $\phi \mathfrak{g}_u = \mathfrak{g}'_u$, $\phi \mathfrak{g}(M) = \mathfrak{g}(M')$ and $\phi Z = Z'$. Thus we get $\phi \mathfrak{g} = \mathfrak{g}'$ and $\phi \mathfrak{k} = \mathfrak{k}'$. Therefore ϕ gives an isomorphism $(\mathfrak{g}, \tau) \to (\mathfrak{g}', \tau')$, and so $(\mathfrak{g}, \tau)$ is isomorphic to $(\mathfrak{g}', \tau')$.

Now we have $\Psi \circ \Phi = I_{\mathscr{S}}$ by definitions, and $\Phi \circ \Psi = I_{\mathscr{T}}$ by Remark 1, where I indicates the identity map. Thus our map Φ is a bijection.

q.e.d.

PROOF OF LEMMA 1.3. (1) Since (M, g) is a compact connected symmetric space, $I^0(M, g)$ is generated by symmetries. Thus the map $\mathfrak{k} \to \mathfrak{k}(M)$ is surjective, because M is totally geodesic in $(\bar{M}, \bar{g})$. So it suffices to show

$$X \in \mathfrak{k},\ X|M = 0 \Rightarrow X = 0 \ . \tag{1.10}$$

We fix a point $p \in M$ and define an endomorphism $\tilde{X}_p$ of $T_p\bar{M}$ by

$$\tilde{X}_p(y) = \bar{\nabla}_y X \quad \text{for} \quad y \in T_p\bar{M} \ ,$$

where $\bar{\nabla}$ is the Riemannian connection of $(\bar{M}, \bar{g})$. It suffices to show $\tilde{X}_p = 0$ since X is a Killing vector field on $(\bar{M}, \bar{g})$. For any $y \in T_pM$ we have

$$\tilde{X}_p(y) = \bar{\nabla}_y X = \nabla_y X = 0 \ ,$$
$$\tilde{X}_p(Jy) = \bar{\nabla}_{Jy} X = J\bar{\nabla}_y X = 0 \ ,$$

where ∇ is the Riemannian connection of (M, g). Here we have used the facts that M is totally geodesic, $X|M = 0$ and X is a holomorphic vector field on the Kähler manifold $(\bar{M}, \bar{g})$. Now $T_p\bar{M} = T_pM \oplus JT_pM$ implies $\tilde{X}_p = 0$.

(2) Let $X \in \mathfrak{g}_u$ and decompose $X|M$ as $X|M = X^T + X^N$, where X^T is tangent to M and X^N is normal to M. Then

$$0 = \langle \bar{\nabla}_y X, z \rangle + \langle \bar{\nabla}_z X, y \rangle = \langle \nabla_y X^T, z \rangle + \langle \nabla_z X^T, y \rangle$$

for any $y, z \in T_pM$, $p \in M$, and thus $X^T \in \mathfrak{k}(M)$. Now by (1) there is $X' \in \mathfrak{k}$ such that $X'|M = X^T$. Put $X'' = X - X' \in \mathfrak{g}_u$. Then $X''|M = X^N$ and $(JX'')|M = JX^N$ which is tangent to M. Therefore $JX'' \in \mathfrak{g}(M) \cap J\mathfrak{g}_u = \mathfrak{p}$, and hence $X = X' + X'' \in \mathfrak{k} + J\mathfrak{p}$. Thus we have shown that $\mathfrak{g}_u \subset \mathfrak{k} + J\mathfrak{p}$ and so $\mathfrak{g}_u = \mathfrak{k} + J\mathfrak{p}$. On the other hand, any $X \in \mathfrak{k} \cap J\mathfrak{p}$ satisfies $X|M = 0$, and hence $X = 0$ by (1.10). This shows $\mathfrak{k} \cap J\mathfrak{p} = 0$. q.e.d.

REMARK 3. Actually the subalgebra $\mathfrak{g}(M)$ of $\bar{\mathfrak{g}}$ in Theorem 1.2 coincides with $\mathfrak{g}$. In fact, for each point p of a symmetric R-space $M \subset \bar{M}$ there exists a holomorphic coordinate (z^α) of $\bar{M}$ around p such that M is given by $\operatorname{Im} z^\alpha = 0$ around p. Therefore we get

$$X \in \bar{\mathfrak{g}},\ X|M = 0 \Rightarrow X = 0\ ,$$

which implies $\mathfrak{g}(M) \cap J\mathfrak{g}(M) = 0$ and so $\mathfrak{g}(M) = \mathfrak{g}$.

REMARK 4. For any connected Hermitian symmetric space $(\bar{M}, \bar{g})$ of compact type, there exists at least one involutive anti-holomorphic isometry of $(\bar{M}, \bar{g})$ (Satake [10]).

2. First eigenvalues of symmetric R-spaces. In this section we shall compute the first eigenvalue of the Laplacian on smooth functions of an irreducible symmetric R-space.

Let $(\mathfrak{g}, \tau)$ be a positive definite symmetric graded Lie algebra and $M = G/U = K/K_0$ be the symmetric R-space associated to $(\mathfrak{g}, \tau)$. We use the same notation as in §1.

LEMMA 2.1. *Let $C_\mathfrak{p}$ be the Casimir operator on the $\mathfrak{k}$-module $\mathfrak{p}$ relative to the $\mathfrak{k}$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{k}$. Then $C_\mathfrak{p} = (1/2)I_\mathfrak{p}$.*

PROOF. Let $\{E_\alpha\}$ be an orthonormal basis for $\mathfrak{k}$ with respect to $\langle\ ,\ \rangle$. Then, by definition

$$C_\mathfrak{p} = -\sum_\alpha (\operatorname{ad}(E_\alpha)|\mathfrak{p})^2\ .$$

For each $X \in \mathfrak{p}$ we have

$$\begin{aligned} -([E_\alpha, [E_\alpha, X]], X) &= ([E_\alpha, X], [E_\alpha, X]) \\ &= (E_\alpha, [X, [E_\alpha, X]]) = -(E_\alpha, [X, [X, E_\alpha]]) \\ &= -(\operatorname{ad}(X)^2 E_\alpha, E_\alpha) = \langle \operatorname{ad}(X)^2 E_\alpha, E_\alpha \rangle\ . \end{aligned}$$

Therefore $(C_\mathfrak{p}X, X) = \operatorname{Tr}(\operatorname{ad}(X)^2|\mathfrak{k})$. On the other hand, from $\operatorname{ad}(X)\mathfrak{k} \subset \mathfrak{p}$, $\operatorname{ad}(X)\mathfrak{p} \subset \mathfrak{k}$ we get $(X, X) = \operatorname{Tr}(\operatorname{ad}(X)^2) = 2\operatorname{Tr}(\operatorname{ad}(X)^2|\mathfrak{k})$. Thus we obtain $(C_\mathfrak{p}X, X) = (X, X)/2$ for each $X \in \mathfrak{p}$, and hence

$$(C_\mathfrak{p}X, Y) = (X, Y)/2 \quad \text{for any } X, Y \in \mathfrak{p}\ .$$

This implies the assertion. q.e.d.

