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DIMENSION OF SPACES OF VECTOR VALUED AUTOMORPHIC FORMS ON THE UNITARY GROUP $SU(2, 1)$

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Introduction. The purpose of this paper is to investigate the dimension of the spaces of the vector valued holomorphic automorphic forms defined on the domain $D = \{(z, w) \in \boldsymbol{C}^2 \mid \delta(\bar{z} - z) - |w|^2 > 0\}$, where δ is an element of an imaginary quadratic field F with $\bar{\delta} = -\delta(\neq 0)$. Let $\Gamma(N)$ be an arithmetic subgroup of $G_{\boldsymbol{R}}$ defined in §1. Let ρ be an irreducible polynomial representation of $GL_2(\boldsymbol{C})$ of degree $m + 1$. Consider a $\boldsymbol{C}^{m+1}$-valued holomorphic function $f(Z)$ on D satisfying

$$f(\gamma(Z)) = \rho(J(\gamma, Z))f(Z)$$

for every $Z \in D$ and for every $\gamma \in \Gamma(N)$, where $J(\gamma, Z)$ is the canonical automorphy factor on $\Gamma(N) \times D$. Denote by $S_\rho(\Gamma(N))$ the space of all such forms. In [3], Cohn calculated the dimension of $S_\rho(\Gamma')$ in the case where $F = \boldsymbol{Q}(\sqrt{-1})$, $\delta = \sqrt{-1}$, $\rho(g) = \det(g)^k$ and $\Gamma' = G_{\boldsymbol{Q}} \cap M_3(\mathfrak{O}_F)$ (see §1 for $G_{\boldsymbol{Q}}$). In this paper we try to extend his results to the case where F is an imaginary quadratic field of class number one, ρ is an arbitrary irreducible representation and $\Gamma(N)$ is a principal congruence subgroup of $\Gamma(1)$.

§1 is devoted to classifying the elements of $\Gamma(N)$, using several methods of Cohn. In §2, we construct a good fundamental domain for $\Gamma(1)$. In §3, applying the method of Selberg [8] and Godement [4], we reduce the computation of $\dim S_\rho(\Gamma(N))$ to that of certain integrals. In the last section, using a method similar to those of Shimizu [9] and Morita [7], we establish the following theorem:

THEOREM. *Suppose that F is an imaginary quadratic field of class number one and $k \geqq m + 6$. Then*

$$\begin{aligned}
&\dim S_\rho(\Gamma(N)) \\
&= \Big\{2^{k+m-1}\pi^2(-i\delta)(2k+2m-3)!!((2k+2m-2)!)^{-1} \sum_{l=0}^{m} {}_mC_l(m-l)!(l+k-3)!\Big\}^{-1} \\
&\quad |\Gamma/\Gamma(N)| \Big\{(m+1)\operatorname{vol}(\Gamma\backslash D) + \delta^2 n_0(|\delta|^2 n_1^2)^{-1}\zeta(2)\operatorname{vol}(\boldsymbol{C}/\delta\mathfrak{m})\,|E(F)|^{-1} \\
&\quad \times \sum_{j=0}^{m}((k+j-1)(k+j-2))^{-1}\Big\}\,.
\end{aligned}$$

Various symbols used here will be explained in §4.

We note that we owe our results in §1 to those of Cohn. We also note that Tsushima [11] has succeeded in computing the dimension of the space of holomorphic vector valued Siegel modular forms of degree two, and Kato [6] has derived the dimension formula of the space of holomorphic automorphic forms on $SU(p, 1)$ of automorphy factor defined by Jacobian.

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NOTATION. We denote, as usual, by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field. For a ring A, we denote by A^n_m the set of all $n \times m$ matrices with entries in A, and denote A^n_1(resp. A^n_n) by A^n(resp. $M_n(A)$). For $z \in \boldsymbol{C}$, we put $e[z] = \exp(2\pi i z)$ with $i = \sqrt{-1}$ $(\operatorname{Im} i > 0)$.

1. Classification of conjugacy classes. This section is devoted to summarizing several facts which we need later. Throughout this paper we denote by F an imaginary quadratic field of class number one. Let $E(F)$ denote the unit group of F. Let δ be a non-zero element of F such that $\bar{\delta} = -\delta$ and $\operatorname{Im} \delta > 0$, where the bar means the complex conjugate. Let

$$G_{\boldsymbol{Q}} = \{g \in SL_3(F) \mid {}^t\bar{g}Hg = H\} \quad (\text{resp. } G_{\boldsymbol{R}} = \{g \in SL_3(\boldsymbol{C}) \mid {}^t\bar{g}Hg = H\}) ,$$

where $H = \begin{pmatrix} 0 & 0 & \delta \\ 0 & -1 & 0 \\ -\delta & 0 & 0 \end{pmatrix}$ and ${}^tg$ denotes the transpose of g. Then $G_{\boldsymbol{Q}}$ is a linear algebraic group defined over $\boldsymbol{Q}$, and $G_{\boldsymbol{R}}$ is its group of $\boldsymbol{R}$-rational points. Introduce a domain D in $\boldsymbol{C}^2$ determined by

$$D = \{Z = {}^t(z, w) \in \boldsymbol{C}^2 \mid \delta(\bar{z} - z) - |w|^2 > 0\} .$$

We note that $G_{\boldsymbol{R}} \cong SU(2, 1)$ and $D \cong SU(2, 1)/S(U(2) \times U(1))$. Define an action of $G_{\boldsymbol{R}}$ on D by

$$Z \mapsto g(Z) = {}^t\!\left(\frac{a_1z + a_2w + a_3}{c_1z + c_2w + c_3}, \frac{b_1z + b_2w + b_3}{c_1z + c_2w + c_3}\right) ,$$

where $Z = {}^t(z, w) \in D$ and $g = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in G_{\boldsymbol{R}}$.

We say that the non-zero vector $x \in \boldsymbol{C}^3$ is *positive, isotropic*, or *negative* according as $\langle x, x\rangle$ is positive, zero, or negative, where $\langle x, y\rangle = {}^t\bar{y}Hx$

for $x, y \in C^3$. By Lemma 1 of Cohn [3, Chap. 111], we can classify the elements of $G'_R = G_R - \{\alpha E_3 | \alpha^3 = 1\}$ as follows:

(i) an element g of G'_R is *elliptic* if g has a positive eigenvector and has no *isotropic* eigenvector,

(ii) an element g of G'_R is *hyperelliptic* if there exists a two-dimensional non-degenerate subspace W containing an isotropic vector such that $gW \subset W$ and $g|_W = \lambda 1d_W$ ($\lambda \neq 1$, $|\lambda| = 1$),

(iii) an element g of G'_R is *hyperbolic* if there exist linearly independent isotropic vectors v_1 and v_2 in C^3 such that $gv_i = \gamma_i v_i$ ($i = 1, 2$) with $\gamma_1 \neq \gamma_2$,

(iv) an element g of G'_R is *parabolic* if g has an isotropic eigenvector and is neither hyperelliptic nor hyperbolic. Here we note that an eigenvalue λ of a non-isotropic eigenvector of $g \in G_R$ satisfies $|\lambda| = 1$. The following proposition can be proved by using the result of [3, pp. 21-22].

