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A REMARK ON THE RICCI CURVATURE OF ALGEBRAIC SURFACES OF GENERAL TYPE

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Introduction and Preliminaries. In Aubin [1], it was shown that if M is a compact complex manifold with negative first Chern class, then there is a unique Einstein-Kaehler metric on M. On the other hand, manifolds with negative first Chern class belong to the class of algebraic manifolds of general type. By the Kodaira embedding theorem, the negativity of the first Chern class is equivalent to the ampleness of the canonical bundle K, i.e., $|mK|$ gives a projective embedding for m large. Now let M be a projective manifold of dimension n. M is said to be of general type if the plulicanonical bundles have sufficiently many sections in the sense that the dimension of the image under the rational map given by the linear system $|mK|$ for m large is equal to the dimension of M, or equivalently

$$\limsup_{n\to\infty} \dim H^0(M, \mathcal{O}(mK))/m^n > 0 .$$

The aim of this note is to extend Aubin's theorem to the case of general type in dimension two. As an application, we give a differential geometric proof of the Miyaoka inequality: $3c_2 \geqq c_1^2$, for surfaces of general type. Our proof implies that if M is a surface of general type whose canonical bundle is not ample, then the strict inequality $3c_2 > c_1^2$ holds. Hence an algebraic surface of general type M is covered by the ball in $\boldsymbol{C}^2$ holomorphically if and only if the equality $3c_2(M) = c_1(M)^2$ holds. Miyaoka also proved this result in [9] by showing that there are no rational curves in M if $3c_2(M) = c_1(M)^2$ holds, using algebro-geometric methods.

In our proof, the following observation due to Kodaira is essential: "If M is a minimal surface of general type, then the canonical bundle K is ample if and only if there are no (-2)-curves", where a (-2)-curve means a non-singular rational curve with self-intersection number -2. Now let $\mathscr{E}$ be the union of all (-2)-curves in M. $\mathscr{E}$ is characterized as the set of all irreducible curves in M which do not meet the canonical divisor. Hence $\mathscr{E}$ is an obstruction to the existence of a smooth Einstein-Kaehler metric on M. So the "Einstein-Kaehler metric" we will show the existence of is singular along $\mathscr{E}$. We see an example of this phenomenon at the end of this introduction. This example will become a model for our later arguments. To make the later arguments easy to understand, we give some examples of algebraic surfaces of general type.

(1) Every non-singular hypersurface in $\boldsymbol{P}^3$ of degree greater than or equal to 5 has ample canonical bundle. They are simply connected and have Einstein-Kaehler metric with negative Ricci curvature. [10].

(2) Every compact Kaehler surface of negative sectional curvature has ample canonical bundle. Its fundamental group is infinite. One of the examples is a compact quotient of the ball with the Bergman metric.

(3) Next, we give an example of a surface of general type whose canonical bundle is not ample. Let M be a surface in $\boldsymbol{P}^3$ given by the equation $\sum_{i=0}^{3} z_i^5 = 0$. Let $g: M \to M$ be an automorphism defined by $g(z_0: z_1: z_2: z_3) = (z_0: \varepsilon z_1: \varepsilon^4 z_2: z_3)$, where $\varepsilon = \exp(2\pi i/5)$. Then the group G generated by g is $\boldsymbol{Z}/5\boldsymbol{Z}$. The quotient variety $N = M/G$ has 5 singularities of type A_4 (the minimal resolution of which is a chain of four (-2)-curves) corresponding to $(1: 0: 0: -\varepsilon^k)$, $0 \leqq k \leqq 4$. Let $\tilde{N}$ be the desingularization of N. Then $\tilde{N}$ is a minimal surface of general type with $\mathscr{E} = \sum_{i=0}^{4} \mathscr{E}_i$, $\mathscr{E} \simeq A_4$. [Proof: Let Ω be a volume form on M which is invariant under the action of G whose Ricci form satisfies the following properties; it is non-positive everywhere on M and vanishes in a small neighborhood of each fixed point of G. By the arguments in the proof of the Lemma 1 of Section 1, Ω projects down to be a smooth volume form on $\tilde{N}$ with non-positive Ricci curvature. Hence the first Chern class of $\tilde{N}$ is represented by a nonpositive and somewhere negative (1, 1)-form. By the adjunction formula: $-c_1(M)\cdot C + C^2 = 2\pi(C) - 2$, $\tilde{N}$ has no nonsingular rational curves of self-intersection number greater than or equal to -1. Then Kodaira's classification of surfaces tells us that $\tilde{N}$ must be a minimal surface of general type.] By [1], M possesses an Einstein-Kaehler metric which is invariant under automorphisms. Hence it projects to an "Einstein-Kaehler metric" $\tilde{\omega}$ on $\tilde{N}$ singular along $\mathscr{E}$. The singularity of $\tilde{\omega}$ along $\mathscr{E}$ looks like as follows. Pick a connected component A_4 of $\mathscr{E}$. Consider a complex manifold M_4 which is covered by five complex planes $(U_i = \boldsymbol{C}^2; u_i, v_i)$, $0 \leqq i \leqq 4$, with transition rules

$$\begin{aligned}
&\text{on } U_0 \cap U_1 = \{u_0 \neq 0\}, \quad u_1 = u_0^{-1},\ v_1 = v_0 u_0^2,\\
&\text{on } U_1 \cap U_2 = \{v_1 \neq 0\}, \quad u_2 = u_1 v_1^2,\ v_2 = v_1^{-1},\\
&\text{on } U_2 \cap U_3 = \{u_2 \neq 0\}, \quad u_3 = u_2^{-1},\ v_3 = v_2 u_2^2,\\
&\text{on } U_3 \cap U_4 = \{v_3 \neq 0\}, \quad u_4 = u_3 v_3^2,\ v_4 = v_3^{-1}.
\end{aligned}$$

The closure of coordinate lines of U_i's in M_4 forms the configuration A_4. By Brieskorn [3], A_4 determines the complex structure around it. Hence there is a positive number ε such that the neighborhood of A_4 defined by

$$|v_0|^{2/5} + |u_0|^2|v_0|^{8/5} = |v_1|^{2/5}|u_1|^{4/5} + |v_1|^{8/5}|u_1|^{6/5} = \cdots < \varepsilon$$