Let $\mathfrak{h}^- \subset \mathfrak{p}$ be a maximal abelian subalgebra in $\mathfrak{p}$ with $Z \in \mathfrak{h}^-$ and take an abelian subalgebra $\mathfrak{h}^+$ of $\mathfrak{k}$ such that $\mathfrak{h} = \mathfrak{h}^+ + \mathfrak{h}^-$ is a Cartan subalgebra of $\mathfrak{g}$. Then the complexification $\bar{\mathfrak{h}}$ of $\mathfrak{h}$ is a Cartan subalgebra of $\bar{\mathfrak{g}}$, whose real part $\mathfrak{h}_R$ is given by $\mathfrak{h}_R = \sqrt{-1}\mathfrak{h}^+ + \mathfrak{h}^-$. Let $\Sigma \subset \mathfrak{h}_R$ be the root system of $\bar{\mathfrak{g}}$ relative to $\bar{\mathfrak{h}}$ and put

$$\bar{\Sigma}_0 = \{\alpha \in \bar{\Sigma};\ (\alpha, Z) = 0\}\ .$$

Choose a σ-order on $\mathfrak{h}_R$ in the sense of Satake [9] such that $(\alpha, Z) \geqq 0$ for each α in $\bar{\Sigma}^+$, the set of positive roots. Then we have

$$\bar{\Sigma}^+ - \bar{\Sigma}_0 = \{\alpha \in \bar{\Sigma};\ (\alpha, Z) = 1\}\ .$$

In what follows in this section we assume that $\mathfrak{g}$ is simple. Then the followings are known (Takeuchi [12]):

There exists a maximal system $\{\gamma_1, \cdots, \gamma_s\}$, $s = \mathrm{rank}(\bar{M}, \bar{g})$, of strongly orthogonal roots in $\bar{\Sigma}^+ - \bar{\Sigma}_0$ with the same length such that $\sigma\{\gamma_1, \cdots, \gamma_s\} = \{\gamma_1, \cdots, \gamma_s\}$. Moreover, if $r = \mathrm{rank}(M, g)$, we have

(a) $r = s$, $\sigma\gamma_i = \gamma_i$ $(1 \leqq i \leqq r)$; or

(b) $2r = s$, $\sigma\gamma_i = \gamma_{r+i}$ $(1 \leqq i \leqq r)$, changing indices of γ_j's if necessary.

We define $\beta_i \in \mathfrak{h}^-(1 \leqq i \leqq r)$ by

$$\beta_i = \begin{cases} \gamma_i & \text{if} \quad r = s\ , \\ (1/2)(\gamma_i + \sigma\gamma_i) & \text{if} \quad 2r = s . \end{cases}$$

Then

$$(\beta_i, \beta_i) = \begin{cases} (\gamma_i, \gamma_i) & \text{if} \quad r = s\ , \\ (\gamma_i, \gamma_i)/2 & \text{if} \quad 2r = s\ . \end{cases} \tag{2.1}$$

Let $\mathfrak{a}^- = \{\beta_1, \cdots, \beta_r\}_R$ be the $\boldsymbol{R}$-span of $\{\beta_1, \cdots, \beta_r\}$, and $\pi_{\mathfrak{a}^-}\colon \mathfrak{h}_R \to \mathfrak{a}^-$ denote the orthogonal projection with respect to $(\ ,\ )$. By Satake [10] (cf. also Moore [7]) we have then

$$\begin{aligned} \pi_{\mathfrak{a}^-}(\bar{\Sigma}) - \{0\} &= \{\pm(1/2)(\beta_i \pm \beta_j)\ (1 \leqq i < j \leqq r),\ \pm\beta_i\ (1 \leqq i \leqq r)\}\ , \\ \text{or}\quad &\{\pm(1/2)(\beta_i \pm \beta_j)\ (1 \leqq i < j \leqq r),\ \pm\beta_i,\ \pm(1/2)\beta_i\ (1 \leqq i \leqq r)\}\ . \end{aligned} \tag{2.2}$$

We may choose (cf. Takeuchi [12]) root vectors $X_\alpha \in \bar{\mathfrak{g}}$ $(\alpha \in \bar{\Sigma})$ in such a way that

$$[X_\alpha, X_{-\alpha}] = -\frac{2}{(\alpha, \alpha)}\alpha\ , \quad \tau X_\alpha = X_{-\alpha}\ , \quad \sigma X_\alpha = X_{\sigma\alpha}\ .$$

We put $U_{\gamma_j} = X_{\gamma_j} + X_{-\gamma_j} \in \mathfrak{m}_u$ $(1 \leqq j \leqq s)$ and define $S_i \in \mathfrak{m}$ $(1 \leqq i \leqq r)$ by

$$S_i = \begin{cases} U_{\gamma_i} & \text{if} \quad r = s\ , \\ U_{\gamma_i} + U_{\sigma\gamma_i} & \text{if} \quad 2r = s\ , \end{cases}$$

whose length with respect to $\langle\ ,\ \rangle$ are the same. Then $\mathfrak{t}^- = \{S_1, \cdots, S_r\}_R$ is a maximal abelian subalgebra in $\mathfrak{m}$. We define elements V_i, V_i' $(1 \leqq i \leqq r)$ of $\bar{\mathfrak{g}}$ by

$$V_i = \begin{cases} X_{\gamma_i} & \text{if} \quad r = s\ , \\ X_{\gamma_i} + X_{\sigma\gamma_i} & \text{if} \quad 2r = s\ , \end{cases}$$

$$V_i' = \begin{cases} \dfrac{1}{2}\Big(X_{\gamma_i} - X_{-\gamma_i} + \dfrac{2\sqrt{-1}}{(\beta_i, \beta_i)}\beta_i\Big) & \text{if} \quad r = s\,, \\ \dfrac{1}{2}\Big(X_{\gamma_i} + X_{\sigma\gamma_i} - X_{-\gamma_i} - X_{-\sigma\gamma_i} + \dfrac{2\sqrt{-1}}{(\beta_i, \beta_i)}\beta_i\Big) & \text{if} \quad 2r = s\,. \end{cases}$$

Note that the V_i''s are non-zero elements of the complexification $\bar{\mathfrak{p}}$ of $\mathfrak{p}$. Moreover we define $c' \in G_u$ by

$$c' = \prod_{j=1}^{s} \exp \frac{\pi}{4\sqrt{-1}}\Big(X_{\gamma_j} - X_{-\gamma_j}\Big)\,.$$

LEMMA 2.2. (1) *For each* i $(1 \leqq i \leqq r)$ *we have*

$$\mathrm{Ad}(c')\Big(\frac{2}{(\beta_i, \beta_i)}\beta_i\Big) = \sqrt{-1}S_i\,, \tag{2.3}$$

$$\mathrm{Ad}(c')\,V_i = V_i'\,. \tag{2.4}$$

(2) *We have*

$$[H, V_i] = (\beta_i, H)\,V_i \quad \textit{for each } H \in \mathfrak{a}^-, 1 \leqq i \leqq r\,.$$

