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TSUCHIHASHI'S CUSP SINGULARITIES ARE BUCHSBAUM SINGULARITIES

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Introduction. In his paper [T], Tsuchihashi studied a kind of "elliptic" isolated singularities which are generalizations, in higher dimensions, of the two-dimensional cusp singularities in the sense of Karras [K] and Nakamura [N1]. However his cusp singularity is not Cohen-Macaulay. He showed that some of his cusp singularities of higher dimensions are quasi-Buchsbaum. This fact suggests that all his cusp singularities are Buchsbaum. Recall that a noetherian local ring A of dimension d is said to be Buchsbaum if there exists an integer k such that, for any system of parameters $\{u_1, \cdots, u_d\}$ of A, the difference of the colength length(A/I) and the multiplicity $e_A(I)$ of the ideal $I = (u_1, \cdots, u_d)$ is equal to k. It is well known that A is Cohen-Macaulay if and only if it is Buchsbaum and the above constant k is equal to 0. Buchsbaum local rings are quasi-Buchsbaum. But the converse is not true. Actually, S. Goto showed that there are many examples of non-Buchsbaum quasi-Buchsbaum local domains [G].

The purpose of this paper is to prove that Tsuchihashi's cusp singularities are always Buchsbaum. For the proof, we use Schenzel's characterization of Buchsbaum local rings in terms of Grothendieck's dualizing complexes [S]. Actually, we determine the truncation below $\tau_{-1}(K_V^{\cdot})$ of the dualizing complex $K_V^{\cdot}$ of Tsuchihashi's cusp singularity $P \in V$ of dimension n.

This problem was raised by Dr. Kimio Watanabe and Mr. Y. Koyama as is mensioned in the introduction of [T].

1. Preparation. Let N be a free $\boldsymbol{Z}$-module of rank $n > 1$. Tsuchihashi considers a pair (C, Γ) of an open convex cone in $N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ which contains no lines in $N_{\boldsymbol{R}}$ and a subgroup Γ of the automorphism group $\mathrm{GL}(N)$ of N such that C is Γ-invariant, the action of Γ on $D = C/\boldsymbol{R}_+$ is properly discontinuous and free, and has the compact quotient D/Γ.

There exists a rational partial polyhedral decomposition Σ of $N_{\boldsymbol{R}}$ such that

(1) $\bigcup_{\sigma \in \Sigma \setminus \{0\}} (\sigma \setminus \{0\}) = C,$

(2) for any compact subset F of C, the cardinality $^{\#}\{\sigma \in \Sigma; \sigma \cap F \neq \emptyset\}$ is finite,

(3) Σ is Γ-invariant,

(4) the action of Γ on $\Sigma \backslash \{\mathbf{0}\}$ is free, and

(5) the quotient $(\Sigma \backslash \{\mathbf{0}\})/\Gamma$ is finite,

where $\mathbf{0}$ is the cone $\{0\}$.

An example of such an r.p.p. decomposition Σ is obtained as follows: Let θ be the convex hull of $C \cap N$ in $N_{\boldsymbol{R}}$. Then θ is a closed convex set of $N_{\boldsymbol{R}}$. The compactness of the quotient D/Γ implies that θ is locally given by a finite number of linear inequalities and each face of θ is a compact polytope. Let Σ be the collection of $\mathbf{0}$ as well as cones over the faces of θ. Then Σ satisfies the condition (c.f. [SC, Chap. 2]).

Furthermore, by taking a Γ-invariant subdivision of Σ, we may assume,

(6) for any elements $\sigma, \tau \in \Sigma$, the cardinality $^{\#}\{g \in \Gamma; g(\sigma) \cap \tau \neq \mathbf{0}\}$ is at most one, and

(7) every $\sigma \in \Sigma$ is a nonsingular cone, i.e., σ is spanned by a part of a $\boldsymbol{Z}$-basis of N (see [TE, Chap. 1, Th. 4]).

Actually, the condition (6) is satisfied if we take some Γ-invariant barycentric subdivision of Σ twice. We can subdivide it further so that the condition (7) is satisfied [TE, Chap. 1, Th. 11] (see also [N2, Th. 7.20]).

