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STABILITY OF A MECHANICAL SYSTEM WITH UNBOUNDED DISSIPATIVE FORCES

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In this article we shall be concerned with a mechanical system described by the Lagrangian equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} - B(t, q)\dot{q} + G(t, q)\dot{q} , \tag{1}$$

with generalized coordinates $q \in R^n$ and generalized velocities $\dot{q} \in R^n$. Salvadori [5] gave sufficient conditions under which the equilibrium of (1) is asymptotically stable in the case where B and G are time-independent. Recently, Hatvani [2] gave the conditions of the (partial) asymptotic stability and instability for more general systems. To obtain a result of the asymptotic stability, he considered some familiar conditions and furthermore, the following:

(*) For any compact subset L of R^n,
$\gamma_L(t) := \sup\{\|G(t, q) - B(t, q)\| : q \in L\} \in F$,

where F is the set of all measurable functions $\xi(t) = \xi_1(t) + \xi_2(t)$, ξ_1, ξ_2: $[0, \infty) \to [0, \infty)$, such that ξ_1 is bounded on $[0, \infty)$ and $\int_0^\infty \xi_2(t)dt < \infty$. If $B(t, q) \equiv tE$ (E is the unit matrix in $R^{n\times n}$) and $G(t, q) \equiv 0$, however, the condition (*) does not hold. In this article, by employing the manner developed in [3] we shall overcome this difficulty for the dissipation B which is unbounded. That is, we shall show that the equilibrium $q = \dot{q} = 0$ of (1) is weakly uniformly asymptotically stable under some familiar conditions and the following; for any bounded continuous function $\psi(s)$ on $[0, \infty)$ there exist a sequence of positive numbers $\{s_n\}$ and a positive constant d, $s_{n+1} \geqq s_n + d$, such that $\operatorname{tr} B(s, \psi(s)) \not\equiv 0$ on $[s_n, s_n + d]$ for all n and that

$$\sum_{n=1}^{\infty}\left[\int_{s_n}^{s_n+d} \operatorname{tr} B(s, \psi(s))ds\right]^{-1} = \infty ,$$

where $\operatorname{tr} B(s, \psi(s))$ denotes the trace of $B(s, \psi(s))$. Thus, our result is applicable to a mechanical system with unbounded B satisfying $0 < \operatorname{tr} B \leqq Mt \cdot \log(1 + t) + N$, $t \geqq 0$, for some positive constants M and N. In this article, only the asymptotic stability is treated for the sake of simplicity; our theorem can be easily modified to obtain a result of the partial stability as in [2].

We denote by R^n the n-dimensional real Euclidean space and by $|x|$ the Euclidean norm of $x \in R^n$, and it is supposed that the elements of R^n are column vectors, and v^T denotes the transposed of $v \in R^n$. Furthermore, for any matrix $A = (a_{ij})$ in $R^{n\times n}$, define $\|A\| = \sup\{|Av|: v \in R^n$ with $|v| \leqq 1\}$ and $\operatorname{tr} A = \sum_{i=1}^n a_{ii}$.

Consider an ordinary differential equation

$$\dot{x} = f(t, x) \qquad (f(t, 0) \equiv 0) , \tag{2}$$

where $f: I \times R^n \to R^n$ is continuous, $I = [0, \infty)$. Denote by $x(t, t_0, x_0)$ a noncontinuable solution of (2) through (t_0, x_0) in $I \times R^n$.

The zero solution of (2) is said to be;

uniformly stable if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for all $t_0 \in I$ and $t \geqq t_0$, $|x_0| < \delta(\varepsilon)$ implies $|x(t, t_0, x_0)| < \varepsilon$;

weakly uniformly asymptotically stable if it is uniformly stable and there exists a $\delta_0 > 0$ such that for all $t_0 \in I$, $|x_0| < \delta_0$ implies $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$.