PROOF. (1) If we put

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then $\{X_+, X_-, H\}$ is a basis for $\mathfrak{sl}(2, C)$ with relations $[X_+, X_-] = -H$, $[H, X_\pm] = \pm 2X_\pm$. On the other hand we have relations $[X_{\gamma_j}, X_{-\gamma_j}] = -(2/(\gamma_j, \gamma_j))\gamma_j$, $[(2/(\gamma_j, \gamma_j))\gamma_j, X_{\pm\gamma_j}] = \pm 2X_{\pm\gamma_j}$ $(1 \leqq j \leqq s)$. Thus the correspondence $X_\pm \mapsto X_{\pm\gamma_j}$, $H \mapsto (2/(\gamma_j, \gamma_j))\gamma_j$ defines an injective Lie homomorphism $\mathfrak{sl}(2, C) \to \bar{\mathfrak{g}}$ such that $U \mapsto U_{\gamma_j}$, where $U = X_+ + X_-$. Since the element c_0' of $SU(2)$ defined by

$$c_0' = \exp \frac{\pi}{4\sqrt{-1}}(X_+ - X_-) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix}$$

satisfies $\mathrm{Ad}(c_0')H = \sqrt{-1}U$, $\mathrm{Ad}(c_0')X_+ = (1/2)(X_+ - X_- + \sqrt{-1}H)$, we get for each j $(1 \leqq j \leqq s)$

$$\mathrm{Ad}(c')\Big(\frac{2}{(\gamma_j, \gamma_j)}\gamma_j\Big) = \sqrt{-1}U_{\gamma_j}\,, \tag{2.3$'$}$$

$$\mathrm{Ad}(c')X_{\gamma_j} = \frac{1}{2}\Big(X_{\gamma_j} - X_{-\gamma_j} + \frac{2\sqrt{-1}}{(\gamma_j, \gamma_j)}\gamma_j\Big)\,. \tag{2.4$'$}$$

Thus we obtain (2.2), (2.3) in case $r = s$. In case $2r = s$, we have for each i $(1 \leqq i \leqq r)$

$$(2.3)'' \qquad \mathrm{Ad}(c')\Big(\frac{2}{(\gamma_i, \gamma_i)}\sigma\gamma_i\Big) = \sqrt{-1}\,U_{\sigma\gamma_i}\,,$$

$$(2.4)'' \qquad \mathrm{Ad}(c')X_{\sigma\gamma_i} = \frac{1}{2}\Big(X_{\sigma\gamma_i} - X_{-\sigma\gamma_i} + \frac{2\sqrt{-1}}{(\gamma_i, \gamma_i)}\sigma\gamma_i\Big)\,.$$

Adding (2.3)′ and (2.3)″ (resp. (2.4)′ and (2.4)″) we get (2.3) (resp. (2.4)), by virtue of the equality

$$\frac{2}{(\gamma_i, \gamma_i)}\gamma_i + \frac{2}{(\gamma_i, \gamma_i)}\sigma\gamma_i = \frac{2}{(\beta_i, \beta_i)}\beta_i\,,$$

which follows from (2.1).

(2) This follows from a direct calculation. q.e.d.

The eigenvalues of the Laplacian Δ with respect to the canonical Riemannian metric g acting on the space $C^\infty(M)$ of smooth functions on $M = K/K_0$ are obtained in the following way (cf. Takeuchi [13]).

Take an abelian subalgebra $\mathfrak{t}^+$ of $\mathfrak{k}_0$ such that $\mathfrak{t} = \mathfrak{t}^+ + \mathfrak{t}^-$ is a maximal abelian subalgebra of $\mathfrak{k}$. The complexification $\bar{\mathfrak{t}}$ of $\mathfrak{t}$ is a Cartan subalgebra of the complexification $\bar{\mathfrak{k}}$ of $\mathfrak{k}$ and the real part $\mathfrak{t}_R$ of $\bar{\mathfrak{t}}$ is given by $\mathfrak{t}_R = \sqrt{-1}\mathfrak{t}^+ + \sqrt{-1}\mathfrak{t}^-$. Taking a basis $\{H_{r+1}, \cdots, H_l\}$ for $\sqrt{-1}\mathfrak{t}^+$, we define a lexicographic order $>$ on $\mathfrak{t}_R$ by the basis $\{\sqrt{-1}S_1, \cdots, \sqrt{-1}S_r, H_{r+1}, \cdots, H_l\}$ for $\mathfrak{t}_R$. Let $\Sigma \subset \sqrt{-1}\mathfrak{t}^-$ be the root system of the symmetric pair $(\mathfrak{k}, \mathfrak{k}_0)$ and Σ^+ the set of positive roots in Σ (with respect to $>$). We set

$$\begin{aligned}
\Gamma &= \{H \in \mathfrak{t}^-;\ \exp H \in K_0\}\,,\\
\Gamma^\perp &= \{\lambda \in \sqrt{-1}\mathfrak{t}^-;\ (\lambda, \Gamma) \subset 2\pi\sqrt{-1}\boldsymbol{Z}\}\,,\\
D &= \{\lambda \in \Gamma^\perp;\ (\lambda, \alpha) \geqq 0 \quad \text{for each } \alpha \in \Sigma^+\}\,.
\end{aligned}$$

Let $\delta \in \sqrt{-1}\mathfrak{t}^-$ be the half-sum of all roots in Σ^+ with multiplicity counted. Then the set $\mathrm{Spec}(M, g)$ of eigenvalues of Δ is given by

$$(2.5) \qquad \mathrm{Spec}(M, g) = \{(2\delta + \lambda, \lambda);\ \lambda \in D\}\,.$$

Here the multiplicity of $(2\delta + \lambda, \lambda)$ is equal to the dimension of the irreducible $\bar{\mathfrak{k}}$-module V_λ with the highest weight λ, and $(2\delta + \lambda, \lambda)$ is nothing but the eigenvalue of the Casimir operator on V_λ relative to the inner product $\langle\ ,\ \rangle$. In our case we have (Takeuchi [12])

$$\Gamma = \pi\{S_1, \cdots, S_r\}_{\boldsymbol{Z}}\,,$$

where $\{S_1, \cdots, S_r\}_{\boldsymbol{Z}}$ denotes the subgroup of $\mathfrak{t}^-$ generated by $\{S_1, \cdots, S_r\}$. Thus, if we define $h_i \in \sqrt{-1}\mathfrak{t}^-(1 \leqq i \leqq r)$ by $(h_i, \sqrt{-1}S_j) = \delta_{ij}$, then they have the same length with respect to $(\ ,\ )$ and

$$(2.6) \qquad \Gamma^{\perp} = 2\{h_1, \cdots, h_r\}_Z , \quad h_1 > \cdots > h_r > 0 .$$

LEMMA 2.3. *The highest weight Λ relative to $\bar{\mathfrak{t}}$ of the $\bar{\mathfrak{k}}$-module $\bar{\mathfrak{p}}$ is given by $\Lambda = 2h_1$.*

PROOF. Take an abelian subalgebra $\mathfrak{s}$ in $\mathfrak{p}$ such that $\mathfrak{h}' = \mathfrak{t} + \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{g}$. Then the real part $\mathfrak{h}'_R$ of the complexification $\bar{\mathfrak{h}}'$ of $\mathfrak{h}'$ is given by $\mathfrak{h}'_R = \sqrt{-1}\mathfrak{t} + \mathfrak{s}$. Let $\bar{\Sigma}' \subset \mathfrak{h}'_R$ be the root system of $\bar{\mathfrak{g}}$ relative to $\bar{\mathfrak{h}}'$. Let $\pi_{\mathfrak{t}}: \mathfrak{h}'_R \to \sqrt{-1}\mathfrak{t}$ and $\pi_{\mathfrak{t}^-}: \mathfrak{h}'_R \to \sqrt{-1}\mathfrak{t}^-$ be orthogonal projections with respect to (,).