PROPOSITION 1.1. *If $g \in G_R$ is either elliptic or hyperelliptic, then there exists $g' \in SL_3(C)$ such that*

$$g = g' \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} (g')^{-1} \quad \textit{with} \quad |\lambda_i| = 1 \quad (i = 1, 2, 3) .$$

If g is hyperbolic, then there exists an element g' of G_R such that

$$g = g' \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix} (g')^{-1} \quad \textit{with} \quad |\alpha_2| = 1 \quad \textit{and} \quad \bar{\alpha}_1 \alpha_3 = 1 .$$

PROOF. First we assume that g is elliptic or hyperelliptic. Then, by [3, proof of Lemma 1 (p. 21)], g has eigenvectors x_1, x_2, x_3 such that $C^3 = Cx_1 + Cx_2 + Cx_3$ and x_i ($i = 1, 2, 3$) are not isotropic. Then the eigenvalue λ_i of g attached to x_i satisfies $|\lambda_i| = 1$. Therefore we obtain the first assertion of Proposition 1.1. Next we assume that g is hyperbolic. Then, by [3, proof of Lemma 1 (p. 21)], g has a basis $\{v_1, v_2, v_3\}$ of C^3 such that $gv_i = \lambda_i v_i$ ($i = 1, 2, 3$), v_3 is negative, v_i ($i = 1, 2$) are isotropic and $v_1, v_2 \in \{v_3\}^\perp$. We may assume that $\langle v_3, v_3 \rangle = -1$. Assume that $\langle v_1, v_2 \rangle = 0$. Then we have $\langle v_1 + v_2, v_i \rangle = 0$ ($i = 1, 2, 3$). So $v_1 + v_2 = 0$. This is contrary to the fact that $\{v_1, v_2, v_3\}$ is a basis of C^3. Therefore we can choose vectors v_1, v_2 such that $\langle v_1, v_2 \rangle = -\delta$. Let h be an element of $GL_3(C)$ satisfying $hv_1 = e_1$, $hv_2 = e_3$ and $hv_3 = \mu e_2$, where μ is a complex number with $|\mu| = 1$, $e_1 = {}^t(1, 0, 0)$, $e_2 = {}^t(0, 1, 0)$ and $e_3 = {}^t(0, 0, 1)$. Then we see that $\langle hx, hy \rangle = \langle x, y \rangle$ for all $x, y \in C^3$ and

$$hgh^{-1} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}.$$

Now we have $\det(h) = 1$ with a suitable μ. Therefore we obtain the remainder of Proposition 1.1 and completes the proof.

Let $\mathfrak{O}_F$ be the ring of all integers in F. We consider a lattice L in F^3 determined by $L = A\mathfrak{O}_F^3$, where

$$A = \begin{pmatrix} 0 & 1/\delta & -1/\delta \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}.$$

For a positive integer N, put

$$\tilde{\Gamma} = \{g \in GL_3(F) \mid g^*Rg = R,\ g\mathfrak{O}_F^3 = \mathfrak{O}_F^3\}, \qquad \tilde{\Gamma}(N) = \{g \in \Gamma \mid g \equiv E_3(N)\},$$

where $R = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$,

$$\Gamma = \{g \in G_Q \mid gL = L\} \quad \text{and} \quad \Gamma(N) = \{g \in G_Q \mid (g - E_3)L \subset NL\} (= A\tilde{\Gamma}(N)A^{-1}).$$

By the same method as that of Morita [7, Lemma 2], the following lemma can be easily verified.

LEMMA 1.2. *Let N be a positive integer $N(\geqq 3)$. Suppose that ζ is an eigenvalue of g of $\tilde{\Gamma}(N)$ and that ζ is a root of unity. Then ζ is equal to* 1.

Since $\boldsymbol{R} = A^*HA$, $\Gamma(N) = A\tilde{\Gamma}(N)A^{-1}$, the above lemma holds for $\Gamma(N)$. Let g be an element of $\Gamma(N)$ not belonging to the center of $\Gamma(N)$. We assume that g is elliptic or hyperelliptic. By Proposition 1.1, all eigenvalues of g are complex numbers of absolute value 1. So, by Lemma 1.2, g is equal to E_3. Therefore we have the following corollary.

COROLLARY 1.1. *Under the same assumption as that of Lemma 1.2, an element of $\Gamma(N) - \{\alpha E_3 \mid \alpha^3 = 1\}$ is hyperbolic or parabolic.*

A vector v in L is called *primitive*, if a vector v belongs to aL with $a \in \mathfrak{O}_F$ implies that a is an unit of $\mathfrak{O}_F$. Now we can verify the following.

LEMMA 1.3. *Under the above notation, every primitive isotropic vector $v \in L$ can be embedded in a basis $\{v, \tilde{v}, y\}$ of L such that $\langle v, \tilde{v}\rangle = \langle y, y\rangle = \langle \tilde{v}, \tilde{v}\rangle = -1$ and $y \perp v, \tilde{v}$.*

PROOF. We observe that $\tilde{e}_1 = {}^t(0, 1, 0)$, $\tilde{e}_2 = {}^t(1/\delta, 0, 1/2)$, $\tilde{e}_3 = (2/\delta, 0, 0)$

satisfy $L = \mathfrak{O}_F\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ and $\det(\langle \tilde{e}_i, \tilde{e}_j\rangle_{1\leq i,j\leq 3}) = 1$. According to [3, Remark (3) (p. 24)], there is a vector $v' \in L$ with $\langle v', v\rangle = 1$. Since $\mathfrak{O}_F$ is a principal ideal domain and since $\langle L, v\rangle = \{\langle x, v\rangle \mid x \in L\} = \mathfrak{O}_F$, there exists a basis $\{x_1, x_2, x_3\}$ of L over $\mathfrak{O}_F$ such that $L = L \cap \{v\}^\perp \oplus \{x_3\}$. Set $v' = \alpha + nx_3$ ($\alpha \in L \cap \{v\}^\perp$, $n \in \mathfrak{O}_F$). Since $\langle v', v\rangle = 1$, n belongs to $E(F)$. Thus $L = L \cap \{v\}^\perp + \{v'\}$. Since $L \cap \{v\}^\perp \cap \{v'\} = 0$, we have $L = L \cap \{v\}^\perp \oplus \{v'\}$. By [3, Remark (1) (p. 24)], we can verify $L \cap \{v\}^\perp = \{v, x\}$. Using [3, Remark (2) (p. 24)], we get

$$\det\begin{pmatrix} \langle v', v'\rangle & \langle v', v\rangle & \langle v', x\rangle \\ \langle v, v'\rangle & \langle v, v\rangle & \langle v, x\rangle \\ \langle x, v'\rangle & \langle x, v\rangle & \langle x, x\rangle \end{pmatrix} = -\langle x, x\rangle = 1 .$$

Set $v'' = v' + \langle v', x\rangle x + bv$ ($b \in \mathfrak{O}_F$). Then, $\langle v'', v\rangle = \langle v', v\rangle = 1$ and $\langle v'', x\rangle = 0$. Let d be the discriminant of F. If $d \equiv 1(4)$ or $d \not\equiv 1(4)$ and $\langle v', v'\rangle + \langle v', x\rangle\langle x, v'\rangle \equiv 1(2)$, we can choose an element b of $\mathfrak{O}_F$ satisfying $\langle v'', v''\rangle = -1$. We set $y = x$ and $\tilde{v} = -v''$. If $d \not\equiv 1(4)$ and $\langle v', v'\rangle + \langle v', x\rangle\langle x, v'\rangle \equiv 0(2)$, we can choose an element b of $\mathfrak{O}_F$ satisfying $\langle v'', v''\rangle = 0$. In this case, we set $y = x + v$ and $\tilde{v} = -v'' - x$. Thus $\{v, \tilde{v}, y\}$ is a required basis of L. This completes the proof.