For a function $V: I \times D \to R$ continuous and locally Lipschitzian in x (D is an open subset of R^n), define the derivative of V with respect to (2) by

$$\dot{V}_{(2)}(t, x) = \limsup_{h \to 0+} [V(t + h, x + hf(t, x)) - V(t, x)]/h .$$

Throughout this paper we suppose the following conditions on the system (1):

(H1) $\Pi: q \to \Pi(q) \in R$ is the potential energy, which is a continuously differentiable function with $\Pi(0) = 0$, $(\partial\Pi/\partial q)(0) = 0$;

(H2) $T = T(q, \dot{q}) = \dot{q}^T A(q)\dot{q}/2$ is the kinetic energy where $A: q \to A(q) \in R^{n\times n}$ is a continuously differentiable symmetric matrix function with $A(0)$ positive definite;

(H3) $B: (t, q) \to B(t, q) \in R^{n\times n}$ is a symmetric positive semidefinite matrix of the dissipations, which is continuous and integrally complete, that is,

$$\dot{q}^T B(t, q)\dot{q} \geqq \beta(t)|\dot{q}|^2 , \qquad t \geqq 0 , \qquad q, \dot{q} \in R^n ,$$

where $\beta: I \to I$ is measurable and $\int_J \beta(t)dt = \infty$ on any set $J = \bigcup_{m=1}^\infty [\alpha_m, \beta_m]$ such that $\alpha_m < \beta_m < \alpha_{m+1}$, $\beta_m - \alpha_m \geqq \delta > 0$;

(H4) $G: (t, q) \to G(t, q) \in R^{n\times n}$ is an antisymmetric matrix of the gyroscopic coefficients ($G^T = -G$, where G^T denotes the transposed matrix of G), which is continuous and $\|G(t, q)\|$ is bounded for all $t \geqq 0$ whenever $|q|$ is bounded.

Clearly, $q = \dot{q} = 0$ is an equilibrium of the system (1). Furthermore, by (H2) we can choose an open neighborhood Ω of $0 \in R^n$ so that $A(q)^{-1}$ exists and is positive definite for all $q \in \Omega$ and $c_1|p| \leqq |A(q)^{-1}p| \leqq c_2|p|$, $q \in \Omega$, $p \in R^n$, for some positive numbers c_1 and c_2. Set $p = A(q)\dot{q} = (\partial T/\partial \dot{q})$ and $H(q, p) = p^T A(q)^{-1} p/2 + \Pi(q)$, $q \in \Omega$, $p \in R^n$. Then, the system (1) is transformed to the following Hamilton's equation

$$
\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p} \\
\dot{p} &= -\frac{\partial H}{\partial q} + (G - B)A(q)^{-1}p ,
\end{aligned}
\tag{3}
$$

where $q \in \Omega$, $p \in R^n$ (cf. [4, p. 362]). Then we have:

THEOREM. *In addition to* (H1) *through* (H4), *suppose the following* (H5), (H6) *and* (H7) *hold;*

(H5) *there exist strictly increasing continuous functions* $a, b: I \to I$ *with* $a(0) = 0$ *and* $b(0) = 0$ *such that for all* $q \in \Omega$ *we have*

$$a(|q|) \leqq \Pi(q) \leqq b(|q|) ;$$

(H6) *for every* $\alpha_1, \alpha_2 (0 < \alpha_1 < \alpha_2)$ *there exists an* $\eta > 0$ *such that*

$$|\mathrm{grad}\, \Pi(q)| \geqq \eta \qquad (\alpha_1 \leqq |q| \leqq \alpha_2,\ q \in \Omega) ;$$

(H7) *for any bounded continuous function* $\psi(s)$ *on* I *there exist a sequence of positive numbers* $\{s_n\}$ *and a positive constant* d, $s_{n+1} \geqq s_n + d$, *such that* $\mathrm{tr}\, B(s, \psi(s)) \not\equiv 0$ *on* $[s_n, s_n + d]$ *for all* n *and that*

$$\sum_{n=1}^{\infty} \left[\int_{s_n}^{s_n+d} \mathrm{tr}\, B(s, \psi(s)) ds \right]^{-1} = \infty .$$

Then the equilibrium $q = \dot{q} = 0$ *of* (1) *is weakly uniformly asymptotically stable.*

To prove this Theorem we need the following lemma, which is an extension of [3, Lemma 2] to a matrix valued function.