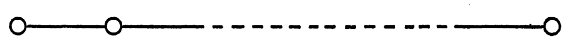
is valid as local coordinate neighborhoods with coordinates (u_i, v_i)'s around A_4 in $\tilde{N}$. The quotient map followed by the blowing up, $M \to \tilde{N}$, can be expressed as $\lambda^5 = v_0$, $\lambda\mu = u_0 v_0$, $\mu^5 = u_0^5 v_0^4$, in a neighborhood of A_4, where (u_0, v_0) is as above and $(\lambda, \mu) = (z_1/z_0, z_2/z_0)$. Hence

$$\begin{aligned} d\lambda \wedge d\bar{\lambda} + d\mu \wedge d\bar{\mu} &= |v_0|^{8/5} du_0 \wedge d\bar{u}_0 + (4/5)\bar{u}_0|v_0|^{-2/5} v_0 du_0 \wedge d\bar{v}_0 \\ &\quad + (4/5)u_0|v_0|^{-2/5}\bar{v}_0 dv_0 \wedge d\bar{u}_0 + (1/25)(|v_0|^{-8/5} + 16|v_0|^{-2/5}|u_0|^2) dv_0 \wedge d\bar{v}_0 , \end{aligned}$$

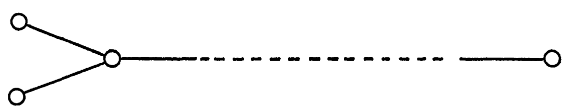
which is equivalent to $\tilde{\omega}$ near A_4.

1. $\mathscr{E}$-volume form with negative Ricci curvature. In this section, M is a minimal surface of general type and $\mathscr{E}$ the union of (-2)-curves in M. According to the Hodge index theorem (Note that $c_1^2 > 0$), and the classification of Cartan matrices, the connected components of $\mathscr{E}$ are divided into the following types:

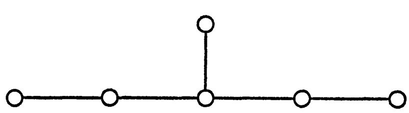
(1) A_n



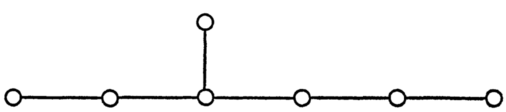
(2) D_n $(n \geqq 4)$



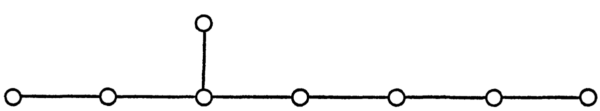
(3) E_6



(4) E_7



(5) E_8



where $\circ$ denotes a (-2)-curve and $\circ$—$\circ$ means that two (-2)-curves meet transversely at one point. Let $\mathscr{E} = \sum_\nu \mathscr{E}_\nu$ be the decomposition of $\mathscr{E}$ into connected components. Each of the above configurations (1)-(5) determines the complex structure around it [3]. The singularities produced by collapsing these are called the rational double points. By [3], each rational double point is realized as a quotient singularity B/Γ, where B is a small ball centered at origin in C^2 and Γ is a finite group consisting of elliptic automorphisms of the unit ball fixing the origin. Hence there exists a neighborhood U_ν of $\mathscr{E}_\nu$ such that $U_\nu - \mathscr{E}_\nu = B - \{0\}/\Gamma_\nu$. Since each Γ_ν is a subgroup of $U(2)$, the function $1 - \|Z\|^2$ on B projects down to a bounded smooth function on $U_\nu - \mathscr{E}_\nu$, which is denoted by h_ν. Then $\pi_\nu^* h_\nu = 1 - \|Z\|^2$ on B. Note that $-\log(1 - \|Z\|^2)$ is the Kaehler potential of the Bergman metric of the unit ball B^2 in C^2. Now let U_ν's be chosen so that $\bar{U}_\nu \cap \bar{U}_\mu = \phi$ if $\nu \neq \mu$. Let ϕ_ν be a smooth function on M such that $\phi_\nu | M - U_\nu \equiv 0$ and $\phi_\nu \equiv 1$ in some neighborhood of $\mathscr{E}_\nu$. Set $\tilde{h}_\nu = \phi_\nu h_\nu$, which is a smooth function on $M - \mathscr{E}_\nu$. Finally we set $\tilde{h}(z) = \max\{1/2, \sum_\nu \tilde{h}_\nu(z)\} \geqq 1/2$. By smoothing $\tilde{h}$ in a suitable manner, we obtain a positive bounded function $h \geqq 1/3$ which is equal to h_ν in some neighborhood of $\mathscr{E}_\nu$ for each ν.

We use the following.

PROPOSITION 1. *Let M be a minimal surface of general type. Then $c_1(M)$ is represented by a real closed $(1, 1)$ form γ with the following properties*:

(1) *γ is positive definite on $M - \mathscr{E}$, and $\gamma(Z, JZ) = 0$ for all Z tangent to $\mathscr{E}$,*

(2) *each $\pi_\nu^* \gamma | U_\nu$ has a smooth Kaehler potential in B.*

PROOF. Let $N + 1 = \dim H^0(M, \mathcal{O}(mK))$. By Kodaira [6], the linear system $|mK|$ has no base points for m large. The plulicanonical map Φ_{mK} has the following property: it is holomorphic and

$$\begin{tikzcd}
M \arrow[rr, "\Phi_{mK}"] & & \mathbf{P}^N \\
U_\nu \arrow[u, phantom, sloped, "\subset"] \arrow[d, "\text{contract } \mathscr{E}_\nu"'] \arrow[drr] & & \\
B/\Gamma_\nu \arrow[rr, "\sim"] & & \Phi_{mK}(U_\nu) \arrow[uu, phantom, sloped, "\subset"] \\
B \arrow[u, "\pi_\nu"']
\end{tikzcd}$$

$$\Phi_{mK}{}^{-1}(\Phi_{mK}(z)) = \begin{cases} Z & \text{if} \quad z \in M - \mathscr{E} \\ \mathscr{E}_\nu & \text{if} \quad z \in \mathscr{E}_\nu \subset \mathscr{E}\,, \end{cases}$$

moreover, $\Phi_{mK} | M - \mathscr{E} \colon M - \mathscr{E} \to \Phi_{mK}(M - \mathscr{E})$ is a biholomorphic mapping. Now define $\gamma = (1/m)\Phi_{mK}{}^*$ (the Fubini-study form on $\boldsymbol{P}^N$) then γ has the required properties, where the Fubini-Study form on $\boldsymbol{P}^N$ is $(i/2\pi)\partial\bar{\partial}\log \|Z\|^2$, which represents the Chern form of the hyperplane bundle over $\boldsymbol{P}^N$.

q.e.d.