For a positive integer k, we denote by Σ_k the set of k-dimensional cones in Σ. For each one-dimensional cone $\gamma \in \Sigma_1$, we denote by $v(\gamma)$ the primitive element of N with $\boldsymbol{R}_0 v(\gamma) = \gamma$ where $\boldsymbol{R}_0 = \{c \in \boldsymbol{R}; c \geqq 0\}$. Since each element of Σ is nonsingular, an element $\sigma \in \Sigma_k$ has exactly k one-dimensional faces. We denote by S_σ the $(k-1)$-simplex in $N_{\boldsymbol{R}}$ spanned by $\{v(\gamma); \gamma \in \Sigma_1, \gamma \prec \sigma\}$ for each $\sigma \in \Sigma_k$, $k = 1, \cdots, n$. If we set $\tilde{K} = \{S_\sigma; \sigma \in \Sigma \backslash \{\mathbf{0}\}\}$, then we know $|\tilde{K}| = \bigcup_{\sigma \in \Sigma \backslash \{0\}} S_\sigma$ is isomorphic to D through $\pi: C \to C/\boldsymbol{R}_+ = D$, and $\tilde{K}$ give rise to a triangulation of D. By (4), Γ acts freely on $\tilde{K}$. Let K be the quotient $\tilde{K}/\Gamma$. Then by (6), we know K is a triangulation of the $(n-1)$-dimensional compact topological manifold D/Γ into a finite simplicial complex. Let Φ be the associated abstract simplicial complex consisting of subsets of the set $\{1, \cdots, s\}$ of indices of the vertices $\{v_1, \cdots, v_s\}$ of K.

Let T_N be the algebraic torus $N \otimes_{\boldsymbol{Z}} \boldsymbol{C}^*$. Since we assume that every cone in Σ is nonsingular, the associated T_N-embedding Z is nonsingular. Since Σ is Γ-invariant, the group Γ acts on Z. Let ord: $T_N = N \otimes_{\boldsymbol{Z}} \boldsymbol{C}^* \to N_{\boldsymbol{R}} = N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ be the homomorphism $1_N \otimes (-\log|\ \ |)$. Then the union $\tilde{W} = \mathrm{ord}^{-1}(C) \cup (Z \backslash T_N)$ is a Γ-invariant open set (in the classical topology) of Z. The action of Γ on $\tilde{W}$ is free and properly discontinuous, and the reduced analytic subspace $\tilde{Y} = Z \backslash T_N \subset \tilde{W}$ is invariant by the action. We denote by W the quotient analytic manifold $\tilde{W}/\Gamma$, and we denote $Y = \tilde{Y}/\Gamma$. By the construction, they have the following properties:

(i) Y is the union of s compact irreducible analytic subspaces $X_1, \cdots, X_s$ of W.

(ii) For a subset I of $\{1, \cdots, s\}$, the intersection $X_I = \bigcap_{i \in I} X_i$ is nonempty if and only if $I \in \Phi$.

(iii) For each $I \in \Phi$, the analytic space X_I is isomorphic to a nonsingular torus embedding of dimension $n - |I|$.

In particular, Y is a simple normal crossing divisor on W.

We define a complex $A(Y)^{\cdot}$ of O_Y-modules as follows. Set $A(Y)^p = \bigoplus_{I \in \Phi_{p+1}} O_{X_I}$ and let $i(I)\colon O_{X_I} \to A(Y)^p$ and $j(I)\colon A(Y)^p \to O_{X_I}$ be the natural injection and projection for each $I \in \Phi_{p+1}$, where $\Phi_p = \{I \in \Phi; |I| = p\}$. The coboundary homomorphism $\delta\colon A(Y)^{p-1} \to A(Y)^p$ is given by

$$\delta = \sum_{\substack{I \in \Phi_p \\ J \in \Phi_{p+1}}} i(J) \circ \lambda_{I/J} \circ j(I) ,$$

where $\lambda_{I/J} = 0$ if $I \not\subset J$ and $(-1)^q$ times the restriction map $O_{X_I} \to O_{X_J}$ if $J = \{i_0, \cdots, i_p\}$, $1 \leqq i_0 < \cdots < i_p \leqq s$ and $I = J \backslash \{i_q\}$. Let $\nu\colon O_Y \to A(Y)^0 = O_{X_1} \oplus \cdots \oplus O_{X_s}$ be the homomorphism obtained as the sum of the restriction maps. Then we get an exact sequence of O_Y-modules:

$$0 \to O_Y \xrightarrow{\nu} A(Y)^0 \to A(Y)^1 \to \cdots \to A(Y)^{n-1} \to 0 .$$

Thus the O_W-module O_Y is equal to $A(Y)^{\cdot}$ in the derived category $D_c^+(W)$ of complexes of O_W-modules bounded below and with coherent cohomology sheaves.