Now we can prove the following proposition.

PROPOSITION 1.2. *Let g be a parabolic element of Γ. Then $[g]_\Gamma \cap P_Q \neq \phi$, where $[g]_\Gamma = \{\gamma g \gamma^{-1} \mid \gamma \in \Gamma\}$ and* $P_Q = \left\{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \Gamma\right\}$. *Furthermore, every eigenvalue of g is a root of unity.*

PROOF. Since g is parabolic, there are only the following two cases (see [3, proof of Lemma 1 (p. 21)]):

(i) g has no positive eigenvector but has a negative eigenvector;

(ii) Every eigenvector of g is isotropic.

By the same method as that of [3, Lemma 1 (p. 24)], we see that every eigenvalue of a parabolic element of $\tilde{\Gamma}$ belongs to $\mathfrak{O}_F$. Therefore, since $\Gamma = A\tilde{\Gamma}A^{-1}$, every eigenvalue of g belongs to $\mathfrak{O}_F$, and every component of g belongs to F. Let $\{\lambda_j\}_{j=1}^3$ be the set of all eigenvalues of g. Then, there exists an isotropic eigenvector x of g belonging to F^3. Indeed, there is an eigenvector x of g in F^3. If x is isotropic, x is a required vector. So we suppose that every eigenvector x of g in F^3 is negative. By the first remark, x is negative. Set $\{x\}_F^\perp = \{y \in F^3 \mid \langle x, y\rangle = 0\}$. Then, $\{x\}_F^\perp$ is a 2-dimensional vector space over F. Since $\langle x, x\rangle = \langle gx, gx\rangle = \langle \gamma_j x, \gamma_j x\rangle = |\gamma_j|^2\langle x, x\rangle$, we have $\gamma_j \neq 0$. So it is easily seen that $g\{x\}_F^\perp \subset \{x\}_F^\perp$. Therefore there exists an eigenvector $x' \in \{x\}_F^\perp$ of g such that

$x' \perp x$. By [3, proof of Lemma 1 (p. 21)], x' is isotropic, which contradicts the assumption on x. This shows the existence of the required isotropic vector x. We can choose $n \in F - \{0\}$ such that $v = nx \in L$ and v is primitive. By Lemma 1.3, we can write $L = \mathfrak{O}_F\{v, y, \tilde{v}\}$ with $\langle \tilde{v}, v\rangle = \langle y, y\rangle = \langle \tilde{v}, \tilde{v}\rangle = -1$ and $y \perp \tilde{v}, v$. Let h be an element of $GL_3(\boldsymbol{C})$ satisfying $h\tilde{e}_1 = y$, $h\tilde{e}_2 = \tilde{v}$, $h\tilde{e}_3 = v$. Then, a simple calculation shows that $\langle hx, hy\rangle = \langle x, y\rangle$ holds for every $x, y \in \boldsymbol{C}^3$, $h(L) = L$ and $h^{-1}gh \in P_Q$. Set $\nu = \det(h)$. Then, ν belongs to $E(F)$ because $h(L) = L$. We put $h' = h\begin{pmatrix}1 & 0 & 0\\ 0 & \nu & 0\\ 0 & 0 & 1\end{pmatrix}$. We see that $h'^{-1}gh' \in P_Q$ and $h' \in \Gamma$. Set $h'^{-1}gh' = \begin{pmatrix}\alpha_1 & * & *\\ 0 & \alpha_2 & *\\ 0 & 0 & \alpha_3\end{pmatrix}$. Since every eigenvalue of g belongs to $\mathfrak{O}_F$, α_i $(i = 1, 2, 3)$ belong to $\mathfrak{O}_F$. Since $\det(g) = \alpha_1\alpha_2\alpha_3 = 1$, α_i is a root of unity. Thus our proposition is proved.

Applying Lemma 1.3 and the method used to prove Proposition 1.2, we can prove the following (cf. [3, p. 26]).

PROPOSITION 1.3. *The group G_Q coincides with ΓP_Q.*

2. **Fundamental domain for Γ.** For $(\alpha, n) \in F \times \boldsymbol{Q}$, put $[\alpha, n] = \begin{pmatrix}1 & \alpha & n + \delta\alpha\bar{\alpha}/2\\ 0 & 1 & \bar{\alpha}\delta\\ 0 & 0 & 1\end{pmatrix}$. We define two groups Γ_∞ and $\Gamma_\infty^{(1)}$ by $\Gamma_\infty = \{[\alpha, n] \in \Gamma\}$ and $\Gamma_\infty^{(1)} = \Gamma \cap P_Q$. Put $\mathfrak{m} = \{\alpha \in F \mid [\alpha, n] \in \Gamma_\infty$ for some $n \in \boldsymbol{Q}\}$. We note that $[\alpha, n] \in \Gamma_\infty$ if and only if $\alpha \in (2/\delta)$ and $n + \delta\alpha\bar{\alpha}/2 \in (4/\delta)$. Therefore we have the following lemma.

LEMMA 2.1. *Let the notation be as above. Then, $\mathfrak{m}$ is an ideal in F. Moreover, if $[\alpha, n]$ and $[\alpha, n']$ belong to Γ_∞, then $n - n'$ belongs to $(4/\delta) \cap \boldsymbol{Q}$.*

Let L be a positive number. We can take the following set $\mathfrak{F}_\infty$ (resp. $\mathfrak{F}_\infty^{(1)}$) as a fundamental domain in D for Γ_∞ (resp. $\Gamma_\infty^{(1)}$):

$$\mathfrak{F}_\infty = \{(z, w) \in D \mid \mathrm{Re}(z) \in (4/\delta) \cap \boldsymbol{Q} \quad \text{and} \quad w \in \boldsymbol{C}/\delta\mathfrak{m}\},$$

$$\mathfrak{F}_\infty^{(1)} = \{(z, w) \in D \mid \mathrm{Re}(z) \in (4/\delta) \cap \boldsymbol{Q} \quad \text{and} \quad w \in (\boldsymbol{C}/\delta\mathfrak{m})/E(F)\},$$

$$V_\infty(L) = \{Z \in \mathfrak{F}_\infty \mid \delta(\bar{z} - z) - |w|^2 > L\},$$

$$V'_\infty(L) = \mathfrak{F}_\infty^{(1)} \cap V_\infty(L)$$

and

$$\mathfrak{F} = \{Z \in \mathfrak{F}_\infty^{(1)} \mid |j(\gamma, Z)| \geqq 1 \quad \text{for every} \quad \gamma \in \Gamma\},$$

with

$$j(\gamma, Z) = (c_1 z + c_2 w + c_3)\left(\gamma = \begin{pmatrix} * & * & * \\ * & * & * \\ c_1 & c_2 & c_3 \end{pmatrix} \in \Gamma\right).$$

By Proposition 1.3 and by Borel's reduction theory of algebraic groups defined over $\boldsymbol{Q}$, we can obtain the following (cf. [1, 2]).