Combining Proposition 1 with the above observation, we obtain the main result of this section.

PROPOSITION 2. *There exists a smooth volume element Ω on M and a positive number p such that the volume form $\Psi = h^p\Omega$ (which is continuous on M and smooth outside of $\mathscr{E}$) has the following properties.*

(1) $\omega = -\mathrm{Ric}(\Psi)$ *is a Kaehler metric on $M - \mathscr{E}$, and $\pi_\nu^*(\omega | U_\nu)$ is uniquely extended to a smooth form in B.*

(2) $f = \log\{\Psi/(-\mathrm{Ric}\,\Psi)^2\}$ *is a bounded continuous function on M which is smooth outside of $\mathscr{E}$ and $\pi_\nu^*(f | U_\nu)$ is uniquely extended to a smooth function in B for each ν.* (*We call this Ψ an $\mathscr{E}$-volume form.*)

For the proof of Proposition 2, we need some facts.

FACT 1 (Brieskorn [3]). Each rational double point is expressed as B/Γ, where B is a small ball centered at the origin of $\boldsymbol{C}^2$ and Γ is a finite subgroup of $SU(2)$ as follows.

(1) type A_n: $\Gamma = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi^n \end{pmatrix} \right\rangle_{\text{gen.}}$ where ξ is a primitive $(n+1)$-th root of unity,

(2) type D_n: $\Gamma =$ the binary dihedral group, $\tilde{\mathfrak{D}}_{2(n-2)}$,

(3) type E_6: $\Gamma =$ the binary tetrahedral group, $\tilde{\mathfrak{A}}_4$,

(4) type E_7: $\Gamma =$ the binary octahedral group, $\tilde{\mathfrak{S}}_4$,

(5) type E_8: $\Gamma =$ the binary icosahedral group, $\tilde{\mathfrak{A}}_5$.

DEFINITION. Let M be an n-dimensional complex manifold and ω a holomorphic n-form. We say that $\omega \in L^2(M)$ (square-integrable) if $\int_M \omega \wedge \bar{\omega}$ is finite.

FACT 2 (See, for example, Laufer [7]). Let A be a codimension 1 analytic subset in an n-dimensional complex manifold M and ω a holomorphic n-form defined in a deleted neighborhood $U - A$ of A. If $\omega \in L^2(U - A)$ then ω can be extended across A to be a holomorphic n-form on M.

LEMMA 1. *Let $\pi \colon (\tilde{X}, A) \to (X, x)$ be the minimal resolution of a* *rational double point. Then there exists a neighborhood U of x such that the canonical bundle of $\tilde{X}$ is trivial in $\pi^{-1}U$.*

PROOF. There is a neighborhood U of x such that $U = B/\Gamma$ as in Fact 1. Let (λ, μ) be the standard coordinate of C^2. Then $d\lambda \wedge d\mu$ is invariant under the action of $SU(2) = \left\{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}; |\alpha|^2 + |\beta|^2 = 1\right\}$, because $d(\alpha\lambda + \beta\mu) \wedge d(-\bar{\beta}\lambda + \bar{\alpha}\mu) = (|\alpha|^2 + |\beta|^2)(d\lambda \wedge d\mu) = d\lambda \wedge d\mu$. Hence if p is the quotient map $B \to U$, $\tilde{\omega} = p^*(d\lambda \wedge d\mu)$ is a non-vanishing holomorphic 2-form on $U - \{x\}$ and is square-integrable. By Fact 2, it is extended to a holomorphic 2-form ω on $\pi^{-1}U \supset A$. The zero divisor of ω is written as $\sum_i \lambda_i A_i$, where λ_i's are non-negative integers and A_i's are irreducible components of A. By the adjunction formula and $A_j^2 = -2$, $K \cdot A_j = 0$ for all j. On the other hand, the intersection matrix $(A_j \cdot A_k)$ is negative definite. Hence $0 \geqq \sum \lambda_j \lambda_k A_j \cdot A_k = \sum \lambda_j K \cdot A_j = 0$, and all λ_j's must be zero. q.e.d.

PROOF OF PROPOSITION 2. We take a volume form Ω on M such that $\gamma = -\mathrm{Ric}\,\Omega$ satisfies (1) and (2) of Proposition 1. Since $h = h_\nu$ in a neighborhood U_ν of $\mathscr{E}_\nu$, there is a positive number p such that the minus of Ric of $\Psi = h^p\Omega$ defines a Kaehler metric on $M - \mathscr{E}$. For the proof of (2), it suffices to verify that $\pi_\nu^*\Omega$ is smooth and non-vanishing in B. By the proof of Lemma 1, there is a non-vanishing holomorphic 2-form η_ν defined in U_ν with $\pi_\nu^*\eta_\nu = d\lambda \wedge d\mu$. If we set $\Omega = g \cdot \eta \wedge \bar{\eta}$, $\pi_\nu^*\Omega = (g \circ \pi_\nu) \cdot (d\lambda \wedge d\mu) \wedge \overline{(d\lambda \wedge d\mu)}$. Now take a smooth Kaehler potential g_ν of $\pi_\nu^*(\gamma | U_\nu)$ in B. Since $\partial\bar{\partial} g_\nu = \partial\bar{\partial} \log (g \circ \pi_\nu)$, $g \circ \pi_\nu$ is also smooth in B. q.e.d.

2. Equation $\Delta_\omega u - u = v$. In this section, we consider the Kaehler manifold $(M - \mathscr{E}, -\mathrm{Ric}\,\Psi)$ introduced in Proposition 2. Set $\omega = -\mathrm{Ric}\,\Psi$, and Δ_ω the Laplacian with respect to ω. In this note, the sign convention is $\Delta = \sum g^{i\bar{j}} \partial^2/\partial z_i \partial \bar{z}_j$. In our proof of the existence of an Einstein-Kaehler metric in some sense, we need to solve the equation $\Delta_\omega u - u = v$ in a suitable function space.

Now we define a function space fitting our purpose. Let $(M, \mathscr{E})$ be as in Section 1.

DEFINITION. A continuous function f on M is said to be $\mathscr{E} - C^{k,\alpha}$ if and only if f is $C^{k,\alpha}$ outside of $\mathscr{E}$ and $\pi_\nu^*(f | U_\nu - \mathscr{E}_\nu)$ is extendable to be $C^{k,\alpha}$ in B, where k is a non-negative integer and $0 < \alpha < 1$.