Tsuchihashi's cusp singularity Cusp(C, Γ) is obtained by contraction of the divisor $Y \subset W$ to a normal point $P \in V$. Let this contraction morphism be $\pi\colon W \to V$.

$$\begin{array}{ccc} W & \supset & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ V & \ni & P \end{array}$$

Now, consider the direct image $\boldsymbol{R}\pi_*\colon D_c^+(W) \to D_c^+(V)$ into the derived category of complexes of O_V-modules bounded below and with coherent cohomology sheaves. Since X_I is a nonsingular rational variety, we have $\boldsymbol{R}\pi_* O_{X_I} = \boldsymbol{C}(P)$, the residue field of the local ring $O_{V,P}$, regarded as an O_V-module. In particular, the O_W-module O_{X_I} is π_*-acyclic in the sense of [RD, Chap. 1, § 5]. Hence $\boldsymbol{R}\pi_* O_Y = \boldsymbol{R}\pi_* A(Y)^{\cdot}$ is represented by the complex $\pi_* A(Y)^{\cdot} = C^{\cdot}(\Phi, \boldsymbol{C}) \otimes_{\boldsymbol{C}} \boldsymbol{C}(P)$ of $\boldsymbol{C}(P)$-vector spaces, where $C^{\cdot}(\Phi, \boldsymbol{C})$ is the usual cochain complex of the abstract simplicial complex Φ with coefficients in $\boldsymbol{C}$. Since Φ is the associated abstract simplicial complex of the triangulation K of D/Γ, the p-th cohomology $H^p(\Phi, \boldsymbol{C})$ of the complex $C^{\cdot}(\Phi, \boldsymbol{C})$ is isomorphic to the cohomology $H^p(D/\Gamma, \boldsymbol{C})$ for every $p \in \boldsymbol{Z}$.

2. The higher direct images for the cusp singularities. Let M be the dual $\boldsymbol{Z}$-module $\mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ of N, and let $\langle\ ,\ \rangle$: $M \times N \to \boldsymbol{Z}$ be the natural pairing. For each element $m \in M$, we denote by $e(m)$ the character $\langle m,\ \rangle \otimes 1_{\boldsymbol{C}^*}$: $T_N = N \otimes_{\boldsymbol{Z}} \boldsymbol{C}^* \to \boldsymbol{C}^*$. Let $\{m_1, \cdots, m_n\}$ be a basis of M. Then we have an isomorphism $(e(m_1), \cdots, e(m_n))$: $T_N \xrightarrow{\sim} (\boldsymbol{C}^*)^n$. We set $x_i = e(m_i)$, $i = 1, \cdots, n$. Then $\tilde{\omega} = (dx_1/x_1) \wedge \cdots \wedge (dx_n/x_n)$ is an n-form on $\tilde{W}$ with pole of order one along $\tilde{Y}$. For an element $h \in \Gamma \subset \mathrm{GL}(N)$, we have $h(\tilde{\omega}) = \det(h) \circ \tilde{\omega}$. Hence, if the group Γ is contained in $\mathrm{SL}(N)$, then the n-form $\tilde{\omega}$ descends to W. In particular, we have $\omega_W \simeq O_W(-Y)$, where ω_W is the canonical invertible sheaf of the nonsingular analytic space W. Clearly, the direct image $\pi_* O_W(-Y)$ is equal to the maximal ideal $\mathfrak{m}_P$ of $P \in V$, for any Γ. Hence by the Grauert-Riemenschneider vanishing theorem [GR], we have $\boldsymbol{R}\pi_* O_W(-Y) = \mathfrak{m}_P$ in case $\Gamma \subset \mathrm{SL}(N)$.

PROPOSITION 2.1. *Even if Γ is not contained in* $\mathrm{SL}(N)$, *we have* $\boldsymbol{R}\pi_* O_W(-Y) = \mathfrak{m}_P$.

PROOF. Let $\Gamma' = \Gamma \cap \mathrm{SL}(N)$ and define $W' \supset Y'$ and the contraction π': $W' \to V' \ni P'$ similarly for Γ' using the same r.p.p. decomposition Σ. Then since $[\Gamma : \Gamma'] = 2$, there exist ρ_W: $W' \to W$, ρ_V: $V' \to V$ with ρ_W an unramified covering of degree two. Since $\rho_W^*(O_W(-Y)) = O_{W'}(-Y')$

$$\begin{array}{ccc} W' & \xrightarrow{\rho_W} & W \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ V' & \xrightarrow{\rho_V} & V \end{array}$$

and since ρ_W is an unramified covering, we know $O_W(-Y)$ is a direct summand of the direct image $(\rho_W)_* O_{W'}(-Y)'$. Hence the higher direct image $R^i\pi_* O_W(-Y)$ is a direct summand of $R^i(\pi \circ \rho_W)_* O_{W'}(-Y') = R^i(\rho_V \circ \pi')_* O_{W'}(-Y') = R^i(\rho_V)_* \mathfrak{m}_{P'}$. Since ρ_V is a finite morphism, we know all higher direct images by it to be zero. Hence the direct summand $R^i\pi_* O_W(-Y)$ is also zero for every $i > 0$. q.e.d.