Since $\mathrm{Ad}(c')\mathfrak{a}^- = \sqrt{-1}\mathfrak{t}^-$ by (2.3), both $\mathrm{Ad}(c')\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}'$ are Cartan subalgebras of the centralizer in $\bar{\mathfrak{g}}$ of $\mathfrak{t}^-$. Thus there exists an element c'' of the centralizer in $\bar{G}$ of $\mathfrak{t}^-$ such that $\mathrm{Ad}(c'')\mathrm{Ad}(c')\bar{\mathfrak{h}} = \bar{\mathfrak{h}}'$. Put $c = c''c' \in \bar{G}$. Then $\mathrm{Ad}(c)\bar{\mathfrak{h}} = \bar{\mathfrak{h}}'$, and hence

$$(2.7) \qquad \mathrm{Ad}(c)\mathfrak{h}_R = \mathfrak{h}'_R , \quad \mathrm{Ad}(c)\bar{\Sigma} = \bar{\Sigma}' ,$$

$$(2.8) \qquad \pi_{\mathfrak{t}^-} \circ \mathrm{Ad}(c) = \mathrm{Ad}(c) \circ \pi_{\mathfrak{a}^-} \quad \text{on} \quad \mathfrak{h}_R .$$

Moreover, by (2.3) we have

$$(2.9) \qquad \mathrm{Ad}(c)((1/2)\beta_i) = h_i \quad (1 \leqq i \leqq r) .$$

Next we show

$$(2.10) \qquad \Lambda = \mathrm{Max}\{\pi_{\mathfrak{t}^-}(\alpha);\ \alpha \in \bar{\Sigma}',\ \exists V \in \bar{\mathfrak{p}} - \{0\} \ \text{with} \ [H, V] = (\alpha, H)V \ \text{for each} \ H \in \sqrt{-1}\mathfrak{t}^-\} .$$

In fact, the set of weights relative to $\bar{\mathfrak{t}}$ of the $\bar{\mathfrak{k}}$-module $\bar{\mathfrak{p}}$ coincides with the set of $\pi_{\mathfrak{t}}(\alpha)$ such that $\alpha \in \bar{\Sigma}' \cup \{0\}$ and that there exists $V \in \bar{\mathfrak{p}} - \{0\}$ with $[H, V] = (\alpha, H)V$ for each $H \in \mathfrak{t}_R$. Since $\mathfrak{p}$ is K-isomorphic with a K-submodule of $C^\infty(M)$, we have $\Lambda \in \sqrt{-1}\mathfrak{t}^-$ (cf. Takeuchi [13]). On the other hand, from the definition of the order $>$ on $\mathfrak{t}_R$ we have

$$\mu, \mu' \in \mathfrak{t}_R , \quad \pi_{\mathfrak{t}^-}(\mu) > \pi_{\mathfrak{t}^-}(\mu') \Rightarrow \mu > \mu' .$$

These imply the assertion (2.10). Finally we show that

$$(2.11) \qquad [H', V'_i] = (2h_i, H')V'_i \quad \text{for each } H' \in \sqrt{-1}\mathfrak{t}^- , \quad 1 \leqq i \leqq r .$$

Put $H = \mathrm{Ad}(c)^{-1}H' \in \mathfrak{a}^-$, so $\mathrm{Ad}(c')H = \mathrm{Ad}(c)H$. Applying $\mathrm{Ad}(c')$ to the equality in Lemma 2.2, (2) we get

$$[\mathrm{Ad}(c)H, \mathrm{Ad}(c')V_i] = (\beta_i, H)\mathrm{Ad}(c')V_i ,$$

and hence by (2.4), (2.9)

$$[H', V'_i] = (\beta_i, \mathrm{Ad}(c)^{-1}H')V'_i = (2h_i, H')V'_i .$$

Now, by (2.7), (2.8), (2.9) and (2.2) we have

$$\pi_{t^-}(\bar{\Sigma}') - \{0\} = \{\pm(h_i \pm h_j)\ (1 \leqq i < j \leqq r),\ \pm 2h_i\ (1 \leqq i \leqq r)\}\,, \quad \text{or}$$
$$\{\pm(h_i \pm h_j)\ (1 \leqq i < j \leqq r),\ \pm 2h_i,\ \pm h_i\ (1 \leqq i \leqq r)\}\,,$$

and thus $\Lambda = 2h_1$ by (2.10) and (2.11). q.e.d.

It is known (Takeuchi [12], [15]) that irreducible symmetric R-spaces are devided into the following five classes.

(I) Hermitian type

$2r = s$, $\bar{\Sigma}$ is reducible, $\pi_1(M) = 0$.

$\Sigma = \{\pm(h_i \pm h_j)(1 \leqq i < j \leqq r),\ \pm 2h_i(1 \leqq i \leqq r)\}$, or

$\{\pm(h_i \pm h_j)\ (1 \leqq i < j \leqq r),\ \pm 2h_i,\ \pm h_i\ (1 \leqq i \leqq r)\}$.

(II) type $Sp(r)$

$2r = s$, $\bar{\Sigma}$ is irreducible, $\pi_1(M) = 0$.

Σ is the same as (I).

(III) type $SO(2r+1)$

$r = s$, $\bar{\Sigma}$ is irreducible, $\pi_1(M) = \boldsymbol{Z}_2$.

$\Sigma = \{\pm(h_i \pm h_j)(1 \leqq i < j \leqq r),\ \pm h_i(1 \leqq i \leqq r)\}$.

(IV) type $SO(2r)$

$r = s \geqq 2$, $\bar{\Sigma}$ is irreducible, $\pi_1(M) = \boldsymbol{Z}_2$.

$\Sigma = \{\pm(h_i \pm h_j)\ (1 \leqq i < j \leqq r)\}$.

(V) type $U(r)$

$r = s$, $\bar{\Sigma}$ is irreducible, $\pi_1(M) = \boldsymbol{Z}$.

$\Sigma = \{\pm(h_i - h_j)\ (1 \leqq i < j \leqq r)\}$.

REMARK 1. If M is of Hermitian type, then (M, g) is an irreducible Hermitian symmetric space of compact type and the canonical imbedding f is given as follows. Let M^* be the complex manifold which is the same as M as smooth manifold, but with the complex structure such that the identity map $M \to M^*$, denoted by $p \mapsto p^*$, is anti-holomorphic. We put $\bar{M} = M \times M^*$ and $\bar{g} = (1/2)(g \times g)$. Then the map $f\colon (M, g) \to (\bar{M}, \bar{g})$ defined by $f(p) = p \times p^*(p \in M)$ is the canonical imbedding.