PROPOSITION 2.1. *Let notation be as above. Then $\mathfrak{F}$ is a fundamental domain for Γ and $\mathfrak{F}$ satisfies the relation $V'_\infty(L') \subset \mathfrak{F} \subset V_\infty(L)$ for some L and L'.*

3. Automorphic forms and the Selberg trace formula. First we summarize the fundamental facts about the representations of $GL_2(\boldsymbol{C})$ on finite dimensional vector spaces. We denote by $\rho_m(g)$ the symmetric tensor representation of degree m of $GL_2(\boldsymbol{C})$, i.e.

$$(\rho_m(g)f)(z_1, z_2) = f(az_1 + cz_2, bz_1 + dz_2)$$

for every $f \in V_m$ and for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\boldsymbol{C})$, where V_m is the vector space of homogeneous polynomials of degree m in (z_1, z_2). Put $f_k(z_1, z_2) = (\sqrt{k!(m-k)!})^{-1} z_1^k z_2^{m-k}$ $(0 \leqq k \leqq m)$. We represent $\rho_m(g)$ with respect to a fixed basis $\{f_k(z_1, z_2)\}_{k=0}^m$ in V_m and denote the corresponding matrix by the same symbol. For each positive integer k, put $\rho(g) = (\det(g))^k \rho_m(g)$ for every $g \in GL_2(\boldsymbol{C})$. It is well known that any irreducible polynomial representation of $GL_2(\boldsymbol{C})$ is given in the above way. For each f and f' in V_m, define an inner product $\langle f, f' \rangle = \sum_{k=1}^m k!(m-k)! a_k \bar{b}_k$ with $f(z_1, z_2) = \sum_{k=0}^m a_k z_1^k z_2^{m-k}$ and $f'(z_1, z_2) = \sum_{k=0}^m b_k z_1^k z_2^{m-k}$. It is easily seen that $\langle \rho_m(g)f, f' \rangle = \langle f, \rho_m(g^*)f' \rangle$ for each f and f' in V_m, where $A^* = {}^t(\bar{A})$. Therefore we have

$$\rho(g)^* = \rho(g^*) .$$

For each $g \in G_{\boldsymbol{R}}$ and for each $Z \in D$, we define an automorphy factor $J(g, Z)$ by

$$J(g, Z) = \begin{pmatrix} \bar{b}_2 - \bar{\delta}^{-1}\bar{b}_1 w & (1/\delta)\bar{b}_3 + (1/\delta)\bar{b}_1 z \\ -\bar{c}_2\bar{\delta} + \bar{c}_1 w & \bar{c}_3 + \bar{c}_1 z \end{pmatrix}, \quad \text{where } g = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

and $Z = (z, w)$. For each $(Z, Z') \in D \times D$, put

$$M(Z, Z') = \begin{pmatrix} \delta & w \\ -\bar{w}' & z - \bar{z}' \end{pmatrix}$$

with $Z = (z, w)$ and $Z' = (z', w')$. For each $g \in G_{\boldsymbol{R}}$ and for each $\boldsymbol{C}^{m+1}$-

valued function f on D, we define a C^{m+1}-valued function $f|_\rho g$ on D by

$$(f|_\rho g)(Z) = \rho(J(g, Z))^{-1} f(g(Z)) .$$

We call a C^{m+1}-valued holomorphic function $f(Z)$ on D a *cusp form of weight ρ with respect to $\Gamma(N)$* if the following conditions are satisfied:

(i) $\|\rho(\sqrt{-i^t M(Z, \bar{Z})}) f(Z)\|$ is bounded on D,

(ii) $f|_\rho \gamma = f$ for every $\gamma \in \Gamma(N)$,

where $\|{}^t(a_1, a_2, \cdots, a_{m+1})\| = (\sum_{i=1}^{m+1} |a_i|^2)^{1/2}$. We denote the space of all such functions by $S_\rho(\Gamma(N))$.

For each $(Z, Z') \in D \times D$ and for each $g \in G_R$, put

$$K_{\rho,g}(Z, Z') = \rho({}^t M(Z, g(Z'))^{-1} \rho(J(g, Z')^*)^{-1} \rho({}^t M(Z', Z')) .$$

Define a measure dZ on D by $dZ = (i(-\delta^{-1}|w|^2 + \bar{z} - z))^{-3} dxdydudv$ with $z = x + iy$, $w = u + iv$. It is well known that dZ is a G_R invariant volume element on D. We consider the Hilbert space $\mathfrak{H}^2_\rho(D)$ consisting of all holomorphic C^{m+1}-valued functions f on D satisfying

$$\int_D \|(\rho(\sqrt{-i^t M(Z, \bar{Z})}) f(Z)\|^2 dZ < \infty .$$

Now we can prove the following lemma (cf. [4]).

LEMMA 3.1. *Let the notation be as above. Then*

$$f(Z) = c(\rho) \int_D \rho({}^t M(Z, Z'))^{-1} \rho({}^t M(Z', Z')) f(Z') dZ'$$

holds for every $f \in \mathfrak{H}^2_\rho(D)$ with $c(\rho)^{-1} = 2^{k+m-1}\pi^2(-i\delta)(2k + 2m - 3)!!$ $((2k + 2m - 2)!)^{-1} \sum_{l=0}^m {}_mC_l(m - l)!(l + k - 3)!$, where $(2n - 1)!!$ means $1 \cdot 3 \cdots (2n - 1)$.

PROOF. Put $f(Z) = {}^t(0, \cdots, 0, (\overline{(-2\delta)^{-1} - \bar{z}})^{-(k+m)})$. We can easily check that $f \in \mathfrak{H}^2_\rho(D)$. By the same fashion as in Godement [4], we can show that

$$f(Z) = c(\rho) \int_D \rho({}^t M(Z, Z'))^{-1} \rho({}^t M(Z', Z')) f(Z') dZ'$$

for every $f \in \mathfrak{H}^2_\rho(D)$, where $c(\rho)$ is a constant not depending upon a choice of f. Therefore we have

$$f(Z_0) = c(\rho) \int_D \rho({}^t M(Z_0, Z'))^{-1} \rho({}^t M(Z', Z')) f(Z') dZ'$$

with $Z_0 = (-1/2\delta, 0)$. Thus we obtain

$$c(\rho) \int_D \rho({}^t M(Z_0, Z'))^{-1} \rho({}^t M(Z', Z')) f(Z') dZ' \tag{3.1}$$

$$= c(\rho)\int_D {}^t(*, \cdots, *, \delta^{-k}\{|w'|^2+\delta(z'-\bar{z}')\}^k(z'-\bar{z}')^m|(-1/2\delta)-\bar{z}'|^{-2(k+m)})dZ'$$

$$= c(\rho)(-2^m i\delta^{k+m+1})\int_{2y-|w|^2>0} {}^t(*, \cdots, (2y-|w|^2)^{k-3}y^m(x^2+(y+1/2)^2)^{-(k+m)})$$
$$\times\, dxdydudv\ .$$

A direct calculation shows that

$$-2^m i\delta^{k+m+1}\int_{2y-|w|^2>0}(2y-|w|^2)^{k-3}y^m(x^2+(y+1/2)^2)^{-(k+m)}dxdydudv$$
$$= -i(4\delta)^{k+m+1}4^{-2}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y')^{k-3}(y'+|w|^2)^m(x^2+(y'+|w|^2+1)^2)^{-(k+m)}$$
$$\times\, dxdy'dudv$$
$$= -i\delta^{k+m+1}4^{k+m-1}\pi\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}y^{k-3}(y+r)^m(x^2+(y+r+1)^2)^{-(k+m)}dxdydr$$
$$= -i\delta^{k+m+1}4^{k+m-1}\pi\sum_{l=0}^{m}{}_mC_l(m-l)!(2(k+m)-2)^{-1}(2(k+m)-3)^{-1}\cdots(2(k+m)$$
$$-2-(m-l))^{-1}\int_0^{\infty}y^{l+k-3}(y+1)^{-2(k+m)+2+(m-l)}dy\int_{-\infty}^{\infty}(x^2+1)^{-(k+m)}dx\ .$$