By taking the direct image, in the derived category, of the short exact sequence $0 \to O_W(-Y) \to O_W \to O_Y \to 0$, we have a triangle of objects in $D_c^+(V)$:

$$
*)\qquad
\begin{array}{ccc}
 & R\pi_* O_Y = C^{\cdot}(\Phi, C) \underset{C}{\otimes} C(P) & \\
 {\scriptstyle (+1)}\swarrow & & \nwarrow \\
 \mathfrak{m}_P = R\pi_* O_W(-Y) & \longrightarrow & R\pi_* O_W \, .
\end{array}
$$

Here recall that a sextuple (A, B, C, u, v, w) of elements $A, B, C \in D_c^+(V)$ and homomorphisms $u: A \to B$, $v: B \to C$, $w: C \to A[1]$ is called a triangle and is written

$$
\begin{array}{ccc}
 & C & \\
 {\scriptstyle (+1)}\swarrow & & \nwarrow \\
 A & \longrightarrow & B
\end{array}
$$

if A, B, C are represented by complexes $A^{\cdot}$, $B^{\cdot}$, $C^{\cdot}$ such that $C^{\cdot}$ is the mapping cone of $u: A^{\cdot} \to B^{\cdot}$, and $v: B^{\cdot} \to C^{\cdot}$ and $w: C^{\cdot} \to A^{\cdot}[1]$ are natural homomorphisms (see [RD, Chap. 1, §0]).

The following theorem is the generalization of Tsuchihashi's result [T, Th. 2.3] to the general case.

THEOREM 2.2. *Let $f: \tilde{V} \to V$ be a resolution of Tsuchihashi's cusp singularity. Then we have $R^p f_* O_{\tilde{V}} \simeq H^p(D/\Gamma, C) \otimes_C C(P)$ for every $p > 0$.*

PROOF. Since the higher direct image $R^p f_* O_{\tilde{V}}$ does not depend on the choice of the resolution, we may assume $\tilde{V} = W$ and $f = \pi$. Then since $\mathscr{H}^p(\mathfrak{m}_P) = 0$ for every $p > 0$, we have the proposition by the long exact sequence obtained from the triangle *). q.e.d.

For a complex $A^{\cdot}$ of O_V-modules and for an integer q, we denote by $\tau_q(A^{\cdot})$ the complex

$$\cdots \to 0 \to \operatorname{Im}(d_A^q) \to A^{q+1} \to A^{q+2} \to \cdots ,$$

i.e., the truncation below of $A^{\cdot}$ at the degree q, where $\operatorname{Im}(d_A^q)$ is the image of the coboundary map $d_A^q: A^q \to A^{q+1}$. If an element A in the derived category $D_c^+(V)$ is represented by the complex $A^{\cdot}$, then we denote by $\tau_q(A)$ the element in $D_c^+(V)$ represented by the complex $\tau_q(A^{\cdot})$. In particular, $\mathscr{H}^p(\tau_q(A)) = \mathscr{H}^p(\tau_q(A^{\cdot}))$ is equal to 0 for $p \leqq q$ and equal to $\mathscr{H}^p(A)$ for $p > q$.

LEMMA 2.3. *Let*

$$
\begin{array}{ccc}
 & C & \\
 {\scriptstyle (+1)}\swarrow & & \nwarrow \\
 A & \underset{u}{\longrightarrow} & B
\end{array}
$$

be a triangle of objects in $D_c^+(V)$. *If the homomorphism* $\mathscr{H}^q(B) \to \mathscr{H}^q(C)$ *is surjective for an integer* q, *then the truncations at* q *give rise to a triangle*

$$\begin{array}{ccc} & \tau_q(C) & \\ {}^{(+1)}\swarrow & & \nwarrow \\ \tau_q(A) & \longrightarrow & \tau_q(B)\,. \end{array}$$

PROOF. Let $A^\cdot$ and $B^\cdot$ be complexes of O_V-modules which represent A and B, respectively, and let u be represented by a homomorphism $u^\cdot\colon A^\cdot \to B^\cdot$ of complexes. Then we have a natural homomorphism $\tau_q(u^\cdot)$: $\tau_q(A^\cdot) \to \tau_q(B^\cdot)$, and its mapping cone is

$$\cdots \to 0 \to \mathrm{Im}\,(d_A^q) \xrightarrow{d^{q-1}} A^{q+1} \oplus \mathrm{Im}\,(d_B^q) \xrightarrow{d^q} A^{q+2} \oplus B^{q+1} \to \cdots .$$