EXAMPLE. If $\mathscr{E}$ consists of only one (-2)-curve, the corresponding singularity is B/Γ, $\Gamma = \{I, -I\}$, where I is the $(2, 2)$ identity matrix. There is a neighborhood of this (-2)-curve isomorphic to a neighborhood of the zero section of $T^*\boldsymbol{P}^1$. Cover $T^*\boldsymbol{P}^1$ by two complex planes (u, v) and (u', v') such that $u' = u^{-1}$, $v' = u^2 v$. Then π_ν is written as $\lambda^2 = v = u'^2 v'$, $\lambda\mu = uv = u'v'$, $\mu^2 = u^2 v = v'$. The function $f(u, v) = |v|(1 + |u|^2) = |v'|(1+|u'|^2)$ is continuous in U but not $C^{2,\alpha}$ near $\mathscr{E}$. $\pi_\nu^* f(\lambda, \mu) = f(\lambda_2, \mu/\lambda) = |\lambda|^2 + |\mu|^2$ is a C^∞ function in B, i.e., f is $\mathscr{E} - C^\infty$.

Let $M = \bigcup_\nu U_\nu \cup \bigcup_\alpha U_\alpha$ be a finite open covering of M such that U_ν's are as in Section 1 and $\overline{\bigcup U_\alpha} \cap \mathscr{E} = \varnothing$.

DEFINITION. Let f be $\mathscr{E} - C^{k,\alpha}$. The $\mathscr{E} - C^{k,\alpha}$ norm of f is defined by

$$\|f\|_{\mathscr{E},k,\alpha} = \max(A, B)\,,$$

$$\begin{aligned} A = \sup_\alpha \Big\{ & \sup_{z \in U_\alpha} \sum_{|p|+|q| \leq k} |\partial^{|p|+|q|} f(z)/\partial z_\alpha{}^p \partial \bar{z}_\alpha{}^q| \\ & + \sup_{\substack{z, z' \in U_\alpha \\ z \neq z'}} \sum_{|p|+|q| = k} |z - z'|^{-\alpha} |\partial^k f(z)/\partial z_\alpha{}^p \partial \bar{z}_\alpha{}^q - \partial^k f(z')/\partial z_\alpha{}^p \partial \bar{z}_\alpha{}^q| \Big\}\,, \end{aligned}$$

$$\begin{aligned} B = \sup_\nu \Big\{ & \sup_{\zeta \in B} \sum_{|p|+|q| \leq k} |\partial^{|p|+|q|} \pi_\nu^* f(\zeta)/\partial \zeta^p \partial \bar{\zeta}^q| \\ & + \sup_{\substack{\zeta, \zeta' \in B \\ \zeta \neq \zeta'}} \sum_{|p|+|q| = k} |\zeta - \zeta'|^{-\alpha} |\partial^k \pi_\nu^* f(\zeta)/\partial \zeta^p \partial \bar{\zeta}^q - \partial^k \pi_\nu^* f(\zeta')/\partial \zeta^p \partial \bar{\zeta}^q| \Big\}\,. \end{aligned}$$

DEFINITION. $\mathscr{E} - C^{k,\alpha}$ is the set of all $\mathscr{E} - C^{k,\alpha}$ functions on M. It is a Banach space with respect to the norm $\|\cdot\|_{\mathscr{E},k,\alpha}$.

This definition of $\mathscr{E} - C^{k,\alpha}$ is independent of the choice of $\{U_\nu\}$ and $\{U_\alpha\}$. The norm arising from a different choice of $\{U_\nu\}$ and $\{U_\alpha\}$ is equivalent to the old one.

DEFINITION. A singular Riemannian metric g on M is $\mathscr{E} - C^{k,\alpha}$ Riemannian metric on M if and only if g is positive and $C^{k,\alpha}$ outside of $\mathscr{E}$, $\pi_\nu^*(g|U_\nu - \mathscr{E}_\nu)$ is extendable to be positive and $C^{k,\alpha}$ in B.

Let g be an $\mathscr{E} - C^{k,\alpha}$ Riemannian metric on M, with $k \geqq 1$. We consider the equation

$$\Delta_g u - b(x)u = f(x)\,, \tag{1}$$

where Δ_g is the Laplacian of g, and we suppose that $b(x)$ and $f(x)$ are $\mathscr{E} - C^{k,\alpha}$.

PROPOSITION 3. *If $b(x) > 0$, equation* (1) *has a unique solution belonging to $\mathscr{E} - C^{k+1,\alpha}$. If in addition g is $\mathscr{E} - C^{k,\alpha}$ Kaehler, the solution belongs to $\mathscr{E} - C^{k+2,\alpha}$.*

PROOF. This can be proved in the almost same way as in the Theorem 4.18 of [2]. The outline is as follows. We use the direct method in the calculous of variations to produce a weak solution. We define a Hilbert space $\mathscr{E} - H_1^2$ with respect to the fixed $\mathscr{E}$-Riemannian metric g as the completion of $\mathscr{E}-C^1$ with respect to the norm $\|f\|_{\mathscr{E}-H_1^2} = \int_M |f|^2 dv_g + \int_M |df|^2 dv_g$. Firstly we assume that g, b, f are $\mathscr{E}-C^\infty$. Consider the functional $J(\phi) = \int_M (|d\phi|^2 + b\phi^2 + 2f\phi) dv_g$ for $\phi \in \mathscr{E}-H_1^2$. By the argument in p. 104 of [2], there is an element $\bar{\phi}$ of $\mathscr{E}-H_1^2$ that minimizes J. $\bar{\phi}$ satisfies the Euler-Lagrange equation of J, i.e., $\int_M [(d\bar{\phi}, d\psi) + b\bar{\phi}\psi + f\psi] dv_g = 0$ for any $\psi \in \mathscr{E}-H_1^2$. Hence the regularity theorem (p. 85 of [2]) guarantees that $\bar{\phi}$ belongs to $\mathscr{E}-C^\infty$ because $\pi_\nu^* \bar{\phi}$ is a weak solution of the equation $\Delta_{\pi_\nu^* g} u + \pi_\nu^* b(x) u = \pi_\nu^* f(x)$ in B. On the other hand, $\int_M \Delta_g f = 0$, is valid for any $\mathscr{E}-C^{k,\alpha}$ Riemannian metric and $\mathscr{E}-C^2$ function f. It follows that $\bar{\phi}$ satisfies the equation (1). In the general case we approximate g, b, and f by $\mathscr{E}-C^\infty$ functions with respect to the $\mathscr{E}-C^{k,\alpha}$ norm. The remainder of the proof is the same as [2] except that we use U_α's and B (a small ball in $\boldsymbol{C}^2$) to apply the interior Schauder estimates. If $k \geqq 1$ and g is $\mathscr{E}-C^{k,\alpha}$ Kaehler, the coefficients in (1) with respect to holomorphic coordinates are $\mathscr{E}-C^{k,\alpha}$. By the above argument, u belongs to $\mathscr{E}-C^{k+2,\alpha}$. Now we can use the maximum principle to obtain the uniqueness of the solution of (1). q.e.d.

