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A THEOREM OF BERNSTEIN TYPE FOR MINIMAL SURFACES IN R^4

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1. Introduction. Let $u: \boldsymbol{R}^2 \to \boldsymbol{R}$ be a function the graph of which is a minimal surface in $\boldsymbol{R}^3$. Then u is a linear function by the classical theorem of Bernstein.

On the other hand, the same conclusion does not hold in the case of codimension greater than one. The graph of an entire holomorphic function on $\boldsymbol{C}$ is a minimal surface of $\boldsymbol{R}^4 = \boldsymbol{C}^2$ which is not necessarily a plane.

Recently a generalization of Bernstein's theorem was proved.

THEOREM (do Carmo and Peng [1], Fischer-Colbrie and Schoen [3]). *Complete orientable stable minimal surfaces in $\boldsymbol{R}^3$ are planes.*

A minimal surface is called *stable* if the second variation is non-negative for every normal vector field on M with compact support. This theorem suggests that the essential property is the stability. In fact, the minimal graphs of codimension one are stable (Federer [2]), whereas those of codimension greater than one are not necessarily stable (Lawson and Osserman [5], Kawai [4]).

Since the graph of a holomorphic function on $\boldsymbol{C}$ is stable, it is quite natural to ask whether or not the complete orientable stable minimal surfaces in $\boldsymbol{R}^4$ are congruent to the complex submanifolds of $\boldsymbol{C}^2 = \boldsymbol{R}^4$, i.e., transformed to the complex submanifolds of $\boldsymbol{C}^2 = \boldsymbol{R}^4$ by the isometries of $\boldsymbol{R}^4$.

The purpose of this paper is to give a partial answer to this question, i.e., to prove the following theorem.

THEOREM. *Let M be a minimal surface in $\boldsymbol{R}^4$ which is a graph of a function defined on the whole plane $\boldsymbol{R}^2$. Suppose that M is stable. Then M is a plane or the graph of a holomorphic function or the graph of an antiholomorphic function with respect to a fixed identification $\boldsymbol{R}^2 = \boldsymbol{C}$. Hence M is congruent to a complex submanifold of $\boldsymbol{C}^2 = \boldsymbol{R}^4$.*

To prove this theorem, we shall show that the second variation is negative for some normal vector field on M with compact support if M

is not of one of the three types. The description of minimal graphs in $\boldsymbol{R}^4$ by Osserman [6] will be used.

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2. **The second variations.** Let M be a surface minimally immersed in a flat Riemannian manifold N. Suppose that M has global isothermal coordinates x, y, and the induced metric is $ds^2 = \lambda^2(dx^2 + dy^2)$ for a positive C^∞ function λ. We make use of the same method as Kawai [4]. But we consider the complexified second variation instead of the sum of the second variations in the directions of a pair of normal vector fields.

We write $X = \partial/\partial x$, $Y = \partial/\partial y$, $Z = (X - iY)/2$, $\bar{Z} = (X + iY)/2$ for simplicity, and denote various operators and their complexifications by the same letters. We denote by $\langle V, W\rangle$ the symmetric product of vectors V and W. Hence $\|V\|^2 = \langle V, \bar{V}\rangle$ for a complex vector V.

PROPOSITION 1. *Suppose a C^∞ cross-section ξ of the complexified normal bundle ν^C of M satisfies the differential equation $\nabla_{\bar{Z}}\xi = 0$, where ∇ denotes the covariant differentiation in the normal bundle ν. Then for every $\boldsymbol{R}$-valued C^∞ function φ on M with compact support, we have*

$$I(\varphi\xi, \varphi\bar{\xi}) = 4\int_M (Z\varphi)(\bar{Z}\varphi)\|\xi\|^2 dxdy - 4\int_M \|A^\xi(Z)\|^2\varphi^2 dxdy\ ,$$

where I is the index form, and A is the C^∞ cross-section of $\nu^*\otimes T^*M\otimes TM$ defined by the second fundamental form B of M in N.