LEMMA. *Let* $\bar{B}(s)$ *be a symmetric positive semi-definite matrix valued function and* $u(s)$ *a vector valued function on* $[0, \infty)$, *respectively, such that* $\bar{B}(s)$ *satisfies the same condition as* (H7) *for* $B(s, \psi(s))$ *and that*

$$\int_0^{\infty} u(s)^T \bar{B}(s) u(s) ds < \infty . \tag{4}$$

Then there exist a constant $d_0 > 0$ *and a sequence* $\{t_n\}$, $t_{n+1} \geqq t_n + d_0$, *such that* $\int_{t_n}^{t_n+t} \bar{B}(s)u(s)ds \to 0$ *as* $n \to \infty$ *uniformly for* $t \in [0, d_0]$.

PROOF. We shall show that the assertion in lemma holds for a subsequence of $\{s_n\}$ and $d_0 = d$, where $\{s_n\}$ and d are the ones given in the same condition as (H7) on $\bar{B}(s)$. Indeed, suppose that this is not the case. Then there exist a $\delta > 0$, a positive integer k_0 and a sequence $\{v_k\}$, $0 \leqq v_k \leqq d$, such that for all $k \geqq k_0$ we have

$$\delta \leqq \left|\int_{s_k}^{s_k+v_k} \bar{B}(s)u(s)ds\right|^2 = \left|\int_{s_k}^{s_k+v_k} \bar{B}(s)^{1/2}\bar{B}(s)^{1/2}u(s)ds\right|^2 .$$

For each i, $1 \leqq i \leqq n$, denote by e_i the vector whose j-th component is 1 if $j = i$ and 0 if $j \neq i$, and set $b_i(s) = \bar{B}(s)^{1/2}e_i$. By the Schwarz inequality we have

$$\begin{aligned}\delta &\leqq \sum_{i=1}^{n} \left|\int_{s_k}^{s_k+v_k} b_i(s)^T\bar{B}(s)^{1/2}u(s)ds\right|^2 \\ &\leqq \sum_{i=1}^{n} \left[\int_{s_k}^{s_k+v_k} b_i(s)^T b_i(s)ds\right]\left[\int_{s_k}^{s_k+v_k} u(s)^T\bar{B}(s)u(s)ds\right] \\ &= \left[\int_{s_k}^{s_k+v_k} \left(\sum_{i=1}^{n} e_i^T\bar{B}(s)e_i\right)ds\right]\left[\int_{s_k}^{s_k+v_k} u(s)^T\bar{B}(s)u(s)ds\right] \\ &\leqq \left[\int_{s_k}^{s_k+d} \operatorname{tr}\bar{B}(s)ds\right]\left[\int_{s_k}^{s_k+d} u(s)^T\bar{B}(s)u(s)ds\right] .\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{k=k_0}^{\infty} \left[\int_{s_k}^{s_k+d} \operatorname{tr}\bar{B}(s)ds\right]^{-1} &\leqq \delta^{-1} \sum_{k=k_0}^{\infty} \left[\int_{s_k}^{s_k+d} u(s)^T\bar{B}(s)u(s)ds\right] \\ &\leqq \delta^{-1} \int_{s_{k_0}}^{\infty} u(s)^T\bar{B}(s)u(s)ds < \infty\end{aligned}$$

by (4). Consequently, $\sum_{k=1}^{\infty}\left[\int_{s_k}^{s_k+d} \operatorname{tr}\bar{B}(s)ds\right]^{-1} < \infty$, which is a contradiction. Hence, the assertion in lemma holds. q.e.d.