3. Existence of an Einstein-Kaehler metric.

In this section we prove the following.

THEOREM 1. *Let M be a minimal surface of general type with non-empty $\mathscr{E}$ where $\mathscr{E}$ denotes the union of all (-2)-curves on M. Then there is a unique $\mathscr{E}-C^\infty$ Einstein-Kaehler metric on M with negative Ricci curvature up to a constant multiple.*

PROOF. By Proposition 2, there is an $\mathscr{E}$-volume form ψ on M such that $\omega = -\mathrm{Ric}\,\psi$ is an $\mathscr{E}$-Kaehler metric. We deform ω into $\omega + i\partial\bar{\partial}u = \tilde{\omega}$ such that $\tilde{\omega}$ is $\mathscr{E}$-Einstein-Kaehler. So the procedure of the proof is almost the same as [11]. Consider the equation $(\omega + i\partial\bar{\partial}u)^2 = \exp(u+f)\omega^2$, where $\exp(f) = \psi/\omega^2$ which is $\mathscr{E}-C^\infty$ by Proposition 2. If u is a solution of this equation belonging to $\mathscr{E}-C^\infty$, then $\tilde{\omega}$ is an $\mathscr{E}-C^\infty$ Einstein-Kaehler metric. Hence Theorem 1 is the special case of the following

THEOREM 2. *Let M be as in Theorem 1. Let ω be an $\mathscr{E}-C^{k,\alpha}$ Kaehler metric on M with $k \geqq 5$. Then for any $\mathscr{E}-C^{k-2,\alpha}$ function f, there is a unique solution u to the equation $(\omega + i\partial\bar{\partial}u)^2 = \exp(u+f)\omega^2$, belonging to $\mathscr{E}-C^{k,\alpha}$.*

PROOF. Let $\Phi: \mathscr{E}-C^{k,\alpha} \to \mathscr{E}-C^{k-2,\alpha}$ be defined by $\Phi(u) = \log\{(\omega + i\partial\bar{\partial}u)^2/\omega^2\} - u$. To solve the equation

$$\Phi(u) = f \tag{2}$$

it suffices to prove that the subset A of $[0, 1]$ defined by $A = \{t \in [0, 1]; \Phi(u) = tf$ has a solution in $\mathscr{E} - C^{k,\alpha}\}$ is non-empty, open and closed. Clearly $0 \in A$. Openness comes from the inverse function theorem. In fact, the Fréchet derivative of Φ at u is a linear map of $C^{k,\alpha}$ into $C^{k-2,\alpha}$ given by $\Phi'(u)h = \Delta_{\tilde{\omega}}h - h$, which is an isomorphism by Proposition 3. Closedness is proved by C^0, C^2 and $C^{2,\alpha}$ a-priori estimates of the solution of (2), which goes as in [11] provided we lift everything up to B using $\pi_\nu: B \to U_\nu$ and represent everything in terms of the Euclidean coordinates (λ, μ) of C^2 in a neighborhood of $\mathscr{E}$. q.e.d.

4. Miyaoka-Yau inequality. In [4] Chen and Ogiue proved that if (M, ω) is an Einstein-Kaehler manifold of dimension n, then the following pointwise inequality: $(nc_1^2 - 2(n+1))c_2)\omega^{n-2} \leqq 0$ holds and the equality occurs if and only if (M, ω) has constant holomorphic sectional curvature. Yau [10] makes use of the existence of Einstein-Kaehler metric to prove that if M is a compact complex manifold with ample canonical bundle then $(-1)^n\{nc_1^n - 2(n+1)c_2c_1^{n-2}\} \leqq 0$ and the equality occurs if and only if M is covered by the ball in C^n holomorphically. On the other hand, Miyaoka [8] [9] proved that if M is an algebraic surface of general type then $3c_2 \geqq c_1^2$ holds and that if the equality happens the canonical bundle is ample. In this section we give a new proof of the Miyaoka inequality making use of our $\mathscr{E}$-Einstein-Kaehler metric. This is an extension of the arguments in [4] and [10] to the case of minimal surfaces of general type.

PROPOSITION 4. *Let M be a minimal algebraic surface of general type and $\mathscr{E}$ the union of all (-2)-curves in M (possibly empty). Let $\tilde{\omega}$ be the $\mathscr{E}$-Einstein-Kaehler metric in Theorem* 1. *If $\tilde{c}_i$ denotes the i-th Chern form of the Hermitian connection of $\tilde{\omega}$, the following equalities hold.*

$$\int_M \tilde{c}_1^2 = \int_M c_1^2 \tag{3}$$

$$\begin{aligned}\int_M \tilde{c}_2 = \int_M c_2 - [&\sum_n \{n(n+2)/(n+1)\} \times \#(\text{type } A_n) \\ &+ \sum_m \{(3/2) + (27/8) + (27/8) + (m-3)(3m^2 - 5m - 1)/4(m-2)\} \\ &\qquad\qquad \times \#(\text{type } D_m,\ m \geqq 4) \\ &+ \{(3/2) + (27/8) + (49/6) + (49/6)\} \times \#(\text{type } E_6) + \{(3/2) \\ &+ (27/8) + (49/6) + (231/16)\} \times \#(\text{type } E_7) + \{(3/2) + (27/8) \\ &+ (49/6) + 111/5)\} \times \#(\text{type } E_8)]\end{aligned} \tag{4}$$

PROOF. We use the same notations as those in Proposition 2. Since $\tilde{\omega} = \omega + i\partial\bar{\partial}u$, $\omega = -\mathrm{Ric}\, h^p\Omega = -\mathrm{Ric}\,\Omega + i\partial\bar{\partial}\log h^p$ and h, u are $\mathscr{E} - C^\infty$, the equality (3) follows from the Stokes' formula. To prove the equality (4), we need to describe how to resolve the rational double points.