From the long exact sequence obtained from the triangle, we know the surjectivity of $\mathscr{H}^q(B^\cdot) \to \mathscr{H}^q(C^\cdot)$ implies that the homomorphism $\mathscr{H}^{q+1}(A^\cdot) \to \mathscr{H}^{q+1}(B^\cdot)$ is injective. Hence in the diagram

$$\begin{array}{ccccc} A^q & \xrightarrow{d_A^q} & A^{q+1} & \xrightarrow{d_A^{q+1}} & A^{q+2} \\ \big\downarrow{\scriptstyle u^q} & & \big\downarrow{\scriptstyle u^{q+1}} & & \big\downarrow{\scriptstyle u^{q+2}} \\ B^q & \xrightarrow{d_B^q} & B^{q+1} & \xrightarrow{d_B^{q+1}} & B^{q+2} \end{array}$$

we have $\mathrm{Ker}\,(d_A^{q+1}) \cap (u^{q+1})^{-1}(\mathrm{Im}\,(d_B^q)) = \mathrm{Im}\,(d_A^q)$. This implies $\mathrm{Im}\,(d^{q-1}) = \mathrm{Ker}\,(d^q)$ for the mapping cone of $\tau_q(u^\cdot)$, and we know it is quasi-isomorphic to $\tau_q(C^\cdot)$. q.e.d.

We denote by $\tilde{C}^\cdot(\Phi, \boldsymbol{C})$ the augmented cochain complex

$$\cdots \to 0 \to \boldsymbol{C} \to C^0(\Phi, \boldsymbol{C}) \to C^1(\Phi, \boldsymbol{C}) \to C^2(\Phi, \boldsymbol{C}) \to \cdots .$$

Since D/Γ is connected, we know $\tilde{H}^0(\Phi, \boldsymbol{C}) = 0$ and $\tilde{C}^\cdot(\Phi, \boldsymbol{C})$ is quasi-isomorphic to $\tau_0(C^\cdot(\Phi, \boldsymbol{C}))$.

PROPOSITION 2.4. *The truncation* $\tau_0(\boldsymbol{R}\pi_* O_W)$ *is isomorphic, in the derived category* $D_c^+(V)$, *to the complex* $\tilde{C}^\cdot(\Phi, \boldsymbol{C}) \otimes_{\boldsymbol{C}} \boldsymbol{C}(P)$ *of* $\boldsymbol{C}(P)$*-vector spaces.*

PROOF. Since the 0-th cohomologies of the three objects in the triangle *) form a short sequence $0 \to \mathfrak{m}_P \to O_{V,P} \to \boldsymbol{C}(P) \to 0$, we can apply Lemma 2.3 for the truncations at degree zero. Since $\tau_0(\mathfrak{m}_P)$ is trivial, we know $\tau_0(\boldsymbol{R}\pi_* O_W)$ is isomorphic to $\tau_0(C^\cdot(\Phi, \boldsymbol{C}) \otimes_{\boldsymbol{C}} \boldsymbol{C}(P)) = \tilde{C}^\cdot(\Phi, \boldsymbol{C}) \otimes_{\boldsymbol{C}} \boldsymbol{C}(P)$. q.e.d.

3. The dualizing complex of the cusp singularity. First assume that the group Γ is contained in $\mathrm{SL}(N)$. Since $\omega_W \simeq O_W(-Y)$ and $R\pi_* O_W(-Y) = \mathfrak{m}_P$, we have an isomorphism by the relative duality theorem [RRV],

$$R\pi_* R\mathcal{H}om_{O_W}(O_W(-Y), O_W(-Y)) \simeq R\mathcal{H}om_{O_V}(\mathfrak{m}_P, K_V^\cdot[-n]) ,$$

where $K_V^\cdot$ is the normalized dualizing complex for V defined in [RR] with non-zero terms only between degrees $-n$ and zero. Hence the degree shift $K_V^\cdot[-n]$ to the right by n has non-zero terms only between zero and n. The left hand side is clearly equal to $R\pi_* O_W$. Hence by taking $R\mathcal{H}om_{O_V}(\ , K_V^\cdot[-n])$ for the short exact sequence $0 \to \mathfrak{m}_P \to O_V \to C(P) \to 0$, we get a triangle

$$\text{**)} \qquad \begin{array}{ccc} & R\mathcal{H}om_{O_V}(\mathfrak{m}_P, K_V^\cdot[-n]) = R\pi_* O_W & \\ {\scriptstyle (+1)}\swarrow & & \nwarrow \\ C(P)[-n] = R\mathcal{H}om_{O_V}(C(P), K_V^\cdot[-n]) & \longrightarrow & K_V^\cdot[-n] , \end{array}$$

where we get the equality $R\mathcal{H}om_{O_V}(C(P), K_V^\cdot[-n]) = C(P)[-n]$ by [RD, Chap. V, Prop. 3.4] since the stalk $(K_V^\cdot)_P$ is the dualizing complex of the local ring $O_{V,P}$ [RR, Prop. 1] and $C(P)$ is the residue field.