PROOF OF THEOREM. It suffices to show that the solution $q = p = 0$ of (3) is weakly uniformly asymptotically stable. Set $V(q, p) = H(q, p)$. Then, by (H2) and (H5) we have

$$(5) \qquad a(|q|) + m|p|^2 \leqq V(q, p) \leqq b(|q|) + M|p|^2$$

for some positive constants m and M whenever $q \in \Omega$, $p \in R^n$. Moreover, for any $q \in \Omega$ and $p \in R^n$ we have $\dot{V}_{(3)}(q, p) = (\partial H/\partial q)^T\dot{q} + (\partial H/\partial p)^T\dot{p} = (A(q)^{-1}p)^T \times (G - B)A(q)^{-1}p = -(A(q)^{-1}p)^T BA(q)^{-1}p \leqq -\beta(t)\,|A(q)^{-1}p\,|^2 \leqq -c_1^2\beta(t)|p|^2 \leqq 0$ by (H3), since G is antisymmetric. Consequently, the solution $q = p = 0$ of (3) is uniformly stable. Hence, we can choose a

$\delta_0 > 0$ so that $(q(t), p(t)) \in S \times S$, $t \geqq t_0$, whenever $|(q_0, p_0)| < \delta_0$, where $S: = \{q \in R^n: |q| \leqq r\}$, $S \subset \Omega$, for a constant $r > 0$, and $p(t): = p(t, t_0, q_0, p_0)$ and $q(t): = q(t, t_0, q_0, p_0)$. It remains only to show that $(q(t), p(t)) \to (0, 0)$ as $t \to \infty$. First, we shall show that $p(t) \to 0$ as $t \to \infty$. Suppose that this is not the case. Then there exist a constant $k > 0$ and a sequence $\{T_n\}$, $T_n \to \infty$ as $n \to \infty$, sucn that $|p(T_n)| \geqq k$ for all n. Consider a function $W(q, p)$ defined by $W(q, p) = p^T A(q)^{-1} p/2 = V(q, p) - \Pi(q)$, q, $p \in S$. Set $c = \inf\{W(q, p): |q| \leqq r, k \leqq |p| \leqq r\}$. Since $A(q)^{-1}$, $q \in \Omega$, is positive definite, we have $c > 0$. Furthermore, we have

$$(6) \qquad \frac{d}{dt} V(q(t), p(t)) \leqq -[A(q(t))^{-1} p(t)]^T B A(q(t))^{-1} p(t) \leqq -c_1^2 \beta(t) |p(t)|^2 \leqq 0 \,,$$

and consequently

$$(7) \qquad \int^{\infty} \beta(s) |p(s)|^2 ds < \infty \,.$$

Then, by (7) we can easily conclude that there exist an $\varepsilon > 0$, $\varepsilon < k$, and a sequence $\{\tau_n\}$, $T_{n-1} < \tau_n < T_n$ for all n, such that $\sup\{W(q, p): |q| \leqq r, |p| = \varepsilon\} < c/2$ and that $|p(\tau_n)| = \varepsilon$ and $\varepsilon \leqq |p(t)|$ on $[\tau_n, T_n]$ for all n. Now, we have $\dot{W}_{(3)}(q, p) = \dot{V}_{(3)}(q, p) - (\partial \Pi/\partial q)^T \dot{q} \leqq -(\partial \Pi/\partial q)^T A(q)^{-1} p \leqq N$, $q, p \in S$, for a constant N. Hence $c/2 \leqq W(q(T_n), p(T_n)) - W(q(\tau_n), p(\tau_n)) \leqq N(T_n - \tau_n)$ for all n, and consequently $\sum_{n=1}^{\infty} \left[\int_{\tau_n}^{T_n} \beta(s) ds\right] = \infty$ by (H3). On the other hand, $\sum_{n=1}^{\infty} \left[\int_{\tau_n}^{T_n} \beta(s) ds\right] \leqq \varepsilon^{-2} \int^{\infty} \beta(s) |p(s)|^2 ds < \infty$ by (7), a contradiction. Thus, $p(t) \to 0$ as $t \to \infty$. Next, we shall show that $q(t) \to 0$ as $t \to \infty$. By (6) we have