The resolution of the rational double points [3] (Summary). Let $P = (p, q)$ be a regular polyhedron in $\boldsymbol{R}^3$, where (p, q) means that each face is p-gon and each vertex is the intersection of q faces. The posibilities of (p, q)'s are $(n, 2)$, $(3, 3)$, $(3, 4)$, $(4, 3)$, $(3, 5)$, $(5, 3)$. Each regular p-gonal face of P can be divided into $2p$ triangles by lines joining the center of the face to its vertices and mid-points of its edges. Pick a concentric sphere S and cut S by planes containing the center and these lines. Then we get a triangulation Δ of S by geodesic triangles. The polyhedral group $G(p, q)$ consists of rotations which preserves Δ and whose axis are lines joining the center ane vertices of Δ. If $P = (p, q)$, $G(p, q)$ consists of elements of order $p, q, 2$ (See Figure 1).

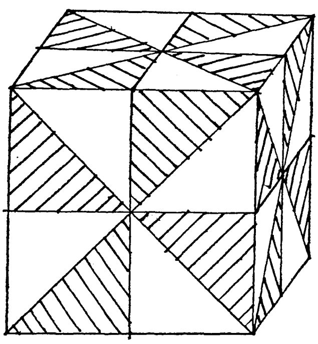


FIGURE 1

LEMMA 2. *There is an exact sequence*

$$1 \to Z_2 \to SU(2) \underset{\phi}{\to} SO(3) \to 1$$

where ϕ *is given by*

$$SU(2) \ni \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mapsto \text{“}S^2 \underset{s}{\to} P^1 \ni u \mapsto (\alpha u + \beta)/(-\bar{\beta}u + \bar{\alpha}) \in P^1 \underset{s}{\to} S^2\text{”}$$

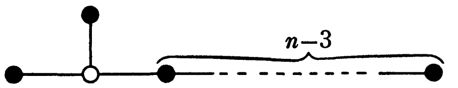
where s is the stereo-graphic projection. The lifting of polyhedral groups are binary polyhedral groups stated in Fact 1.

Let Γ be a binary polyhedral group and G the corresponding polyhedral group. The action of Γ on $\boldsymbol{C}^2$ induces the action of G on $E' = T^*\boldsymbol{P}^1$. The action of $\phi\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ is given by $E' \ni (u, v) \mapsto ((\alpha u + \beta)/(-\bar{\beta}u + \bar{\alpha}),$

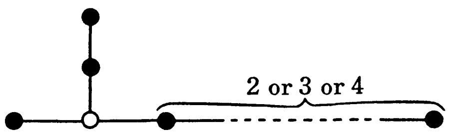
$v(-\bar{\beta}u+\bar{\alpha})^2)\in E'$ or $E'\ni(u',v')\mapsto((-\bar{\beta}+\bar{\alpha}u')/(\alpha+\beta u'),\ v'(\alpha+\beta u')^2)\in E'$ where $v'=vu^2$, $u'=u^{-1}$. If $(u,v)=(0,0)$ is fixed by $\phi\begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}$ then $\beta=0$ and the resulting action is $(u,v)\mapsto(\zeta u,\zeta^{-1}u)$ where $\zeta=\alpha/\bar{\alpha}$. From this expression, the quotient variety $E=E'/G$ has the three singularities of types

$$\begin{array}{ll} A_1,\ A_1,\ A_{n-3} & \text{if}\quad \Gamma=\tilde{\mathfrak{D}}_{2n}\ , \\ A_1,\ A_2,\ A_2 & \text{if}\quad \Gamma=\tilde{\mathfrak{A}}_4\ , \\ A_1,\ A_2,\ A_3 & \text{if}\quad \Gamma=\tilde{\mathfrak{S}}_4\ , \\ A_1,\ A_2,\ A_4 & \text{if}\quad \Gamma=\tilde{\mathfrak{A}}_5\ , \end{array}$$

corresponding to the order of the isotropy subgroups of the vertices of the triangulation Δ. Let C' be the zero section of E' and C the quotient curve C'/G, which is $\boldsymbol{P}^1$ by the Hurwitz formula. By [3], the exceptional divisor of the desingularization $\tilde{E}$ of E consists of (-2)-curves and looks like as follows:



, if $\Gamma=\tilde{\mathfrak{D}}_{2(n-2)}$



, if $\Gamma=\tilde{\mathfrak{A}}_4$ or $\tilde{\mathfrak{S}}_4$ or $\tilde{\mathfrak{A}}_5$,

where $\bullet$ denotes the component of the exceptional divisor of $\tilde{E}\to E$. Moreover, $\circ$ denotes the proper transform of C which is also a (-2)-curve.

LEMMA 3. *Let $\tilde{\omega}'$ be a Hermitian metric on $M-\mathscr{E}$ such that $\pi_\nu^*(\tilde{\omega}'|U_\nu-\mathscr{E}_\nu)$ can be extended to a smooth metric of B which is equal to the flat metric $|d\lambda|^2+|d\mu|^2$ in a neighborhood of the origin*; (*Such a metric exists by the Fact* 1). *Let $\tilde{c}_2{}'$ be the second Chern form of the Hermitian connection of $\tilde{\omega}'$. Then $\int_M\tilde{c}_2{}'=\int_M\tilde{c}_2$.*

PROOF. By pp. 400-406 of [5], there is an $\mathscr{E}-C^\infty$ 3-form $\tilde{\eta}$ such that $\tilde{c}_2{}'-\tilde{c}_2=d\tilde{\eta}$. By Stokes, $\int_{M-\cup_\nu U_\nu}\tilde{c}_2{}'-\int_{M-\cup_\nu U_\nu}\tilde{c}_2=-\sum_\nu\int_{\partial U_\nu}\tilde{\eta}$. Therefore we obtain the desired result by choosing U_ν smaller and smaller.

q.e.d.

Now we prove the assertion (4). We pick a smooth Hermitian metric

ω of M and denote by θ and Θ the connection form and the curvature form of the Hermitian connection of ω. Let $\tilde{\theta}$ be the connection form of $\tilde{\omega}'$ By pp. 400-406, [5], there is a smooth 3-form on $M - \mathscr{E}$, η such that $\tilde{c}_2{}' - c_2 = d\eta$, and since $\tilde{\omega}'$ is flat near $\mathscr{E}$, η is written as

$$\det\begin{pmatrix} i(\tilde{\theta} - \theta)_1^1/2\pi & i(\tilde{\theta} - \theta)_2^1/2\pi \\ i\Theta_1^2/4\pi & i\Theta_2^2/4\pi \end{pmatrix} + \det\begin{pmatrix} i\Theta_1^1/4\pi & i\Theta_2^1/4\pi \\ i(\tilde{\theta} - \theta)_1^2/2\pi & i(\tilde{\theta} - \theta)_2^2/2\pi \end{pmatrix}$$

in a small neighborhood of $\mathscr{E}$. Since $\int_M \tilde{c}_2 - \int_M c_2 = \int_M \tilde{c}_2{}' - \int_M c_2 = -\lim \sum_\nu \int_{\partial U_\nu} \eta$, the difference of Chern numbers is obtained by computing each term of the right hand side of this equality.