Observe that

$$\int_{-\infty}^{\infty}(1+x^2)^{-(k+m)}dx = B(1/2,\, k+m-1/2)$$

and

$$\int_0^{\infty}y^{l+k-3}(y+1)^{-(2k+m+l-2)}dy = B(l+k-2,\, k+m)\ ,$$

where $B(x, y)$ is the beta function. Thus the value of the integral (3.1) equals

$$-i\delta^{k+m+1}4^{k+m-1}\pi\sum_{l=0}^{m}{}_mC_l(m-l)!(2k+m+l-3)!((2k+m-2)!)^{-1}$$
$$\times\, B(1/2,\, k+m-1/2)B(l+k-2,\, k+m)\ .$$

Therefore we get the explicit value of $c(\rho)$.

Now we prove the following lemma.

LEMMA 3.2. *Let E and E' be compact subsets in D. Suppose that $k-m\geqq 6$. Then the series*

$$\sum_{\gamma\in\Gamma}\|\rho({}^tM(Z, \gamma(Z')))^{-1}\rho(J(\gamma, Z')^*)^{-1}\|$$

converges absolutely and uniformly on $(Z, Z')\in E\times E'$, where $\|A\| = \sqrt{\mathrm{tr}\,(AA^*)}$.

PROOF. We can verify that

$$\|\rho_m({}^tM(Z, Z'))^{-1}\| \, \|\rho_m(\sqrt{-i^tM(Z', Z)})\| \leqq \tilde{c}\,|1 - (2z' + \delta^{-1})(2z' - \delta^{-1})^{-1}|^{-m/2} \tag{3.2}$$

for every $Z \in E$ and for every $Z' \in D$, where $\tilde{c}$ is a constant not depending upon Z, Z'. Let r be a sufficiently small positive number such that $E'' = \{Z'' \in E \mid \exists Z' \in E' \text{ and } \sqrt{|z'' - z'|^2 + |w'' - w'|^2} \leqq r\}$ is contained in D with $Z'' = (z'', w'')$ and $Z' = (z', w')$. We fix $Z \in E$. Then we have

$$\begin{aligned}
&|(\rho({}^tM(Z, \gamma(Z')))^{-1})^* J(\gamma, Z)^{-1})_{i,j}| \\
&\quad \leqq c \int_{E''} |(\rho(\sqrt{-i^tM(Z'', Z'')})\rho({}^tM(Z, \gamma(Z'')){}^{-1})^* J(\gamma, Z'')^{-1})_{i,j}| \, dZ'' \\
&\quad \leqq c' \int_{E''} \|\rho({}^tM(Z, \gamma(Z''))^{-1}\| \, \|\rho(\sqrt{-i^tM(Z'', Z'')})\rho(J(\gamma, Z''))^{-1}\| \, dZ'' ,
\end{aligned}$$

where c and c' are constants. Put $\zeta = \gamma(Z'')$. Then

$$|(\rho({}^tM(Z, \gamma(Z')))^{-1*} J(\gamma, Z')^{-1})_{i,j}| \leqq \tilde{c} \int_{\gamma(E'')} \|\rho(\sqrt{-i^tM(\zeta, \zeta)})\| \, \|\rho({}^tM(Z, \zeta))^{-1}\| d\zeta .$$

By virtue of (3.2), we obtain

$$\int_D \|\rho(\sqrt{-i^tM(\zeta, \zeta)})\| \, \|\rho({}^tM(Z, \zeta))^{-1}\| d\zeta < \infty$$

for any $k \geqq m + 6$. Hence for any $\varepsilon > 0$, there exists a compact subset A in D such that

$$\int_{D-A} \|\rho(\sqrt{-i^tM(\zeta, \zeta)})\| \, \|\rho({}^tM(Z, \zeta))^{-1}\| d\zeta < \varepsilon .$$

Let E be a compact subset in B. Put $p^{-1}(E) = \{g \in G_R \mid g(Z_0) \in E\}$. Then $p^{-1}(E)$ and $B = \cup_{g \in p^{-1}(E)} g(A)$ are compact. Since Γ is discrete, $S = \{\gamma \in \Gamma \mid \gamma(E'') \cap B \neq \emptyset\}$ is a finite set. Put $n = \#\{\gamma \in \Gamma \mid \gamma(E'') \cap E'' \neq \emptyset\}$. Then

$$\begin{aligned}
&\sum_{\gamma \in \Gamma - S} |(\rho({}^tM(Z, \gamma(Z'))^{-1})^* J(\gamma, Z')^{-1})_{i,j}| \\
&\quad \leqq \tilde{c} \sum_{\gamma \in \Gamma - S} \int_{\gamma(E'')} \|\rho(\sqrt{-i^tM(\zeta, \zeta)})\| \, \|\rho({}^tM(Z, \zeta))^{-1}\| \, d\zeta \\
&\quad \leqq n\tilde{c} \int_{D-B} \|\rho({}^tM(Z, \zeta))^{-1}\| \, \|\rho(\sqrt{-i^tM(\zeta, \zeta)})\| \, d\zeta \\
&\quad \leqq n\tilde{c} \int_{D-g(A)} \|\rho({}^tM(Z, \zeta))^{-1}\| \, \|\rho(\sqrt{-i^tM(\zeta', \zeta')})\| d\zeta \\
&\quad \leqq n\tilde{\tilde{c}} \int_{D-A} \|\rho({}^tM(Z_0, \zeta'))^{-1}\| \, \|\rho(\sqrt{-i^tM(\zeta', \zeta')})\| \, \|\rho(\sqrt{-i^tM(Z, Z)})^{-1}\| \, d\zeta' \\
&\quad \leqq n\tilde{\tilde{c}}\varepsilon M ,
\end{aligned}$$

where $Z = g(Z_0)$ and $\zeta = g(\zeta')$ and $M = \max_{Z \in E} \|\rho(\sqrt{-i^tM(Z, Z)})^{-1}\|$. Hence we have proved the assertion:

Put

$$K_\rho^\Gamma(Z, Z') = c(\rho) \sum_{\gamma \in \Gamma(N)} K_{\rho,\gamma}(Z, Z') .$$

Applying Lemma 3.1 and Lemma 3.2 as in Godement [4], we can show the following (3.3)-(3.5):

$$f(Z) = \int_{\mathfrak{F}(N)} K_\rho^\Gamma(Z, Z')f(Z')dZ' \tag{3.3}$$

holds for every $f \in S_\rho(\Gamma(N))$, where $\mathfrak{F}(N)$ is a fundamental domain for $\Gamma(N)$;

$$\|K_\rho^\Gamma(Z, Z)\| \quad \text{is bounded on} \quad \mathfrak{F}(N) ; \tag{3.4}$$

$$\dim S_\rho(\Gamma(N)) = \int_{\mathfrak{F}(N)} \operatorname{tr} K_\rho^\Gamma(Z, Z)dZ . \tag{3.5}$$

By (3.5), we have

$$\dim S_\rho(\Gamma(N)) = c(\rho) \sum_{\beta \in \Gamma/\Gamma(N)} \int_{\mathfrak{F}} \sum_{\gamma'} \operatorname{tr} K_{\rho,\gamma'}(Z, Z)dZ ,$$

where $\gamma' = \beta^{-1}\gamma\beta$ and γ runs over $\Gamma(N)$.