PROOF. By the result of Simons [8], we have

$$I(\varphi\xi, \varphi\bar{\xi}) = \int_M [-\langle\Delta(\varphi\xi), \varphi\bar{\xi}\rangle - \langle A^{\varphi\xi}, A^{\varphi\bar{\xi}}\rangle]^*1\ ,$$

where $\Delta = \operatorname{trace}\nabla\nabla$ is the Laplacian in the normal bundle ν of M in N, and ^{*1} is the volume form of the induced metric of M.

Since x, y are isothermal coordinates, we get

$$\begin{aligned}\Delta(\varphi\xi) &= (1/\lambda^2)[\nabla_X\nabla_X(\varphi\xi) + \nabla_Y\nabla_Y(\varphi\xi)] \\ &= (2/\lambda^2)[\nabla_Z\nabla_{\bar{Z}}(\varphi\xi) + \nabla_{\bar{Z}}\nabla_Z(\varphi\xi)] \\ &= (4/\lambda^2)[(Z\bar{Z}\varphi)\xi + (\bar{Z}\varphi)\nabla_Z\xi] \\ &\quad - (2/\lambda^2)\varphi R(Z, \bar{Z})\xi\ ,\end{aligned}$$

where R denotes the curvature of the normal bundle ν. Hence we have

$$\begin{aligned}-\langle\Delta(\varphi\xi), \varphi\bar{\xi}\rangle &= -(4/\lambda^2)[(Z\bar{Z}\varphi)\varphi\|\xi\|^2 + \varphi(\bar{Z}\varphi)(Z\|\xi\|^2)] \\ &\quad + (2/\lambda^2)\varphi^2\langle R(Z, \bar{Z})\xi, \bar{\xi}\rangle \\ &= -(1/\lambda^2)[(\Delta_0\varphi)\varphi\|\xi\|^2 + \varphi(X\varphi)(X\|\xi\|^2) \\ &\quad + \varphi(Y\varphi)(Y\|\xi\|^2) + i\{(Y\varphi)(X\|\xi\|^2) - (X\varphi)(Y\|\xi\|^2)\}] \\ &\quad + (2/\lambda^2)\varphi^2\langle R(Z, \bar{Z})\xi, \bar{\xi}\rangle ,\end{aligned}$$

where $\Delta_0 = XX + YY = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

By the theorem of Stokes, we get

$$\begin{aligned}\int_M &-(1/\lambda^2)[(\Delta_0\varphi)\varphi\|\xi\|^2 + \varphi(X\varphi)(X\|\xi\|^2) + \varphi(Y\varphi)(Y\|\xi\|^2) \\ &\quad + i\{(Y\varphi)(X\|\xi\|^2) - (X\varphi)(Y\|\xi\|^2)\}]^*1 \\ &= \int_M [(X\varphi)^2 + (Y\varphi)^2]\|\xi\|^2 dxdy .\end{aligned}$$

By the identity of Ricci, we have

$$\begin{aligned}&(2/\lambda^2)\varphi^2\langle R(Z, \bar{Z})\xi, \bar{\xi}\rangle - \langle A^{\varphi\xi}, A^{\varphi\bar{\xi}}\rangle \\ &\quad = (2/\lambda^2)\varphi^2[\langle A^{\bar{\xi}}(Z), A^{\xi}(\bar{Z})\rangle - \langle A^{\xi}(Z), A^{\bar{\xi}}(\bar{Z})\rangle \\ &\qquad - \langle A^{\xi}(Z), A^{\bar{\xi}}(\bar{Z})\rangle - \langle A^{\xi}(\bar{Z}), A^{\bar{\xi}}(Z)\rangle] \\ &\quad = -(4/\lambda^2)\varphi^2\|A^{\xi}(Z)\|^2 .\end{aligned}$$

Hence we obtain the desired result.