Since $\Gamma \subset \mathrm{SL}(N)$, the $(n-1)$-dimensional manifold D/Γ is orientable. Hence we know $H^{n-1}(\Phi, C) \simeq C$, and there exists a non-trivial homomorphism $\varepsilon\colon C^{n-1}(\Phi, C) \to C$ such that the connection

$$\cdots \to 0 \to C \to C^0(\Phi, C) \to \cdots \to C^{n-1}(\Phi, C) \xrightarrow{\varepsilon} C \to 0 \to \cdots$$

with the cochain complex $\tilde{C}^\cdot(\Phi, C)$ is a complex. We denote this complex by $\bar{C}^\cdot(\Phi, C)$.

THEOREM 3.1. *If $\Gamma \subset \mathrm{SL}(N)$, then $\mathcal{H}^0(K_V^\cdot[-n])$ is isomorphic to O_V and the truncation $\tau_0(K_V^\cdot[-n])$ is isomorphic, in the derived category $D_c^+(V)$, to the complex $\bar{C}^\cdot(\Phi, C) \otimes_C C(P)$ of $C(P)$-vector spaces.*

PROOF. Since $\mathcal{H}^p(C(P)[-n]) = 0$ for $p \neq n$, we have an isomorphism $\mathcal{H}^0(K_V^\cdot[-n]) \simeq \pi_* O_W = O_V$ from the long exact sequence obtained from the triangle **). By Lemma 2.3, we have a triangle

$$\begin{array}{ccc} & \tilde{C}^\cdot(\Phi, C) \underset{C}{\otimes} C(P) & \\ {\scriptstyle (+1)\ \mu}\swarrow & & \nwarrow \\ C(P)[-n] & \longrightarrow & \tau_0(K_V^\cdot[-n]) . \end{array}$$

Hence we know $\tau_0(K_V^\cdot[-n])$ is isomorphic to the mapping cone of the homomorphism $\mu\colon \tilde{C}^\cdot(\Phi, C) \otimes_C C(P)[-1] \to C(P)[-n]$. However, every homomorphism $A^\cdot \to B^\cdot$ in the derived category $D_c^+(V)$ is represented by a homomorphism of the complexes by replacing $B^{\cdot}$ by its injective resolution. In this case, we need not replace $C(P)[-n]$ by its injective resolution since the complex $\tilde{C}^{\cdot}(\Phi, C) \otimes_C C(P)[-1]$ has no term for degrees greater than n. Since $O_{V,P}$ is a normal local ring of dimension greater than one, its depth is at least two. Hence we have $\mathscr{H}^{n-1}(K_V^{\cdot}[-n]) = \mathscr{H}^n(K_V^{\cdot}[-n]) = 0$ [RD, Chap. V, Cor. 6.5]. By the long exact sequence, we know the homomorphism $\mathscr{H}^{n-1}(\Phi, C) \otimes_C C(P) \to \mathscr{H}^n(C(P)[-n]) = C(P)$ is isomorphic. Thus we know μ is induced by the homomorphism $C^{n-1}(\Phi, C) \to C$ which kills the $(n-1)$-st cohomology. Then the mapping cone of μ is equal to $\bar{C}^{\cdot}(\Phi, C) \otimes_C C(P)$. q.e.d.

Now consider the case where the group Γ is not contained in SL (N). Let the notations Γ', W', Y', π', ρ_W and ρ_V have the same meaning as in §2. Let K' be the triangulation of D/Γ' induced by the same r.p.p. decomposition Σ. Since the natural map $D/\Gamma' \to D/\Gamma$ is an unramified double covering, we have the two to one map $\rho_K: K' \to K$. Let v'_{2i-1} and v'_{2i} be the vertices of K' with $\rho_K(v'_{2i-1}) = \rho_K(v'_{2i}) = v_i$ for each $i = 1, \cdots, s$. Then the associated abstract simplicial complex Φ' of K' consists of subsets of $\{1, \cdots, 2s\}$. We see the natural homomorphism $\alpha^p: A(Y)^p \to (\rho_W)_* A(Y')^p$ commutes with the coboundary maps, and the homomorphism $\alpha^{\cdot}: A(Y)^{\cdot} \to (\rho_W)_* A(Y')^{\cdot}$ of complexes is equivalent to the inclusion $O_Y \hookrightarrow (\rho_W)_* O_{Y'}$, in the derived category $D_c^+(W)$. Clearly, the direct image of $\alpha^{\cdot}$ by the contraction map $\pi: W \to V$ is equal to $g^{\cdot} \otimes 1_{C(P)}$, where $g^{\cdot}$ is the injective homomorphism $C^{\cdot}(\Phi, C) \to C^{\cdot}(\Phi', C)$ induced by the two to one map $\rho_\Phi: \Phi' \to \Phi$. Since $\Gamma \not\subset$ SL (N) we know D/Γ is a non-orientable $(n-1)$-dimensional topological manifold. Hence we know $H^{n-1}(\Phi, C) = 0$ and $d: C^{n-2}(\Phi, C) \to C^{n-1}(\Phi, C)$ is surjective. This implies that g^{n-1}: $C^{n-1}(\Phi, C) \to C^{n-1}(\Phi', C)$ has its image in