THEOREM 2.4. *Let (M, g) be an irreducible symmetric R-space with the canonical Riemannian metric g. Let λ_1 be the least positive eigenvalue of the Laplacian Δ on $C^\infty(M)$. Suppose that the fundamental group $\pi_1(M)$ of M is finite and g is an Einstein metric. Then $\lambda_1 = 1/2$ with the multiplicity equal to* $\dim \mathfrak{p}$.

PROOF. From the classification of irreducible symmetric R-spaces (cf. §3) we know that the only non-Einstein irreducible symmetric R-spaces M with finite $\pi_1(M)$ are

$$M = Q_{p,q}(\boldsymbol{R}) = \{[x] \in P_{p+q-1}(\boldsymbol{R});\ x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 = 0\}\ ,\quad 3 \leqq p < q\ ,$$

where $[x]$ denotes the line of $\boldsymbol{R}^{p+q}$ through $x = (x_i) \in \boldsymbol{R}^{p+q} - \{0\}$. They are characterized by the property that M is of type $SO(4)$ and the multiplicities of roots $h_1 + h_2$ and $h_1 - h_2$ are different.

We introduce a new inner product $((\ ,\ ))$ on $\mathfrak{t}_{\boldsymbol{R}}$ with $((h_i, h_j)) = \delta_{ij}$ by

$$((H, H')) = \frac{1}{(h_1, h_1)}(H, H') \quad \text{for} \quad H, H' \in \mathfrak{t}_{\boldsymbol{R}}\ .$$

We shall show that $\Lambda = 2h_1$ is the unique element of $D - \{0\}$ such that

$$((2\delta + \Lambda, \Lambda)) = \mathrm{Min}\{((2\delta + \lambda, \lambda));\ \lambda \in D - \{0\}\}\ .$$

If M is of Hermitian type, we have

$$\Sigma^+ = \{h_i \pm h_j\ (1 \leqq i < j \leqq r),\ 2h_i\ (1 \leqq i \leqq r)\}\ , \quad \text{or}$$
$$\{h_i \pm h_j\ (1 \leqq i < j \leqq r),\ 2h_i,\ h_i\ (1 \leqq i \leqq r)\}\ ,$$

and hence by (2.6)

$$D = \{\lambda = 2(m_1h_1 + \cdots + m_rh_r);\ m_i \in \boldsymbol{Z},\ m_1 \geqq \cdots \geqq m_r \geqq 0\}\ .$$

Since the Weyl group W of Σ consists of transformations $h_i \mapsto \varepsilon_i h_{s(i)}$, $\varepsilon_i = \pm 1$, $s \in \mathfrak{S}_r$, and leaves the multiplicities of roots invariant, 2δ is of the form

$$2\delta = n_1h_1 + \cdots + n_rh_r\ , \quad n_i \in \boldsymbol{Z},\ n_1 > \cdots > n_r > 0\ .$$

Thus, for $\lambda \in D - \{0\}$ as above, we have

$$\begin{aligned} ((2\delta + \lambda, \lambda)) &= ((2\delta, \lambda)) + ((\lambda, \lambda)) \\ &= 2\Sigma n_i m_i + 4\Sigma m_i^2 \\ &\geqq 2n_1 + 4 = ((2\delta + 2h_1, 2h_1))\ . \end{aligned}$$

If $\lambda \neq 2h_1$, then $((2\delta, \lambda)) \geqq 2n_1$, $((\lambda, \lambda)) > 4$ and so $((2\delta + \lambda, \lambda)) > 2n_1 + 4$. Thus $\Lambda = 2h_1$ has the required property. In the same way we can show the assertion for a space M of type $Sp(r)$ or of type $SO(2r + 1)$. If M is of type $SO(2r)$, we have

$$\Sigma^+ = \{h_i \pm h_j\ (1 \leqq i < j \leqq r)\}\ ,$$

and hence

$$D = \{\lambda = 2(m_1h_1 + \cdots + m_rh_r);\ m_i \in \boldsymbol{Z},\ m_1 \geqq \cdots \geqq m_{r-1} \geqq |m_r|\}\ .$$

The Weyl group W consists of transformations $h_i \mapsto \varepsilon_i h_{s(i)}$, $\varepsilon_i = \pm 1$, $\prod_i \varepsilon_i =$

1, $s \in \mathfrak{S}_r$. Moreover the multiplicities of $h_1 + h_2$ and $h_1 - h_2$ are the same if $r = 2$. Therefore 2δ is of the form

$$2\delta = n_1 h_1 + \cdots + n_r h_r\,, \quad n_i \in \boldsymbol{Z},\, n_1 > \cdots > n_{r-1} > n_r = 0\,.$$

For $\lambda \in D - \{0\}$ as above, we have

$$((2\delta + \lambda, \lambda)) = 2\Sigma n_i m_i + 4\Sigma m_i^2\,.$$

Theorefore the assertion for M of type $SO(2r)$ follows in the same way as above. Thus the assertion is proved for each (M, g) in consideration.

Now, since $\pi_1(M)$ is finite, K is semi-simple, and hence the $\bar{\mathfrak{k}}$-module $\bar{\mathfrak{p}}$ is irreducible. Thus Lemmas 2.1 and 2.3 imply that $(2\delta + \Lambda, \Lambda) = 1/2$. The theorem follows from this and (2.5). q.e.d.

REMARK 2. The first eigenvalues λ_1 for the other irreducible symmetric R-spaces are calculated in the same way as follows.

(i) $M = Q_{p,q}(\boldsymbol{R})$ $(3 \leqq p < q)$, $\pi_1(M) = Z_2$.

$$\lambda_1 = \begin{cases} 1/2 \quad \text{with multiplicity} = p(p+1) = \dim \mathfrak{p} & \text{if} \quad q = q + 1\,, \\ 1/2 \quad \text{with multiplicity} = (p+2)(3p-1)/2 & \text{if} \quad q = p + 2\,, \\ p/(p+q-2)(<1/2) \quad \text{with multiplicity} = (p+2)(p-1)/2 & \\ & \text{if} \quad q \geqq p + 3\,. \end{cases}$$

(ii) M is of type $U(r)$, $\pi_1(M) = \boldsymbol{Z}$.

Let $\nu \geqq 0$ be the multiplicity of the root $h_1 - h_2$. Then

$$\lambda_1 = \begin{cases} 1/2 \quad \text{with multiplicity} = \dim \mathfrak{p} & \text{if} \quad \nu \leqq 1\,, \\ 1/2 \quad \text{with multiplicity} = \dim \mathfrak{p} + 2 & \text{if} \quad \nu = 2\,, \\ r/(\nu(r-1)+2)(<1/2) \quad \text{with multiplicity} = 2 & \text{if} \quad \nu \geqq 3\,. \end{cases}$$

3. Ricci curvatures of symmetric R-spaces. In this section we shall study the Ricci curvature tensor of an irreducible symmetric R-space.

In general, for a symmetric space (M, g) expressed as $M = K/K_0$ by a symmetric pair (K, K_0) with a K-invariant Riemannian metric g, the Ricci curvature tensor S is given at the origin $o = K_0 \in M$ by

$$S(X, Y) = -(X, Y)_{\mathfrak{k}}/2 \quad \text{for} \quad X, Y \in \mathfrak{m} = T_o M\,, \tag{3.1}$$

where $(\ ,\ )_{\mathfrak{k}}$ is the Killing form of $\mathfrak{k} = \operatorname{Lie} K$ and $\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{m}$ is the Cartan decomposition (cf. Takeuchi-Kobayashi [16]).