3. Minimal surfaces in R^4.

Consider a minimal surface M in R^n defined by

$$f(x, y) = (f_1(x, y), \cdots, f_n(x, y))$$

with respect to isothermal coordinates x, y. Let us define functions ϕ_k of $z = x + iy$ by

$$\phi_k = \partial f_k/\partial x - i\partial f_k/\partial y \qquad (k = 1, 2, \cdots, n) .$$

Since x, y are isothermal coordinates, ϕ_k are holomorphic in z and they satisfy the following identities.

$$\langle\phi, \phi\rangle = \phi_1^2 + \phi_2^2 + \cdots + \phi_n^2 = 0 , \tag{1}$$

$$\begin{aligned}\|\partial f/\partial x\|^2 = \|\partial f/\partial y\|^2 &= \|\phi\|^2/2 \\ &= (|\phi_1|^2 + |\phi_2|^2 + \cdots + |\phi_n|^2)/2 .\end{aligned} \tag{2}$$

Hence the induced metric of M is $ds^2 = \lambda^2(dx^2 + dy^2)$ with $\lambda^2 = \|\phi\|^2/2$.

We identify X, Y, Z, $\bar{Z}$ with their image by the differential of f.

DEFINITION. $\xi_0 = B(X, X) - iB(X, Y)$.

By Ruh [7], the normal vector fields $B(X, X)$ and $B(X, Y)$ satisfy the following identities

$$\nabla_Y(B(X, X)) = \nabla_X(B(X, Y)) , \qquad \nabla_X(B(X, X)) + \nabla_Y(B(X, Y)) = 0 .$$

Hence ξ_0 satisfies the equation $\nabla_{\bar{Z}}\xi = 0$.

We shall construct a cross-section ξ of ν^C from ξ_0 which also satisfies the equation $\nabla_{\bar{Z}}\xi = 0$, and apply Proposition 1 to ξ. For this purpose we study the properties of ξ_0

LEMMA 1. $\xi_0 = 2B(Z, Z) = Z\phi - (Z\lambda^2)\phi/\lambda^2$.

PROOF. By the minimality of M, we have $B(Z, \bar{Z})=0$. Hence we get

$$\xi_0 = B(Z + \bar{Z}, Z + \bar{Z}) - iB(Z + \bar{Z}, i(Z - \bar{Z})) = 2B(Z, Z) .$$

Since the real (resp. imaginary) part of $B(Z, Z)$ is the normal component of the real (resp. imaginary) part of $ZZf = Z\phi/2$, and $\|Zf\|^2 = \|\bar{Z}f\|^2 = \lambda^2/2$, we have

$$B(Z, Z) = Z\phi/2 - 2\langle Z\phi/2, \bar{Z}f\rangle Zf/\lambda^2 - 2\langle Z\phi/2, Zf\rangle \bar{Z}f/\lambda^2 .$$

Making use of the identities (1) and (2), we obtain

$$B(Z, Z) = Z\phi/2 - (Z\lambda^2)\phi/2\lambda^2 .$$

Thus the lemma is proved.

LEMMA 2.
$$\|A^{\xi_0}(Z)\|^2 = |\langle Z\phi, Z\phi\rangle|^2/2\lambda ,$$
$$\|\xi_0\|^2 = \|Z\phi\|^2 - (2/\lambda^2)|Z\lambda^2|^2 .$$

PROOF. By the definitions of A and ξ_0, we have

$$\begin{aligned} \|A^{\xi_0}(Z)\|^2 &= (2/\lambda^2)|\langle A^{\xi_0}(Z), \bar{Z}\rangle|^2 + (2/\lambda^2)|\langle A^{\xi_0}(Z), Z\rangle|^2 \\ &= (2/\lambda^2)|\langle B(Z, \bar{Z}), \xi_0\rangle|^2 + (2/\lambda^2)|\langle B(Z, Z), \xi_0\rangle|^2 \\ &= (1/2\lambda^2)|\langle \xi_0, \xi_0\rangle|^2 . \end{aligned} \tag{3}$$

By Lemma 1 and the identity (1), we get the first equality. By Lemma 1, we have

$$\|\xi_0\|^2 = \langle Z\phi - (1/\lambda^2)(Z\lambda^2)\phi, \overline{Z\phi} - (1/\lambda^2)(\bar{Z}\lambda^2)\bar{\phi}\rangle .$$

Since $\|\phi\|^2 = 2\lambda^2$, we get the second equality.