Now we have the following lemma.

LEMMA 3.3. *Let L be a sufficiently large positive number. Then for every $Z \in V_\infty(L)$ we have*

$$\sum_{[\alpha,n] \in \Gamma_\infty} |\operatorname{tr} K_{\rho,[\alpha,n]\gamma}(Z, Z)| \leqq J(-2i\delta y - |w|^2)^{-k+m+2}(\det (J(\gamma, Z)^{*-1})({}^tM(Z, Z)))^{k-m} ,$$

where J is a constant not depending upon a choice of Z and γ.

PROOF. A simple calculation yields that

$$\operatorname{tr} K_{\rho,\gamma}(Z, Z) = \operatorname{tr} (\rho({}^tM(Z, \gamma(Z))^{-1}\rho(J(\gamma, Z)^*)^{-1}\rho({}^tM(Z, Z))) .$$

Set $\rho_1(g) = \det (g)^{k-m}$ and $\rho_2(g) = \det (g)^m \rho_m(g)$. Then $\rho(g) = \rho_1(g)\rho_2(g)$. Thus $|\operatorname{tr} K_{\rho,\gamma}(Z, Z)| = |K_{\rho_1,\gamma}(Z, Z)|\,|\operatorname{tr} (K_{\rho_2,\gamma}(Z, Z))|$. Put $Z = g(Z_0)$. A direct calculation shows that

$$\operatorname{tr} K_{\rho_2,\gamma}(Z, Z) = \operatorname{tr} \{\rho_2(J(g^{-1}\gamma g, Z_0)^*)^{-1}\rho_2({}^tM(Z_0, Z_0))\rho_2({}^tM(g^{-1}\gamma g(Z_0), Z_0)^{-1*})\} .$$

Note that $\|\rho_m(\sqrt{A^{-1*}BA^{-1}})\| > j(\varepsilon)\,\|\rho_m(A^{-1})\|$ for all positive Hermitian matrix B satisfying $B > \varepsilon E_2$, where ε is a positive number and $j(\varepsilon)$ is a function of ε satisfying $j(\varepsilon) > 0$. So we obtain

$$\begin{aligned} |\operatorname{tr} K_{\rho_2,\gamma}(Z, Z)| &\leqq c\,\|\rho_2(J(g^{-1}\gamma g, Z_0)^{*-1})\|\,\|\rho_2({}^tM(g^{-1}\gamma g(Z_0), Z_0))^{-1*}\| \\ &\leqq c'\,\|\rho_2({}^tM(g^{-1}\gamma g(Z_0), Z_0))^{-1}\|\,\|\rho_2(\sqrt{-i{}^tM(g^{-1}\gamma g(Z_0), g^{-1}\gamma g(Z_0))}\,)\| , \end{aligned}$$

where c and c' are constants depending only upon Z_0. Therefore, by (3.2),

$$\sup \{|\operatorname{tr} K_{\rho_2,\gamma}(Z, Z)|\,|\,Z \in D, \gamma \in G_R\} < \infty .$$

Thus

$$|\mathrm{tr}\, K_{\rho,\gamma}(Z, Z)| \leqq M(|\det({}^tM(Z, \gamma(Z))^{-1}J(\gamma, Z)^{*-1}\, {}^tM(Z, Z))|)^{k-m}$$

for every $\gamma \in G_R$ and for every $Z \in D$, where M is a constant not depending upon $\gamma \in G_R$ and $Z \in D$. By the same fashion as in Cohn [3], we can obtain

$$\sum_{[\alpha,n]\in\Gamma_\infty} |\mathrm{tr}\, K_{\rho,[\alpha,n]\gamma}(Z, Z)| \leqq C(-2i\delta y - |w|^2)^{-k+m+2}|\det(J(\gamma, Z)^{*-1}\, {}^tM(Z, Z))|^{k-m} .$$

By Lemma 3.3, we can prove the following.

LEMMA 3.4. *Let* $k \geqq m+6$ (resp. $L_0 > 0$) *be a positive integer* (resp. *a sufficient large number*). *Then*

$$\int_{V'_\infty(L_0)} \sum_\gamma |\mathrm{tr}\, K_{\rho,\gamma}(Z, Z)|\, dZ < \infty ,$$

where γ *runs over* $\beta^{-1}\Gamma(N)\beta - \Gamma_\infty^{(1)} \cap \beta^{-1}\Gamma(N)\beta$ *with* $\beta \in \Gamma'$.

PROOF. Put $\Gamma = \beta^{-1}\Gamma(N)\beta - \Gamma_\infty^{(1)} \cap \beta^{-1}\Gamma(N)\beta$, $\Gamma' = \Gamma_\infty^{(1)} \cap \beta^{-1}\Gamma(N)\beta$ and $\Gamma'' = \Gamma_\infty \cap \beta^{-1}\Gamma(N)\beta$. It is seen that

$$S(Z) = \sum_{\gamma\in\Gamma} |\mathrm{tr}\, K_{\rho,\gamma}(Z, Z)| = \sum_{\gamma\in\Gamma/\Gamma''} \sum_{[\alpha,n]\in\Gamma''} |\mathrm{tr}\, K_{\rho,[\alpha,n]\gamma}(Z, Z)| .$$

It follows from Lemma 3.3 that

$$\begin{aligned} S(Z) &\leqq C \sum_{\gamma\in\Gamma/\Gamma''} (-2i\delta y - |w|^2)^{-k+m+2} |\det((J(\gamma, Z)^{*-1})^t M(Z, Z))|^{k-m} \\ &= C \sum_{\gamma\in\Gamma/\Gamma''} \sqrt{(-2i\delta y - |w|^2)}^{-k+m+4} \sqrt{|\det({}^tM(\gamma(Z), \gamma(Z)))|}^{k-m} \\ &\leqq CL^{-(k-m)/2+2} \sum_{\gamma\in\Gamma/\Gamma''} \sqrt{|\det({}^tM(\gamma(Z), \gamma(Z)))|}^{k-m} . \end{aligned}$$

Since $V'_\infty(L)$ is a Siegel domain, $\{\gamma \in \Gamma \mid \gamma(V'_\infty(L)) \cap V'_\infty(L) \neq \varnothing\}$ is a finite set. Thus we obtain

$$\begin{aligned} &\int_{V'_\infty(L)} \sum_{\gamma\in\Gamma/\Gamma''} \sqrt{|\det({}^tM(\gamma(Z), \gamma(Z)))|}^{k-m} dZ \\ &\qquad \leqq C' \int_{\bigcup_{\gamma\in\Gamma/\Gamma''} \gamma(V'_\infty(L))} \sqrt{|\det({}^tM(Z, Z))|}^{k-m}\, dZ . \end{aligned}$$

Note that $\bigcup_{\gamma\in\Gamma/\Gamma''} \gamma(V'_\infty(L)) = \{(z, w) \in \Gamma_\infty\backslash D \mid -2i\delta y - |w|^2 \leqq L\}$. Therefore we have the desired result.