$$dC^{n-2}(\Phi', C) = \mathrm{Ker}\,[\varepsilon: C^{n-1}(\Phi', C) \to C]\,.$$

Thus we know $g^{\cdot}$ induces an injective homomorphism

$$\bar{g}; \tilde{C}^{\cdot}(\Phi, C) \to \bar{C}^{\cdot}(\Phi', C)\,.$$

LEMMA 3.2. *The direct image* $(\rho_W)_* O_{W'}(-Y')$ *is isomorphic to* $O_W(-Y) \oplus \omega_W$.

PROOF. Let $\iota: W' \to W'$ be the involution such that $W'/(\iota) = W$. Since ρ_W is an unramified morphism, we have $(\rho_W)^* \omega_W \simeq \omega_{W'} \simeq O_{W'}(-Y')$. Let ω' be the n-form on W' such that its pull-back on $\tilde{W}$ is $\tilde{\omega} = (dx_1/x_1) \wedge \cdots \wedge (dx_n/x_n)$ in §2, and let $O_{W'} \to \omega_{W'}$ be the injection with $1 \to \omega'$. Since we know $\iota^*(\omega') = -\omega'$, the diagram

$$\begin{array}{ccc} O_{W'} & \xrightarrow{-1} & \iota^* O_{W'} = O_{W'} \\ \downarrow & & \downarrow \\ \omega_{W'} & \xrightarrow{\iota^*} & \iota^* \omega_{W'} \end{array}$$

commutes. Hence we know $\omega_W \simeq L \otimes_{O_W} O_W(-Y)$, where L is the invertible sheaf on W given by the descent data $(\iota, O_{W'} \xrightarrow{-1} \iota^* O_{W'} = O_{W'})$. Since $(\rho_W)_* O_{W'} = O_W \oplus L$, we have $(\rho_W)_* O_{W'}(-Y') \simeq (\rho_W)_*(\rho_W)^* O_W(-Y) \simeq O_W(-Y) \oplus \omega_W$ by the projection formula. q.e.d.

LEMMA 3.3. *The O_V-modules $(\rho_V)_* O_{V'}/O_V$, $\pi_* \omega_W$ and $\mathscr{H}^0(K_V^{\cdot}[-n])$ are isomorphic, and the exact sequence $0 \to O_V \to (\rho_V)_* O_{V'} \to \pi_* \omega_W \to 0$ splits.*

PROOF. In view of Lemma 3.2, we have $(\rho_V)_* \mathfrak{m}_{P'} = (\rho_V \circ \pi')_* O_{W'}(-Y') = (\pi \circ \rho_W)_* O_{W'}(-Y') \simeq \pi_*(O_W(-Y) \oplus \omega_W) = \mathfrak{m}_P \oplus \pi_* \omega_W$. Since V' is a normal analytic space of dimension greater than one, the depth of the local ring $O_{V',P'}$ is at least two. By [LC, Th. 3.8 and Cor. 5.7], we know the O_V-module $(\rho_V)_* O_{V'}$ is also of depth at least two. Since P' is an isolated singularity, we know the double dual of $(\rho_V)_* \mathfrak{m}_{P'}$ is equal to $(\rho_V)_* O_{W'}$, where the dual of an O_V-module A is $\mathscr{H}om_{O_V}(A, O_V)$. Hence the double dual of $\mathfrak{m}_P \oplus \pi_* \omega_W$ is isomorphic to $(\rho_V)_* O_{V'}$. By comparing the colengths, we know $(\rho_V)_* O_{V'} \simeq O_V \oplus \pi_* \omega_W$ and the O_V-module $\pi_* \omega_W$ is reflexive, i.e., it is equal to its double dual. By the Grauert-Riemenschneider vanishing theorem and the relative duality theorem, we have $\pi_* \omega_W = \boldsymbol{R}\pi_* \omega_W = \boldsymbol{R}\mathscr{H}om_{O_V}(\boldsymbol{R}\pi_* O_W, K_V^{\cdot}[-n])$. Hence the natural homomorphism $O_V \to \boldsymbol{R}\pi_* O_W$ induces a homomorphism $\beta \colon \pi_* \omega_W \to \mathscr{H}^0(K_V^{\cdot}[-n])$. Clearly, β is isomorphic on $V \backslash \{P\}$. Since $\pi_* \omega_W$ is reflexive and $\mathscr{H}om_{O_V}(\boldsymbol{C}(P), \mathscr{H}^0(K_V^{\cdot}[-n])) = \mathscr{E}xt^0_{O_V}(\boldsymbol{C}(P), K_V^{\cdot}[-n]) = 0$ implies that $\mathscr{H}^0(K_V^{\cdot}[-n])$ has no torsion, we know β is isomorphic on V. q.e.d.