Now let $(\mathfrak{g}, \tau)$ be a simple positive definite symmetric graded Lie algebra and (M, g) the irreducible symmetric R-space associated to $(\mathfrak{g}, \tau)$ with the canonical Riemannian metric g. We retain the notation in §1.

If (M, g) is an Einstein manifold: $S = cg$, $c \geqq 0$, we can compute the constant c by (3.1).

For example, let M be of Hermitian type. Then there exists a a complex simple Lie algebra $\mathscr{G}$ such that $\mathfrak{g}$ is the scalar restriction to $\boldsymbol{R}$ of $\mathscr{G}$, and $\mathfrak{k}$ is a compact real form of $\mathscr{G}$ and $\mathfrak{p} = J\mathfrak{k}$, where J is the complex structure of $\mathfrak{g}$. Thus we have

$$(X, Y) = 2(X, Y)_{\mathfrak{k}} \quad \text{for} \quad X, Y \in \mathfrak{k} ,$$

and hence by (3.1)

$$S(X, Y) = -(X, Y)_{\mathfrak{k}}/2 = -(X, Y)/4 = \langle X, Y\rangle/4$$

for $X, Y \in \mathfrak{m}$. Therefore (M, g) is an Einstein manifold: $S = cg$ with

$$c = 1/4 . \tag{3.2}$$

If $M = Q_{p,q}(\boldsymbol{R})$ $(3 \leqq p < q)$, we have decompositions

(3.3) $(M, g) \sim (M_1, g_1) \times (M_2, g_2)$ (locally isometric); and
$K \sim K_1 \times K_2$ (locally isomorphic),

where (M_i, g_i) is a compact connected Einstein symmetric space: $S_i = c_i g_i$ $(i = 1, 2)$ with $0 \leqq c_1 < c_2$ and $K_i = I^0(M_i, g_i)$ $(i = 1, 2)$. That is, $M_1 = S^{p-1}$, $M_2 = S^{q-1}$, $K_1 = SO(p)$ and $K_2 = SO(q)$. The remaining irreducible symmetric R-spaces are those of type $U(r)$ $(r \geqq 2)$. In this case we have also the decompositions (3.3) with $M_1 = S^1$, $K_1 = SO(2)$ and $c_1 = 0$. These constants c_1, c_2 are also computed by (3.1).

We give here the constants c or c_1, c_2 for each non-Hermitian irreducible symmetric R-space.

(1) $\bar{M} = G_{p,q}(\boldsymbol{C})$ $(1 \leqq p \leqq q)$, $M = G_{p,q}(\boldsymbol{R})$.

(a) $p = q = 1$. $r = 1$, type $U(1)$, $\nu = 0$, $\pi_1(M) = \boldsymbol{Z}$, Einstein, $c = 0$.

(b) $p = q \geqq 2$. $r = p$, type $SO(2p)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = (p-1)/4p$.

(c) Otherwise. $r = p$, type $SO(2p+1)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = (p+q-2)/4(p+q)$.

(2) $\bar{M} = G_{2p,2q}(\boldsymbol{C})$ $(1 \leqq p \leqq q)$, $M = G_{p,q}(\boldsymbol{H})$. $r = p$, type $\mathrm{Sp}(p)$, $\pi_1(M) = 0$, Einstein, $c = (p+q+1)/4(p+q)$.

(3) $\bar{M} = G_{n,n}(\boldsymbol{C})$ $(n \geqq 2)$, $M = U(n)$. $r = n$, type $U(n)$, $\nu = 2$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = 1/4$.

(4) $\bar{M} = SO(2n)/U(n)$ $(n \geqq 5)$, $M = SO(n)$. $r = [n/2]$, type $SO(n)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = (n-2)/4(n-1)$.

(5) $\bar{M} = SO(4n)/U(2n)$ $(n \geqq 3)$, $M = U(2n)/Sp(n)$. $r = n$, type $U(n)$, $\nu = 4$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = n/2(2n-1)$.

(6) $\bar{M} = Sp(2n)/U(2n)$ $(n \geqq 2)$, $M = Sp(n)$. $r = n$, type $Sp(n)$, $\pi_1(M) = 0$, Einstein, $c = (n+1)/2(2n+1)$.

(7) $\bar{M} = Sp(n)/U(n)$ $(n \geqq 3)$, $M = U(n)/O(n)$. $r = n$, type $U(n)$, $\nu = 1$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = n/4(n+1)$.

(8) $\bar{M} = Q_{p+q-2}(\boldsymbol{C})$ $(p + q \geqq 3, 1 \leqq p \leqq q)$, $M = Q_{p,q}(\boldsymbol{R})$.

(a) $p = 1$, $q \geqq 4$ $(q \neq 5)$. $r = 1$, type $Sp(1)$, $\pi_1(M) = 0$, Einstein, $c = (q-2)/2(q-1)$.

(b) $p = 2$, $q \geqq 3$ $(q \neq 4)$. $r = 2$, type $U(2)$, $\nu = q - 2$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = (q-2)/2q$.

(c) $p = q \geqq 4$. $r = 2$, type $SO(4)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = (p-2)/4(p-1)$.

(d) $3 \leqq p < q$. $r = 2$, type $SO(4)$, $\pi_1(M) = \boldsymbol{Z}_2$, $c_1 = (p-2)/2(p + q - 2)$, $c_2 = (q-2)/2(p+q-2)$.

(9) $\bar{M} = E_6/T\cdot Spin(10)$, $M = G_{2,2}(\boldsymbol{H})/\boldsymbol{Z}_2$. $r = 2$, type $SO(5)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = 5/24$.

(10) $\bar{M} = E_6/T\cdot Spin(10)$, $M = P_2(\boldsymbol{K})$. $r = 1$, type $Sp(1)$, $\pi_1(M) = 0$, Einstein, $c = 3/8$.

(11) $\bar{M} = E_7/T\cdot E_6$, $M = SU(8)/Sp(4)\cdot \boldsymbol{Z}_2$. $r = 4$, type $SO(8)$, $\pi_1(M) = \boldsymbol{Z}_2$, Einstein, $c = 2/9$.

(12) $\bar{M} = E_7/T\cdot E_6$, $M = T\cdot E_6/F_4$. $r = 3$, type $U(3)$, $\nu = 8$, $\pi_1(M) = \boldsymbol{Z}$, $c_1 = 0$, $c_2 = 1/3$.

In the above list,

$G_{p,q}(\boldsymbol{F})$: Grassmann manifold of all p-subspaces in $\boldsymbol{F}^{p+q}$, for $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{C}$ or real quaternion algebra $\boldsymbol{H}$,

$P_2(\boldsymbol{K})$: Cayley projective plane,

$Q_n(\boldsymbol{C})$: Complex quadric of dimension n,

Einstein: (M, g) is an Einstein manifold.

4. **Stability of TRG-pairs.** In this section we shall study the stability as a minimal submanifold of M in $(\bar{M}, \bar{g})$ for a TRG-pair $((\bar{M}, \bar{g}), M)$.