(1) The case of A_l: We cover the configuration A_l by $(l+1)$ coordinate neighborhoods (u_i, v_i) $(0 \leqq i \leqq l)$ with transition rules $u_1 = u_0^{-1}$, $v_1 = v_0 u_0^2$, $u_2 = u_1 v_1^2$, $v_2 = v_1^{-1}$, $\cdots$. The quotient map $\pi_\nu \colon B \to U_\nu$ is expressed as

$$\begin{aligned} \lambda^{l+1} &= u_0^0 v_0^1 = v_1 u_1^2 = u_2^2 v_2^3 = \cdots = u^k v^{k+1} = \cdots, \\ \lambda\mu &= u_0 v_0 = u_1 v_1 = u_2 v_2 = \cdots, \\ \mu^{l+1} &= u_0^{l+1} v_0^l = v_1^l u_1^{l-1} = \cdots = u^{l-k+1} v^{l-k} = \cdots. \end{aligned}$$

If we write the flat metric $|d\lambda|^2 + |d\mu|^2$ in terms of (u, v) as $h_{\bar{v}v} dv d\bar{v} + h_{\bar{v}u} du d\bar{v} + h_{\bar{u}v} dv d\bar{u} + h_{\bar{u}u} du d\bar{u}$, then

$$\begin{aligned} (l+1)^2 h_{\bar{v}v} &= |u|^{2k/(l+1)} |v|^{2(k+1)/(l+1)-2} (k+1)^2 \\ &\quad + |u|^{2(l-k+1)/(l+1)} |v|^{2(l-k)/(l+1)-2} (l-k)^2, \\ (l+1)^2 h_{\bar{v}u} &= \{|u|^{2k/(l+1)-2} |v|^{2(k+1)/(l+1)-2} k(k+1) \\ &\quad + |u|^{2(l-k+1)/(l+1)-2} |v|^{2(l-k)/(l+1)-2} (l-k+1)(l-k)\} v\bar{u}, \\ (l+1)^2 h_{\bar{u}u} &= |u|^{2k/(l+1)-2} |v|^{2(k+1)/(l+1)} k^2 \\ &\quad + |u|^{2(l-k+1)/l+1)-2} |v|^{2(l-k)/(l+1)} (l-k+1)^2. \end{aligned}$$

Hence the connection form is given by

$$\begin{aligned} (l+1)^2 \theta_v{}^v &= -k(l-k+1)(l-2k-1) du/u + (l-k)(k+1)(2k-l-1) dv/v, \\ (l+1)^2 \theta_u{}^v &= (l-k+1)k\{(2k-l+1) dv/u + (2k-l-1) v du/u^2\}, \\ (l+1)^2 \theta_v{}^u &= (l-k)(k+1)\{(l-2k+1) du/v + (l-2k-1) u dv/v^2\}, \\ (l+1)^2 \theta_u{}^u &= (l-k)(k+1)(l+1-2k) dv/v - k(l-k+1)(2k-l+1) du/u, \end{aligned}$$

where $\theta_j{}^i = \sum_k h^{i\bar{k}} \partial h_{\bar{k}j}$. Therefore the contribution of A_l to the integral over M of $\tilde{c}_{2'} - c_2$ is as follows:

$$-(l+1)^{-2}\sum_{k=0}^{l-1}\Big\{\int_{P^1}-(l-k)(k+1)(2k-l-1)(i/4\pi)\Theta_u{}^u{}_{u\bar u}du\wedge d\bar u$$
$$+\int_{P^1}-(l-k)(k+1)(l+1-2k)(i/4\pi)\Theta_u{}^v{}_{u\bar u}du\wedge d\bar u$$
$$-\int_{P^1}(l-k)(k+1)(l-2k+1)(i/4\pi)\Theta_u{}^v{}_{v\bar u}du\wedge d\bar u$$
$$+\int_{P^1}(l-k)(k+1)(l-2k-1)(i/4\pi)(u\partial\Theta_u{}^v{}_{u\bar u}/\partial v)du\wedge d\bar u\Big\}$$

On the other hand the following equalities are valid on the submanifold given by $v=0$: $\bar\partial(\theta_u{}^u{}_u du)=\Theta_u{}^u{}_{u\bar u}du\wedge d\bar u$, $\bar\partial(\theta_v{}^v{}_u du)=\Theta_v{}^v{}_{u\bar u}du\wedge d\bar u$, $\bar\partial(\theta_u{}^v{}_v du)=\Theta_u{}^v{}_{v\bar u}du\wedge d\bar u$, $\bar\partial\{u(\partial\theta_u{}^v{}_u/\partial v)du\}=u(\partial\Theta_u{}^v{}_{u\bar u}/\partial v)du\wedge d\bar u$.

LEMMA 4. *If $\theta_u{}^u{}_u du$ is considered to be a $(1, 0)$-form on $|u|<\infty$, $v=0$, then $\theta_u{}^u{}_u du+2du/u$ is smooth at $u'=0$, where $u'=1/u$. Hence it defines a connection of type $(1, 0)$ of TP^1 on the rational curve given by $v=0$.*

LEMMA 5. *If $\theta_u{}^v{}_v du$ is considered to be a $(1, 0)$-form on $|u|<\infty$, $v=0$, then $\theta_u{}^v{}_v du-2du/u$ is smooth at $u'=0$, where $u'=1/u$. Hence it defines a connection of type $(1, 0)$ of T^*P^1 on the rational curve given by $v=0$.*

LEMMA 6. *If $u(\partial\theta_u{}^v{}_u/\partial v)du$ is considered to be a $(1, 0)$ form on u, $v=0$, then $u(\partial\theta_u{}^v{}_u/\partial v)du+6du/u$ is smooth at $u'=0$, where $u'=1/u$. Hence it defines a connection of type $(1, 0)$ of $6H$ on the rational curve given by $v=0$, where H is the line bundle of degree 1 over P^1.*

PROOF OF LEMMAS. These Lemmas are shown easily by substituting $v=0$ (or $v'=vu^2=0$) in the transition rules of these connection forms under the base change $(\partial/\partial u, \partial/\partial v)\mapsto(\partial/\partial u', \partial/\partial v')$ with $u'=1/u$, $v'=vu^2$. q.e.d.