THEOREM 3.4. *If $\Gamma \not\subset \mathrm{SL}(N)$, then $\mathscr{H}^0(K_V^{\cdot}[-n])$ is isomorphic to $\pi_* \omega_W$, and the truncation $\tau_0(K_V^{\cdot}[-n])$ is equal to $(\bar{C}^{\cdot}(\Phi', \boldsymbol{C})/\tilde{C}^{\cdot}(\Phi, \boldsymbol{C})) \otimes_{\boldsymbol{C}} \boldsymbol{C}(P)$ in the derived category $D_c^+(V)$.*

PROOF. The first part is already proved in Lemma 3.3. For the second part, take $\boldsymbol{R}\mathscr{H}om_{O_V}(\ , K_V[-n])$ for the split exact sequence

$$({*}{*}{*}) \qquad 0 \to O_V \to (\rho_V)_* O_{V'} \to \pi_* \omega_W \to 0 \,.$$

Then we get a triangle

$$\begin{array}{ccc} & \boldsymbol{R}\mathscr{H}om_{O_V}(O_V, K_V^{\cdot}[-n]) = K_V^{\cdot}[-n] & \\ {\scriptstyle (+1)} \swarrow & & \nwarrow \\ \boldsymbol{R}\mathscr{H}om_{O_V}(\pi_* \omega_W, K_V^{\cdot}[-n]) & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{O_V}((\rho_V)_* O_{V'}, K_V^{\cdot}[-n]) \,. \end{array}$$

By the relative duality [RRV], we have $R\mathscr{H}om_{O_V}(\pi_*\omega_W, K_V^{\cdot}[-n]) = R\pi_* R\mathscr{H}om_{O_W}(\omega_W, \omega_W) = R\pi_* O_W$ and

$$R\mathscr{H}mo_{O_V}((\rho_V)_* O_{V'}, K_V^{\cdot}[-n]) = R(\rho_V)_* R\mathscr{H}om_{O_{V'}}(O_{V'}, K_{V'}^{\cdot}[-n]) = (\rho_V)_* K_{V'}^{\cdot}[-n] .$$

Since the exact sequence ***) splits, we know the sequence $0 \to \mathscr{H}^0(R\pi_* O_W) \to \mathscr{H}^0((\rho_V)_* K_{V'}^{\cdot}[-n]) \to \mathscr{H}^0(K_V^{\cdot}[-n]) \to 0$ is also a split exact sequence. In particular, we can apply Lemma 2.3 for the truncation below at the degree zero. We know $\tau_0(R\pi_* O_W) = \tilde{C}^{\cdot}(\Phi, C) \otimes_C C(P)$ and $\tau_0(K_{V'}^{\cdot}[-n]) = \bar{C}^{\cdot}(\Phi', C) \otimes_C C(P')$ by Proposition 2.4 and Theorem 3.1. We see the homomorphism $\tilde{C}^{\cdot}(\Phi, C) \otimes_C C(P) \to \bar{C}^{\cdot}(\Phi', C) \otimes_C C(P)$, which is compatible with the natural homomorphism $A(Y)^{\cdot} \to A(Y')^{\cdot}$, is equal to the injection $\bar{g}$. Hence we know $\tau_0(K_V^{\cdot}[-n])$ is equal to $(\bar{C}^{\cdot}(\Phi', C)/\tilde{C}^{\cdot}(\Phi, C)) \otimes_C C(P)$ in $D_c^+(V)$, since it is quasi-isomorphic to the mapping cone of $\bar{g}$.

q.e.d.

THEOREM 3.5. *Tsuchihashi's cusp singularity* Cusp(C, Γ) *is Buchsbaum.*

PROOF. By Schenzel [S], a local ring $(A, \mathfrak{m})$ of dimension d is Buchsbaum if and only if the truncation below $\tau_{-d}(\omega_A^{\cdot})$ of the normalized dualizing complex $\omega_A^{\cdot}$ is equal to a complex of $(A/\mathfrak{m})$-vector spaces, i.e., $\tau_0(\omega_A^{\cdot}[-d])$ is equal to a complex of $(A/\mathfrak{m})$-vector spaces, in the derived category $D_c^+(A)$. Since the stalk $(K_V^{\cdot})_P$ is the normalized dualizing complex of the local ring $O_{V,P}$ [RR, Prop. 1], this theorem follows from Theorems 3.1 and 3.4. q.e.d.

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