In general, let $f: (M, g) \to (\bar{M}, \bar{g})$ be a minimal isometric immersion of a compact Riemannian manifold (M, g) into a Riemannian manifold $(\bar{M}, \bar{g})$. Let f_t be a smooth variation of f with $f_0 = f$ and $\mathscr{V}(t)$ the volume of $(M, f_t^*\bar{g})$. Then the second derivative of $\mathscr{V}(t)$ is described as follows (cf. Simons [11]). We define a vector field V along f by

$$V_p = \left[\frac{d}{dt} f_t(p)\right]_{t=0} \quad \text{for} \quad p \in M .$$

We define furthermore an elliptic self-adjoint differential operator L of order 2 on the space $C^\infty(NM)$ of all smooth sections of the normal bundle NM for f, called the *Jacobi operator* for f, by

$$L = \Delta^{\perp} + S^{\perp} - \tilde{\alpha} .$$

Here $\Delta^{\perp} = -\mathrm{Tr}_g(\nabla^{\perp})^2$ is the Laplacian on NM; $\tilde{\alpha} \in C^{\infty}(\mathrm{End}\, NM)$ is defined by $\tilde{\alpha} = \alpha \circ {}^t\alpha$ regarding the second fundamental form α of f as $\alpha \in C^{\infty}(\mathrm{Hom}(TM \otimes TM, NM))$; $S^{\perp} \in C^{\infty}(\mathrm{End}\, NM)$ is defined by

$$\langle S^{\perp}(u), v\rangle = \sum_i \langle \bar{R}(e_i, u)e_i, v\rangle \quad \text{for} \quad u, v \in N_pM , \quad p \in M ,$$

where $\bar{R}$ is the curvature tensor of $(\bar{M}, \bar{g})$ and $\{e_i\}$ is an orthonormal basis for T_pM. We have then

$$\frac{d^2\mathscr{V}}{dt^2}(0) = \int_M \langle L V^N, V^N\rangle dv ,$$

where V^N denotes the normal component of V and dv the Riemannian measure of (M, g).

The multiplicity $n(f)$ of the eigenvalue 0 of L is called the *nullity* of f. The sum $i(f)$ of multiplicities of negative eigenvalues of L is called the *index* of f. The minimal immersion f is said to be *stable* if $i(f) = 0$. We define moreover a subspace P of $C^{\infty}(NM)$ by

$$P = \{(X|M)^N ; X \text{ is a Killing vector field on } (\bar{M}, \bar{g})\} ,$$

and call the dimension $n_k(f)$ of P the *Killing nullity* of f. It is known (cf. Simons [11]) that $L|P = 0$, and hence $n_k(f) \leqq n(f)$.

LEMMA 4.1. (Chen-Leung-Nagano [1]) *Let (M, g) be a compact connected symmetric space expressed as $M = K/K_0$ by an almost effective compact symmetric pair (K, K_0). Suppose that g is defined by a K-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{k} = \mathrm{Lie}\, K$ and let C denote the Casimir operator of $\mathfrak{k}$ relative to $\langle\ ,\ \rangle$. Let $f: (M, g) \to (\bar{M}, \bar{g})$ be a totally geodesic isometric immersion of (M, g) into a symmetric space $(\bar{M}, \bar{g})$. Then $\mathfrak{k}$ acts on the normal bundle NM and there exists a $\mathfrak{k}$-invariant symmetric endomorphism Q of NM such that the Jacobi operator L for f is given by*

$$L = C + Q . \tag{4.1}$$

We retain the notation in §1 for symmetric R-spaces. By a method in [1] we prove the following:

THEOREM 4.2. *Let (M, g) be a symmetric R-space with the canonical Riemannian metric g associated to a positive definite symmetric graded Lie algebra $(\mathfrak{g}, \tau)$, and $f: (M, g) \to (\bar{M}, \bar{g})$ the canonical isometric imbedding. Then*

(1) $n_k(f) = \dim \mathfrak{p}$,
(2) $Q = -(1/2)I_{NM}$.

PROOF. (1) Identifying $\mathfrak{p}$ with a space of vector fields on M, we define a linear map $\mathfrak{p} \to P$ by the correspondence $X \mapsto (JX)|M$ $(X \in \mathfrak{p})$. Then it is a K-isomorphism since $\mathfrak{p} = \mathfrak{g} \cap J\mathfrak{g}_u$, and thus the assertion follows.

(2) Let C be the Casimir operator of $\mathfrak{k}$ relative to $\langle X, Y \rangle = -(X, Y)$. By the proof of (1) and Lemma 2.1 we have $C|P = (1/2)I_P$. Thus, by $L|P = 0$ and (4.1) we get $Q|P = -(1/2)I_P$. On the other hand, since G_u is transitive on $\bar{M}$ we have

$$T_p\bar{M} = \{X_p;\ X \in \mathfrak{g}_u\} \quad \text{for any } p \in M .$$

Therefore, by $\mathfrak{g}_u = \mathfrak{k} + J\mathfrak{p}$ we have

$$N_pM = \{X_p;\ X \in P\} \quad \text{for any } p \in M .$$

This and $Q|P = -(1/2)I_P$ imply the assertion. q.e.d.

REMARK 1. Let $f: (M, g) \to (\bar{M}, \bar{g})$ be as in Theorem 4.2. We define an endomorphism $\bar{S}^\perp$ of NM by

$$\langle \bar{S}^\perp(u), v \rangle = \bar{S}(u, v) \quad \text{for} \quad u, v \in N_pM , \quad p \in M ,$$

where $\bar{S}$ denotes the Ricci curvature tensor of $(\bar{M}, \bar{g})$. It can be proved by a direct calculation that then $Q = -\bar{S}^\perp$, and hence the assertion (2) follows also from the formula (3.1) for our $(\bar{M}, \bar{g})$.

Recalling (Ikeda-Taniguchi [3]) that the Laplacian acting on forms on a compact symmetric space M coincides with the Casimir operator, we get the following:

COROLLARY. *Let $\hat{L}$ be the differential operator on $C^\infty(T^*M)$ corresponding to L on $C^\infty(NM)$ under the K-isomorphism:*

$$NM \underset{J\cdot}{\overset{\cong}{\rightarrow}} TM \underset{\hat{g}}{\overset{\cong}{\rightarrow}} T^*M ,$$

where T^*M is the cotangent bundle of M, $J\cdot$ is the multiplication by J and $\hat{g}$ is the duality by means of g. Then

$$\hat{L} = \Delta - (1/2)I_{T^*M} ,$$

where Δ denotes the Laplacian of (M, g) acting on the space $C^\infty(T^*M)$ of 1-forms on M.