Next we show the following lemma.

LEMMA 3.5. *Let* $k \geqq m+6$ (resp. $L_0 > 0$) *be an integer* (resp. *a sufficiently large number*). *Then*

$$\int_{V'_\infty(L)} \sum_\gamma \mathrm{tr}\, K_{\rho,\gamma}(Z, Z)\, dZ = \lim_{s\to 0} \sum_\gamma \int_{V'_\infty(L)} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z)(-2i\delta y - |w|^2)^{-ks} dZ ,$$

where γ runs over $\Gamma_\infty^{(1)} \cap \beta^{-1}\Gamma(N)\beta$.

By Lemma 3.3, we have

$$\sum_{\gamma \in \Gamma'/\Gamma''} \sum_{[\alpha,n] \in \Gamma''} |\mathrm{tr}\, K_{\rho,\gamma}(Z, Z)| \leqq c(-2i\delta y - |w|^2)^{-2}$$

with a constant c. Since $(-2i\delta y - |w|^2)^{-ks} \leqq L^{-ks}(Z \in V'_\infty(L))$ and

$$\int_{V'_\infty(L)} (-2i\delta y - |w|^2)^{-1-ks} dxdydudv < \infty ,$$

it follows from Lebesgue's convergence theorem that

$$\sum_{\gamma \in \Gamma'} \int_{V'_\infty(L)} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z)(-2i\delta y - |w|^2)^{-ks-3} dxdydudv$$
$$= \int_{V'_\infty(L)} \sum_{\gamma \in \Gamma'} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z)(-2i\delta y - |w|^2)^{-ks-3} dxdydudv .$$

By Proposition 2.1, (3.4) and Lemma 3.4, we have

$$\int_{V'_\infty(L)} |\sum_{\gamma \in \Gamma'} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z)| dZ < \infty .$$

It follows that

$$\lim_{s \to 0} \int_{V'_\infty(L)} \sum_{\gamma \in \Gamma'} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z)(-2i\delta y - |w|^2)^{-ks-3} dxdydudv$$
$$= \int_{V'_\infty(L)} \sum_{\gamma \in \Gamma'} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z) dZ .$$

Consequently our lemma is proved.

By Proposition 2.1, (3.4), Lemma 3.4 and Lemma 3.5, we have

PROPOSITION 3.1. *Suppose that $k \geqq m + 6$. Then,*

$$\dim S_\rho(\Gamma(N))$$
$$= c(\rho) \sum_{\beta \in \Gamma/\Gamma(N)} \Big[\int_{\mathfrak{F}} \mathrm{tr}\, K_{\rho,E_3}(Z, Z) dZ + \sum_{\gamma} \int_{\mathfrak{F}} \mathrm{tr}\, K_{\rho,\beta^{-1}\gamma\beta}(Z, Z) dZ$$
$$+ \lim_{s \to 0} \sum_{\gamma'} \Big\{ \int_{\mathfrak{F} - V_\infty(L)} \mathrm{tr}\, K_{\rho,\beta^{-1}\gamma'\beta}(Z, Z) dZ + \int_{V_\infty(L)} \mathrm{tr}\, K_{\rho,\beta^{-1}\gamma'\beta}(Z, Z)$$
$$\times (-2i\delta y - |w|^2)^{-ks} dZ \Big\} \Big] ,$$

where γ(resp. γ') runs over $\{\gamma \in \Gamma(N) | \beta^{-1}\gamma\beta \notin \Gamma_\infty^{(1)}\}$ (resp. $\{\gamma' \in \Gamma(N) | \beta^{-1}\gamma'\beta \in \Gamma_\infty^{(1)} - \{E_3\}\}$).

4. Explicit calculation of integrals. Put $H_N = \{\gamma \in \Gamma(N) | \gamma$ is hyperbolic$\}$ and $U_N = \{\gamma \in \Gamma(N) | \gamma$ is parabolic$\}$. By corollary 1.1, we have $\Gamma(N) = H_N \cup U_N \cup \{E_3\}$ (disjoint union). It follows from Proposition 2.1, and Lemma 3.4 that

$$\int_{\mathfrak{F}} \sum_{\gamma \in H_N} |\mathrm{tr}\, K_{\rho,\gamma}(Z, Z)| dZ < \infty .$$

So

$$\sum_{\gamma \in \Gamma(N)} \int_{\mathfrak{F}} \mathrm{tr}\, K_{\rho,\gamma'}(Z, Z) dZ = \sum_{\tilde{\gamma}} \int_{\mathfrak{F}_{\tilde{\gamma}}} \mathrm{tr}\, K_{\rho,\tilde{\gamma}}(Z, Z) dZ , \tag{4.1}$$

where γ runs over $\{\gamma \in \Gamma(N) \mid \gamma' = \beta^{-1}\gamma\beta \notin \Gamma_{\infty}^{(1)}\}$, $\tilde{\gamma}$ runs over all Γ-conjugacy classes in H_N and $\mathfrak{F}_{\tilde{\gamma}}$ is a fundamental domain for the group $\{\gamma \in \Gamma \mid \gamma\tilde{\gamma} = \tilde{\gamma}\gamma\}$.

To verify that the series (4.1) vanishes, it is sufficient to show that $\int_{\mathfrak{F}_\gamma} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z) dZ$ vanishes for $\gamma \in H_N$. For any $\gamma \in \Gamma$, put $C_\gamma = \{g \in \Gamma \mid g\gamma = \gamma g\}$ and $C_\gamma^{\boldsymbol{R}} = \{g \in G_{\boldsymbol{R}} \mid g\gamma = \gamma g\}$. Assume that γ belongs to H_N. Then we can write

$$\int_{\mathfrak{F}_\gamma} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z) dZ = \int_{C_\gamma \backslash C_\gamma^{\boldsymbol{R}}} dZ^1 \int_{C_\gamma^{\boldsymbol{R}} \backslash D} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z) dZ^2 ,$$

where dZ^1(resp. dZ^2) is the restriction of dZ on $C_\gamma^{\boldsymbol{R}}$(resp. the induced measure on $C_\gamma^{\boldsymbol{R}} \backslash D$) (cf. [5, Chap. X (p. 369)]). It is enough to show

$$\int_{C_\gamma^{\boldsymbol{R}} \backslash D} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z) dZ^2 = 0 .$$

Here we may assume

$$\gamma = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} (|\alpha_2| = 1,\ \alpha_1\bar{\alpha}_3 = 1 \text{ and } \alpha_1 \neq \alpha_3)$$

(cf. Prop. 1.1). A simple calculation shows

$$C_\gamma^{\boldsymbol{R}} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \in G_{\boldsymbol{R}} \right\} \quad \text{and} \quad \{(v + i,\ v') \in D\}$$

is a fundamental domain for $C_\gamma^{\boldsymbol{R}}$ in D. Consequently

$$\begin{aligned} &\int_{C_\gamma^{\boldsymbol{R}} \backslash D} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z) dZ^2 \\ &\quad = c \int_0^{\sqrt{-2i\delta}} dv' \int_{-\infty}^{\infty} \psi(v, v') \{\delta((v + i - |\alpha_1|^2(v - i)) + (\bar{\alpha}_2/\bar{\alpha}_3)v'^2\}^{-k-m} dv , \end{aligned}$$

where c is a constant and $\psi(v, v')$ is a polynomial of degree m in (v, v'). Since

$$\int_{-\infty}^{\infty} v^j/(v+a)^{j'}dv = 0 \quad (a \notin \boldsymbol{R},\ j'-j \geqq 2)\ ,$$

$$\int_{C_\gamma^{\boldsymbol{R}}\setminus D} \mathrm{tr}\, K_{\rho,\gamma}(Z, Z)dZ^2 = 0\ .$$

Therefore we conclude that

$$\int \sum_{\gamma} \mathrm{tr}\, K_{\rho,\gamma'}(Z, Z)dZ = 0\ , \tag{4.2}$$

where γ runs over $\{\gamma \in \Gamma(N) \mid \gamma' = \beta^{-1}\gamma\beta \notin \Gamma_\infty^{(1)}\}$.