Now we consider the minimal surfaces in $\boldsymbol{R}^4$ which are graphs of maps from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$. These objects are investigated by Osserman [6].

PROPOSITION 2 (Osserman). *Let $M: f = (f_1, f_2, f_3, f_4)$ be a minimal surface in $\boldsymbol{R}^4$ where $f_3(f_1, f_2)$ and $f_4(f_1, f_2)$ are functions f_1 and f_2 defined on the whole plane $\boldsymbol{R}^2$. Then there exists a linear transformation of $\boldsymbol{R}^2$*

$$f_1 = x , \qquad f_2 = ax + by \qquad (a, b \in \boldsymbol{R},\ b > 0) ,$$

such that x, y are global isothermal coordinates of M.

With respect to these isothermal coordinates, we have

$$\phi_1 = \partial f_1/\partial x - i\partial f_1/\partial y = 1 ,$$
$$\phi_2 = \partial f_2/\partial x - i\partial f_2/\partial y = a - ib .$$

Putting $d = 1 + (a - ib)^2$, we get $\phi_3^2 + \phi_4^2 = -d$, since $\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = 0$.

In the case $d = 0$, we show that M is either the graph of a holomorphic function or the graph of an antiholomorphic function with respect to a fixed complex structure of $\boldsymbol{R}^2$. Because the condition $d = 0$ implies $a = 0$ and $b = \pm 1$, the linear transformation in Proposition 2 is either $f_1 + if_2 = x + iy$ or $f_1 + if_2 = x - iy$. The identity $\phi_3^2 + \phi_4^2 = 0$ implies that either $\phi_4 = i\phi_3$ or $\phi_4 = -i\phi_3$. Hence $f_3 + if_4$ is either a holomorphic function or an antiholomorphic function of $f_1 + if_2$. Of course M is congruent to a complex submanifold $\boldsymbol{C}^2 = \boldsymbol{R}^4$.

Now we consider the case $d \neq 0$. In this case the condition

$$(\phi_3 + i\phi_4)(\phi_3 - i\phi_4) = -d \neq 0$$

implies that the non-vanishing holomorphic function $\phi_3 - i\phi_4$ is of the form $\phi_3 - i\phi_4 = \mathrm{e}^{H(z)}$ where $H(z)$ is an entire function. Hence we get the identities

$$\phi_3 = (\mathrm{e}^{H(z)} - d\mathrm{e}^{-H(z)})/2 , \qquad \phi_4 = i(\mathrm{e}^{H(z)} + d\mathrm{e}^{-H(z)})/2 , \tag{4}$$

$$Z\phi = (0, 0, H'(\mathrm{e}^{H} + d\mathrm{e}^{-H})/2 , \qquad iH'(\mathrm{e}^{H} - d\mathrm{e}^{-H})/2) . \tag{5}$$

By Lemma 2 and the above identity (5), we have the following proposition.

PROPOSITION 3. $\qquad ||A^{\xi_0}(Z)||^2 = |d|^2|H'|^4/2\lambda^2 .$

Here we give a property of a minimal surface in $\boldsymbol{R}^4$ which is congruent to a complex submanifold in $\boldsymbol{C}^2 = \boldsymbol{R}^4$.

LEMMA 3. *M is congruent to a complex submanifold in $\boldsymbol{C}^2 = \boldsymbol{R}^4$ if and only if $H' \equiv 0$ or $d = 0$ holds.*

PROOF. We show that $||A^{\xi_0}(Z)||^2$ vanishes identically if M is congruent to a complex submanifold. We may suppose that M is a complex submanifold. Let us denote by J the complex structure of $\boldsymbol{C}^2 = \boldsymbol{R}^4$. Then J induces the complex structure J' of M and we have

$$B(J'u, v) = B(u, J'v) = JB(u, v)$$

for every tangent vector u and v of M at $p \in M$.