Here we recall some results on the Laplacian Δ on 1-forms on a general compact connected Riemannian manifold (M, g). For $\lambda \geqq 0$ we put

$$F_\lambda = \{f \in C^\infty(M);\ \Delta f = \lambda f\} ,$$
$$E_\lambda = \{\xi \in C^\infty(T^*M);\ \Delta\xi = \lambda\xi\} ,$$

$$B_\lambda = \{\xi \in E_\lambda;\ d\xi = 0\}\ ,$$
$$C_\lambda = \{\xi \in E_\lambda;\ d^*\xi = 0\}\ ,$$

where d^* denotes the formal adjoint operator of d with respect to the Riemannian measure for g. If $\lambda > 0$, we have

$$E_\lambda = B_\lambda + C_\lambda \quad \text{(direct sum)}\ , \tag{4.2}$$

and d induces an isomorphism

$$d\colon F_\lambda \overset{\cong}{\to} B_\lambda\ . \tag{4.3}$$

THEOREM OF YANO. (cf. Kobayashi [4]) *If (M, g) is an Einstein manifold: $S = cg$, then C_{2c} coincides with the space of all Killing 1-forms on (M, g).*

THEOREM OF NAGANO [8]. *If (M, g) is an Einstein manifold: $S = cg$ with $c > 0$, then $C_\lambda = 0$ for each λ with $0 < \lambda < 2c$.*

THEOREM 4.3. *Let $f\colon (M, g) \to (\bar{M}, \bar{g})$ be the canonical isometric imbedding of an irreducible symmetric R-space (M, g). Then, f is stable if and only if M is simply connected.*

PROOF. By Corollary of Theorem 4.2, f is stable if and only if $E_\lambda = 0$ for each λ with $0 \leqq \lambda < 1/2$. We prove the assertion in the following four cases separately.

(i) M is of Hermitian type.

(ii) M is not of Hermitian type, $\pi_1(M)$ is finite and g is an Einstein metric: $S = cg$.

(iii) M is not of Hermitian type, $\pi_1(M)$ is finite and g is not an Einstein metric.

(iv) M is of type $U(r)$.

In case (i), $\pi_1(M) = 0$ and (M, g) is an Einstein manifold: $S = cg$ with $c = 1/4$ by (3.2). Thus $E_0 = 0$ and $\lambda_1 = 1/2$ by Theorem 2.4. Therefore $B_\lambda = 0$ for $0 < \lambda < 1/2$ by (4.3). Moreover, by Theorem of Nagano $C_\lambda = 0$ for $0 < \lambda < 1/2$. Thus by (4.2) $E_\lambda = 0$ for $0 < \lambda < 1/2$, and hence f is stable.

In case (ii), in the same way as (i) we get $E_0 = 0$ and $B_\lambda = 0$ for $0 < \lambda < 1/2$. From §3 we see that

$$\pi_1(M) = 0 \Leftrightarrow c > 1/4\ ,$$
$$\pi_1(M) \neq 0 \Leftrightarrow 0 < c < 1/4\ .$$

Thus, if $\pi_1(M) = 0$ f is stable by the same reasoning as in case (i). If $\pi_1(M) \neq 0$, we have $0 < 2c < 1/2$ and $\dim E_{2c} = \dim C_{2c} = \dim \mathfrak{k} > 0$ by

Theorem of Yano. Thus f is not stable.

In case (iii), $M = Q_{p,q}(\boldsymbol{R})(3 \leqq p < q)$, $\pi_1(M) = \boldsymbol{Z}_2$ and $0 < c_1 = (p-2)/2(p+q-2) < 1/4$. Thus $0 < 2c_1 < 1/2$ and $\dim E_{2c_1} \geqq \dim C_{2c_1} \geqq \dim SO(p) > 0$ by Theorem of Yano. Thus f is not stable.

In case (iv), $\pi_1(M) = \boldsymbol{Z}$ and so $\dim E_0 = 1$. Hence f is not stable.

q.e.d.

REMARK 2. From the proof we see:

In case (i), $n(f) = \dim_{\boldsymbol{R}} \mathrm{Aut}^0(M)$;

In case (ii), $n(f) = \dim \mathfrak{p}$ if $\pi_1(M) = 0$, and $i(f) \geqq \dim I^0(M, g)$ if $\pi_1(M) \neq 0$.

THEOREM 4.4. *Let $(\bar{M}, \bar{g})$ be a connected Hermitian symmetric space of compact type and M a compact connected totally real totally geodesic submanifold of $(\bar{M}, \bar{g})$ with $\dim M = \dim_{\boldsymbol{C}} \bar{M}$. Then, M is a stable minimal submanifold if and only if M is simply connected.*

PROOF. It is easily seen that the stability of M in $(\bar{M}, \bar{g})$ for a TRG-pair $((\bar{M}, \bar{g}), M)$ is invariant under the equivalence of TRG-pairs and that for the direct product $((\bar{M}, \bar{g}), M) = ((\bar{M}_1, \bar{g}_1), M_1) \times ((\bar{M}_2, \bar{g}_2), M_2)$, M is stable in $(\bar{M}, \bar{g})$ if and only if each M_i is stable in $(\bar{M}_i, \bar{g}_i)$ $(i = 1, 2)$. Thus the assertion follows from Theorems 1.2 and 4.3. q.e.d.

REFERENCES

[1] B. Y. CHEN, P. F. LEUNG AND T. NAGANO, Totally geodesic submanifolds of symmetric spaces III, preprint.

[2] S. HELGASON, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.

[3] A. IKEDA AND Y. TANIGUCHI, Spectra and eigenforms of the Laplacian on S^n and $P^n(C)$, Osaka J. Math. 15 (1978), 515–546.

[4] S. KOBAYASHI, Transformation Groups in Differential Geometry, Springer-Verlag, Berlin, 1972.

[5] S. KOBAYASHI AND T. NAGANO, On filtered Lie algebras and geometric structures I, J. Math. Mech. 13 (1964), 875–908.

[6] H. B. LAWSON, JR. AND J. SIMONS, On stable currents and their application to global problems in real and complex geometry, Ann. Math. 84 (1973), 427–450.

[7] C. C. MOORE, Compactifications of symmetric spaces, Amer. J. Math. 86 (1964), 201–218, 358–378.

[8] T. NAGANO, On the minimum eigenvalues of the Laplacians in Riemannian manifolds, Sci. Papers Coll. Gen. Ed. Univ. Tokyo 11 (1961), 177–182.

[9] I. SATAKE, On representations and compactifications of symmetric Riemannian spaces, Ann. Math. 71 (1960), 77–110.

[10] I. SATAKE, Algebraic Structures of Symmetric Domains, Iwanami Shoten, Publishers and Princeton Univ. Press, Tokyo-Princeton, 1981.

[11] J. SIMONS, Minimal varieties in riemannian manifolds, Ann. Math. 88 (1968), 62–105.

[12] M. TAKEUCHI, Cell decompositions and Morse equalities on certain symmetric spaces, J. Fac. Sci. Univ. Tokyo, I, 12 (1965), 81-192.
[13] M. TAKEUCHI, Polynomial representations associated with symmetric bounded domains, Osaka J. Math. 10 (1973), 441-475.
[14] M. TAKEUCHI, Homogeneous Kähler submanifolds in complex projective spaces, Jap. J. Math. 4 (1978), 171-219.
[15] M. TAKEUCHI, On conjugate loci and cut loci of compact symmetric spaces II, Tsukuba J. Math. 3 (1979),1-29.
[16] M. TAKEUCHI AND S. KOBAYASHI, Minimal imbeddings of R-spaces, J. Diff. Geom. 2 (1969), 203-215.

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
OSAKA UNIVERSITY
TOYONAKA, 560
JAPAN