By Lemmas 4, 5, 6, the above integral equals

$$(1/2(l+1)^{-2}\sum_{k=0}^{l-1}\{(l-k)(k+1)(2k-l-1)(2)+(l-k)(k+1)(l+1-2k)(-2)$$
$$+(l-k)(k+1)(l-2k+1)(-2)-(l-k)(k+1)(l-2k+1)(6)\}$$
$$=-l(l+2)/(l+1)\,.$$

(2) The other cases: In this paragraph, we consider the exceptional divisor of the resolution $\tilde E\to E$. We may assume that this is of type A_e. We must rewrite the flat metric $\partial\bar\partial\{|\lambda|(1+|\mu|^2)$ in terms of (u, v) with $\lambda^{e+1}=u^k v^{k+1}$, $\mu^{e+1}=u^{e-k+1}v^{e-k}$ (See example in Section 2). The result is the same as that of (1) provided we replace l by $2e+1$.

Hence the connection form is given by the following

$$2(e+1)^2\theta_v^{\ v} = -(2e-k+2)k(e-k)du/u-(2e-k+1)(k+1)(e-k+1)dv/v\ ,$$
$$2(e+1)^2\theta_u^{\ v} = k(2e-k+2)\{-(e-k)dv/u-(e-k+1)vdu/u^2\}\ ,$$
$$2(e+1)^2\theta_v^{\ u} = (k+1)(2e-k+1)\{(e-k+1)du/v+(e-k)udv/v^2\}\ ,$$
$$2(e+1)^2\theta_u^{\ u} = (2e-k+1)(k+1)(e-k+1)dv/v+(e-k)(2e-k+2)du/u\ .$$

If $k=e$, then

$$\theta_v^{\ v} = (-1/2)dv/v\ ,\quad \theta_v^{\ u} = (1/2)du/v\ ,\quad \theta_u^{\ u} = (1/2)dv/v\ .$$

As in the paragraph (1) (case of A_l), the contribution of such A_e is

$$\begin{aligned}(-1/4)(e+1)^{-2}\sum_{k=0}^{e-1}\{&(2e-k+1)(k+1)(e-k+1)(2)\\ &+(2e-k+1)(k+1)(e-k+1)(-2)-(2e-k+1)(k+1)(e-k+1)(-2)\\ &+(2e-k+1)(k+1)(e-k)(6)\}\\ = -e(3e^2&+13e+11)/4(e+1)\ .\end{aligned}$$

This number is $-27/8$ if $e=1$, $-49/6$ if $e=2$, $-231/16$ if $e=3$, $-111/5$ if $e=4$. The contribution of the proper transform $\tilde{C}$ of C is $(-1/4)\{2-(-2)-(-2)\}=-3/2$.

(3) Conclusion: If we set $\delta=\int_M \tilde{c}_2-\int_M c_2$, then $\mathscr{E}$ contributes to make the value δ negative. More precisely,

$$\begin{aligned}\delta = -[&\sum_\nu l_\nu(l_\nu+2)/(l_\nu+1)+\sum_\mu\{(3/2)+(27/8)+(27/8)\\ &+(l_\mu-3)(3l_\mu^2-5l_\mu-1)/4(l_\mu-2)\}+\sum_6\{(3/2)+(27/8)+(49/6)\\ &+(49/6)\}+\sum_7\{(3/2)+(27/8)+(49/6)+(231/16)\}\\ &+\sum_8\{(2/3)+(27/8)+(49/6)+(111/5)\}]\ ,\end{aligned}$$

where $\sum_\nu$ means the summation over the connected components of $\mathscr{E}$ of type A_{l_ν}, $\sum_\mu$ over components of type D_{l_μ}, $\sum_6$ over components of type E_6, $\sum_7$ over components of type E_7, $\sum_8$ over components of type E_8. This completes the proof of Proposition 4. q.e.d.

THEOREM 3 (Miyaoka, Yau). *If M is a minimal algebraic surface of general type, then the inequality $3c_2 \geqq c_1^2$ holds. The equality occurs if and only if M is covered by the ball in $\boldsymbol{C}^2$ holomorphically.*

PROOF. $3c_2-c_1^2=3\tilde{c}_2-\tilde{c}_1^2-3\delta\geqq 0$, because δ is non-positive and negative if there is a (-2)-curve on M, and $3\tilde{c}_2-\tilde{c}_1^2\geqq 0$ by the arguments in [4] and [10]. q.e.d.

REMARK. One blowing up decreases c_1^2 by one and increases c_2 by one. Hence the inequality $3c_2 \geqq c_1^2$ is valid for all surfaces of general type.

REMARK. The equality case $3c_2 - c_1^2 = -3\delta$ in Theorem 3 occurs if and only if the universal covering of the smooth part of $\Phi_{mK}(M)$ for m large is the complex hyperbolic 2-ball minus the discrete set of points.

REFERENCES

[1] T. AUBIN, Équations du type Monge-Ampère sur les Variétés Kählériennes compactes, C. R. Acad. Sci. Paris, 283 (1976), 119–121.

[2] T. AUBIN, "Nonlinear Analysis on Manifolds, Monge-Ampère Equations", Springer, Grundlehren der Math., 252.

[3] E. BRIESKORN, Rationale Singularitäten Komplexer Flächen, Inventiones Math. 4 (1968), 336–358.

[4] B. Y. CHEN AND K. OGIUE, Some characterization of complex space forms in terms of Chern classes, Quarterly J. Math. 26 (1975), 459–464.

[5] P. A. GRIFFITHS AND J. HARRIS, "Principles of Algebraic Geometry", Wiley-Interscience Publication, 1978.

[6] K. KODAIRA, Pluricanonical systems on algebraic surfaces of general type, J. Math. Soc. Japan, 20 (1968), 170–192.

[7] H. LAUFER, On rational singularities, Amer. J. Math. 94 (1972), 597–608.

[8] Y. MIYAOKA, On the Chern numbers of surfaces of general type, Inventiones Math. 42 (1977), 225–237.

[9] Y. MIYAOKA, On algebraic surfaces with positive index, to appear.

[10] S. T. YAU, On Calabi's conjecture and some new results in algebraic geometry, Nat. Acad. Sci., U.S.A. 74 (1977), 1798–1799.

[11] S. T. YAU, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, Comm. Pure Appl. Math. 31 (1978), 339–411.

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