Since J is Hermitian, we get by Lemma 1 the equality

$$\begin{aligned}\langle \xi_0, \xi_0\rangle &= 4\langle JB(Z, Z), JB(Z, Z)\rangle\\ &= 4\langle B(J'Z, Z), B(J'Z, Z)\rangle\\ &= 4\langle B(iZ, Z), B(iZ, Z)\rangle\\ &= -\langle \xi_0, \xi_0\rangle .\end{aligned}$$

This shows $\langle \xi_0, \xi_0\rangle = 0$ which implies the desired result by the identity (3). The converse is clear, because $H' \equiv 0$ implies that M is a plane.

PROPOSITION 5. $\|\xi_0\|^2 = [(1 + a^2 + b^2) - 2b^2/\lambda^2]|H'|^2$.

PROOF. Since

$$\lambda^2 = [1 + a^2 + b^2 + (|e^H|^2 + |d|^2|e^{-H}|^2)/2]/2 ,$$

we have

$$\begin{aligned}&|e^H|^2 + |d|^2|e^{-H}|^2 = 2[2\lambda^2 - (1 + a^2 + b^2)] ,\\ &(|e^H|^2 - |d|^2|e^{-H}|^2)^2 = 4[2\lambda^2 - (1 + a^2 + b^2)]^2 - 4|d|^2 .\end{aligned}$$

Hence we get

$$\begin{aligned}(6)\qquad &Z\lambda^2 = H'(|e^H|^2 - |d|^2|e^{-H}|^2)/4 ,\\ &2|Z\lambda^2|^2/\lambda^2 = |H'|^2[\{2\lambda^2 - (1 + a^2 + b^2)\}^2 - |d|^2]/2\lambda^2 .\end{aligned}$$

By Lemma 2, we obtain

$$\begin{aligned}\|\xi_0\|^2 &= |H'|^2|e^H + de^{-H}|^2/4 + |H'|^2|e^H - de^{-H}|^2/4\\ &\quad - |H'|^2[\{2\lambda^2 - (1 + a^2 + b^2)\}^2 - |d|^2]/2\lambda^2\\ &= |H'|^2[(1 + a^2 + b^2) - 2b^2/\lambda^2] .\end{aligned}$$

4. The proof of Theorem. In this section, we show under the conditions $d \neq 0$ and $H' \not\equiv 0$ the existence of a normal vector field with compact support for which the second variation is negative. By Lemma 1 and the identities (5) and (6), we have

$$\begin{aligned}\xi_0 = H'[&(0, 0, (e^H + de^{-H})/2, i(e^H - de^{-H})/2)\\ &- (1/4\lambda^2)(|e^H|^2 - |d|^2|e^{-H}|^2)\phi] .\end{aligned}$$

DEFINITION. We define a C^∞ cross section ξ of ν^C to be the quantity inside the bracket [ ] in the above expression for ξ_0.

LEMMA 4. *Suppose that $H' \not\equiv 0$. Then we have the following equalities.*

$$\begin{aligned}&\nabla_{\bar Z}\xi = 0 , \qquad \|A^\xi(Z)\|^2 = |d|^2|H'|^2/2\lambda^2 ,\\ &\|\xi\|^2 = (1 + a^2 + b^2) - 2b^2/\lambda^2 \leqq 1 + a^2 + b^2 .\end{aligned}$$

PROOF. Since $\xi = \xi_0/H'$ at the points z with $H'(z) \neq 0$, we get the second and the third equalities. Since H' is holomorphic, we get the first equality except at the zero points of H'. These equalities hold on the whole plane $\boldsymbol{R}^2$, because the zero points of H' are isolated.