$$(8) \qquad \int^{\infty} u(s)^T \bar{B}(s) u(s) ds < \infty \,,$$

where $\bar{B}(s): = B(s, q(s))$ and $u(s): = A(q(s))^{-1} p(s)$. Applying Lemma to $\bar{B}(s)$ and $u(s)$, it follows from (H7) and (8) that there exist a positive constant d_0 and a sequence $\{t_n\}$, $t_{n+1} \geqq t_n + d_0$, such that

$$(9) \qquad \int_{t_n}^{t_n + t} \bar{B}(s) u(s) ds \to 0 \text{ as } n \to \infty \text{ uniformly for } t \in [0, d_0] \,.$$

Taking a subsequence if necessary, we may assume that $q(t_n) \to q_0$ as $n \to \infty$ for a point q_0. We shall show $q_0 = 0$. Consider the functions $p_n(t): = p(t_n + t)$ and $q_n(t): = q(t_n + t)$, $n = 1, 2, \cdots$, defined for $t \in [0, d_0]$. Since $|q_n(t)| \leqq r$ and $|\dot{q}_n(t)| \leqq |A(q_n(t))^{-1} p_n(t)| \leqq c_2 r$ on $[0, d_0]$, taking a subsequence if necessary, Ascoli's theorem implies that $q_n(t) \to \psi(t)$ as $n \to \infty$ uniformly on $[0, d_0]$ for some continuous function $\psi(t)$. Integrating the equation of $\dot{p}$ in (3) over $[t_n, t_n + t]$, $0 \leqq t \leqq d_0$, we have

$$
\begin{aligned}
p(t_n+t)-p(t_n) &= -\int_{t_n}^{t_n+t}\frac{\partial H}{\partial q}(q(s),\,p(s))ds \\
&\quad + \int_{t_n}^{t_n+t} G(s,\,q(s))u(s)ds - \int_{t_n}^{t_n+t}\bar{B}(s)u(s)ds\ .
\end{aligned}
\tag{10}
$$

Letting $n\to\infty$ in (10), by (9), (H4) and the fact that $p(t)\to 0$ as $t\to\infty$, we have $\int_0^t \operatorname{grad} \Pi(\psi(s))ds \equiv 0$ on $[0, d_0]$, that is, $\operatorname{grad} \Pi(\psi(t))\equiv 0$ and consequently $\psi(t)\equiv 0$ on $[0, d_0]$ by (H6). Thus, $q_0=\psi(0)=0$. Then, for $t\geqq t_n$ we have $a(|q(t)|)\leqq V(q(t), p(t))\leqq V(q(t_n), p(t_n))$ by (5) and (6). Thus, since $V(q(t_n), p(t_n))\to 0$ as $n\to\infty$, we conclude that $a(|q(t)|)\to 0$ as $t\to\infty$ and hence $q(t)\to 0$ as $t\to\infty$. q.e.d.

REMARK. In order to obtain a result similar to Theorem, Hatvani [2] imposed the condition (*) on (1), which is different from (H7). Under the condition (*), however, without applying Lemma we can directly deduce that $\int_0^t \operatorname{grad} \Pi(\psi(s))ds\equiv 0$ on $[0, d_0]$ from (10) in the proof of Theorem given above, that is, our argument is applicable also to this case. Furthermore, note that the condition (*) does not hold in the case where $G\equiv 0$ and $B=B(t, q)$ with $0<\operatorname{tr} B\leqq Mt\cdot\log(1+t)+N$, $t\geqq 0$, for some positive constants M and N. As easily checked, however, (H7) in Theorem is satisfied in this case. On the other hand, it should be noted that Artstein and Infante [1] obtained a solution of the second order scalar differential equation $\ddot{x}+t^{\alpha}\dot{x}+x=0$, which is an example of (1) with $B=t^{\alpha}$, not tending to 0 as $t\to\infty$ in the case $\alpha>1$.

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