Next we calculate the integral

$$\begin{aligned} &\lim_{s\to 0} \sum_{\gamma} \int_{\mathfrak{F}} \mathrm{tr}\, K_{\rho,\gamma'}(Z, Z)(-2i\delta y - |w|^2)^{-ks}dZ \\ &\quad = \lim_{s\to 0} \sum_{\tilde{\gamma}} \int_{\mathfrak{F}_{\tilde{\gamma}}} \mathrm{tr}\, K_{\rho,\tilde{\gamma}}(Z, Z)(-2i\delta y - |w|^2)^{-ks}dZ\ , \end{aligned} \tag{4.3}$$

where γ runs over $\{\gamma \in \Gamma(N) \mid \gamma' = \beta^{-1}\gamma\beta \in \Gamma_\infty^{(1)} - \{E_3\}\}$, $\tilde{\gamma}$ runs over all Γ-conjugacy classes in U_N and $\mathfrak{F}_{\tilde{\gamma}}$ is a fundamental domain for $C_{\tilde{\gamma}}$. By Lemma 1.2 and Proposition 1.2, we may assume that $\tilde{\gamma} = [\alpha, n]$.

First we treat the case where $\tilde{\gamma} = [0, n]$. A simple calculation yields that $\mathfrak{F}_{\tilde{\gamma}} = \mathfrak{F}_\infty^{(1)}$ (cf. Lemma 2.1). Since

$$\mathrm{tr}\, K_{\rho,\tilde{\gamma}}(Z, Z) = \sum_{i=0}^{m} ((-2i\delta y - |w|^2)/(-2i\delta y + \delta n - |w|^2))^{k+i}\ ,$$

the integral in the sum of the right hand side of (4.3) is equal to

$$\begin{aligned} &\lim_{s\to 0} \int_0^{n_0} dx \int_{\left\{\substack{-2i\delta y - |w|^2 > 0 \\ w \in (\boldsymbol{C}/\delta\mathfrak{m})/E(F)}\right.} (-i\delta^{-3})^{-1} \sum_{j=0}^{m} (-2i\delta y - |w|^2)^{k+j} \\ &\qquad \times (-2i\delta y + \delta n - |w|^2)^{-k-j}(-2i\delta y - |w|^2)^{-3-ks} dy du dv \\ &= \lim_{s\to 0} (-i\delta^{-3})^{-1} \sum_{j=0}^{m} n_0(-2i\delta)^{-1} \mathrm{vol}\,((\boldsymbol{C}/\delta\mathfrak{m})/E(F)) \int_0^{\infty} y^{k+j-ks-3}(y+\delta n)^{-k-j}dy \\ &= (-i\delta^{-3})^{-1} \sum_{j=0}^{m} n_0(-2i\delta)^{-1} \mathrm{vol}\,((\boldsymbol{C}/\delta\mathfrak{m})/E(F))\{|i\delta n|^{2+(k+j)ks(k+j)^{-1}}\}^{-1} \\ &\qquad \times (k+j-1)^{-1}(k+j-2)^{-1}\phi(k(k+j)^{-1}ks) \\ &\qquad \times \exp(-\{\mathrm{sgn}\,(n)\pi i((k+j)ks(k+j)^{-1}+2)\}/2)\ , \end{aligned}$$

where $\mathrm{vol}\,((\boldsymbol{C}/\delta\mathfrak{m})E(F)) = \int_{(\boldsymbol{C}/\delta\mathfrak{m})E(F)} du dv (w = u + iv)$, $(4/\delta) \cap \boldsymbol{Q} = (n_0)$ $(n_0 > 0)$ and $\phi(s) \to 1 (s \to 0)$.

If $\tilde{\gamma} = [\alpha, n] (\alpha \neq 0)$, then, by a simple calculation, we can show that the integral

$$\int_{\mathfrak{F}_{\tilde{\gamma}}} \mathrm{tr}\, K_{\rho,\tilde{\gamma}}(Z, Z)(-2i\delta y - |w|^2)^{-ks}dZ$$

vanishes. Consequently we have established the following theorem.

THEOREM. *Let F be an imaginary quadratic field of class number one. Suppose that $k \geqq m+6$ and $N \geqq 3$. Then*

$$\dim S_\rho(\Gamma(N))$$
$$=\{2^{k+m-1}\pi^2(-i\delta)(2k+2m-3)!!((2k+2m-2)!)^{-1}\sum_{l=0}^{m}{}_mC_l(m-l)!(l+k-3)!\}^{-1}$$
$$|\Gamma/\Gamma(N)|\{(m+1)\operatorname{vol}(\Gamma\backslash D)+\delta^2 n_0(|\delta|^2 n_1^2)^{-1}\zeta(2)\operatorname{vol}(\boldsymbol{C}/\delta\mathfrak{m})|E(F)|^{-1}$$
$$\times\sum_{j=0}^{m}((k+j-1)(k+j-2))^{-1}\},$$

where $\operatorname{vol}(\Gamma\backslash D)=\int_{\mathfrak{F}} dZ$, $\operatorname{vol}(\boldsymbol{C}/\delta\mathfrak{m})=\int_{\boldsymbol{C}/\delta\mathfrak{m}} dudv(w=u+iv)$, $\boldsymbol{Q}\cap(4/\delta)=(n_0)$, $\boldsymbol{Q}\cap(4/\delta)\cap(N)=(n_1)(n_0, n_1>0)$ *and* $\zeta(s)$ *is the Riemann zeta function. The volumes of* $\Gamma\backslash D$ *and* $\boldsymbol{C}/\delta\mathfrak{m}$ *are given as follows*:

$$\operatorname{vol}(\Gamma\backslash D)=2^{-3}\{|\delta|^2(i\delta^{-1})^3\}^{-1}\pi^2 L(-2,\chi)\zeta(-1)\times\begin{cases}1 & \text{if } F\neq \boldsymbol{Q}(\sqrt{-3})\\ 3 & \text{if } F=\boldsymbol{Q}(\sqrt{-3}),\end{cases}$$

where $L(s,\chi)=\zeta_F(s)/\zeta(s)$ *and* $\zeta_F(s)$ *is the Dedekind zeta function of* F *(see* [6, 12]*), and*

$$\operatorname{vol}(\boldsymbol{C}/\delta\mathfrak{m})=\begin{cases}4|\sqrt{d}| & \text{if } d\equiv 2,3(4)\\ |\sqrt{d}| & \text{if } d\equiv 1(4),\end{cases}$$

where d is the discriminant of F.

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ADDED IN PROOF. The referee has informed the author that H. Koseki calculated the traces of Hecke operators acting on the spaces of automorphic forms on $SU(1,2)$ and $SU(3)$.