By Proposition 1 and the above Lemma 4, we have

$$I(\varphi\xi, \varphi\bar{\xi}) \leqq (1 + a^2 + b^2)\int_{\boldsymbol{R}^2}[(X\varphi)^2 + (Y\varphi)^2]dxdy - 2|d|^2\int_{\boldsymbol{R}^2}(|H'|^2/\lambda^2)\varphi^2 dxdy \tag{7}$$

for every $\boldsymbol{R}$-valued C^∞ function φ on $\boldsymbol{R}^2$ with compact support. Since

$$I(\varphi\xi, \varphi\bar{\xi}) = I(\varphi \operatorname{Re}\xi, \varphi \operatorname{Re}\xi) + I(\varphi \operatorname{Im}\xi, \varphi \operatorname{Im}\xi)$$

is the sum of the second variations in the directions of $\varphi \operatorname{Re}\xi$ and $\varphi \operatorname{Im}\xi$, we have only to show the existence of an $\boldsymbol{R}$-valued C^∞ function φ with compact support for which the right hand side of the inequality (7) is negative.

The quantity $|H'|^2$ is positive on an open set of $\boldsymbol{R}^2$. Hence it suffices to prove the following lemma.

LEMMA 5. *Let c be a positive constant. Let F be a non-negative function on $\boldsymbol{R}^2$ which is positive on an open neighborhood of the origin. Then there exists an $\boldsymbol{R}$-valued C^∞ function φ on $\boldsymbol{R}^2$ such that*

$$\int_{\boldsymbol{R}^2}[(X\varphi)^2 + (Y\varphi)^2]dxdy < c\int_{\boldsymbol{R}^2} F\varphi^2 dxdy .$$

PROOF. Let us define a sequence φ_m $(m = 2, 3, \cdots)$ of functions on $\boldsymbol{R}^2$ as follows:

$$\varphi_m(r, \theta) = 1/3 + 1/5 + \cdots + 1/(2m - 1) \qquad (0 \leqq r \leqq 1) ,$$

$$\varphi_m(r, \theta) = 1/(2j + 1) + 1/(2j + 3) + \cdots + 1/(2m - 1) - (r - j)/(2j + 1)$$
$$(j \leqq r \leqq j + 1,\ j = 1, 2, \cdots, m - 1) ,$$

$$\varphi_m(r, \theta) = 0 \qquad (m \leqq r) ,$$

where r, θ are the polar coordinates of $\boldsymbol{R}^2$. Then we have

$$\int_{\boldsymbol{R}^2}[(X\varphi_m)^2 + (Y\varphi_m)^2]dxdy = \pi[1/3 + 1/5 + \cdots + 1/(2m - 1)] ,$$

$$c\int_{\boldsymbol{R}^2}(F\varphi_m^2)dxdy \geqq c\int_{D(1)}(F\varphi_m^2)dxdy = c'[1/3 + 1/5 + \cdots + 1/(2m - 1)]^2$$

where $D(1) = \{(x, y) \in \boldsymbol{R}^2 \,|\, x^2 + y^2 \leqq 1\}$ and $c' = c\int_{D(1)} F dxdy > 0$.

Since $1/3 + 1/5 + \cdots + 1/(2m-1) \to \infty$ $(m \to \infty)$, we have

$$\int_{\boldsymbol{R}^2} [(X\varphi_m)^2 + (Y\varphi_m)^2] dxdy < c\int_{\boldsymbol{R}^2} (F\varphi_m^2) dxdy$$

for sufficiently large m. Approximating φ_m by a C^∞ function, we obtain the desired result.

REMARK. The first proof of Lemma 5 used the following fact due to Fischer-Colbrie and Schoen [3]. For any non-negative function q, the existence of a positive function g on $\boldsymbol{R}^2$ satisfying $-\Delta_0 g - q \cdot g = 0$ is equivalent to the condition that the first eigenvalue of $-\Delta_0 - q$ be non-negative on each bounded domain in $\boldsymbol{R}^2$. The above elementary proof was kindly informed to the author by Professor K. Sugahara.

REMARK. For an orientable parametric minimal surface M in $\boldsymbol{R}^4$, the following can be proved: M is congruent to a complex submanifold in $\boldsymbol{C}^2 = \boldsymbol{R}^4$ if and only if $\|A^{\xi_0}(Z)\|$ vanishes identically, i.e., $\langle \xi_0, \xi_0 \rangle$ vanishes identically on the domain of an isothermal coordinate. This fact may be useful to generalize Theorem for parametric minimal surfaces in $\boldsymbol{R}^4$.

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