Tôhoku Math. Journ.
36 (1984), 555-562.

ON ZETA-FUNCTIONS AND CYCLOTOMIC Z_p-EXTENSIONS OF ALGEBRAIC NUMBER FIELDS

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(Received September 20, 1983)

1. In Tate [5] and Turner [7], the following result is proved:

THEOREM. *Let k, k' be function fields in one variable over a finite constant field $\boldsymbol{F}$ and ζ_k, $\zeta_{k'}$ Dedekind zeta-functions of k, k'. Let C, C' be complete non-singular curves defined over $\boldsymbol{F}$ with function fields isomorphic to k, k' and $J(C)$, $J(C')$ the Jacobian varieties of C, C'. Then the following are equivalent:*

(1) $\zeta_k = \zeta_{k'}$.

(2) *$J(C)$ and $J(C')$ are $\boldsymbol{F}$-isogenous.*

In the present paper, we shall investigate the situation which arises when we replace the function fields by the algebraic number fields. In [2] and [3], Iwasawa discussed analogues of Jacobian varieties in this situation. We shall see that these analogues play some roles in this question.

Let $\boldsymbol{Q}$ be the rational number field, k, k' finite algebraic extensions of $\boldsymbol{Q}$ and ζ_k, $\zeta_{k'}$ the Dedekind zeta-functions of k and k', respectively. Perlis [4] gave interesting consequences from $\zeta_k = \zeta_{k'}$. Using his method, we shall obtain the following results:

Let p be a prime number, $k(p)$ the maximal abelian pro-p-extension of k and $G_k(p)$ the Galois group of $k(p)$ over k. For these and also for other notations which will be introduced afterwards, we adopt similar notations for k'. Let $\boldsymbol{Z}_p$ be the p-adic integer ring and k_∞ the cyclotomic $\boldsymbol{Z}_p$-extension of k. We shall prove that $\zeta_k = \zeta_{k'}$ implies $G_k(p) \cong G_{k'}(p)$ and $G_{k_\infty}(p) \cong G_{k'_\infty}(p)$ for almost all prime numbers p. Let $\tilde{k}_\infty$ the maximal unramified abelian pro-p-extension of k_∞ and $Y_k(p)$ the Galois group of $\tilde{k}_\infty/k_\infty$. Let A and A' be the p-primary subgroups of ideal class groups of k_∞ and k'_∞, respectively. Let $X_k(p)$ be the Pontrjagin dual of the discrete group A. Let α_p be a primitive p-th root of 1. We shall prove that $\zeta_k = \zeta_{k'}$ implies $X_{k(\alpha_p)}(p) \cong X_{k'(\alpha_p)}(p)$ and $Y_{k(\alpha_p)}(p) \cong Y_{k'(\alpha_p)}(p)$ for almost all prime numbers p. The duals of $X_{k(\alpha_p)}(p)$ and $Y_{k(\alpha_p)}(p)$ are regarded as analogies of the Jacobian variety in our situation (cf. [2], [3]), so that this can be interpreted as an analogue of the fact that (1) implies (2) in the case of function fields. We are not in a position now to prove an analogue of (2) implies (1) in our case, but it is conjectured that $k \neq \boldsymbol{Q}$ would imply that there exist some primes p such that $Y_k(p) \neq 0$. (This can be in fact proved in case k is not totally real, as shown below.) In our last paragraph, we shall give such p's for some real quadratic fields k.

In this paper, $\boldsymbol{Z}$ and $\boldsymbol{R}$ denote the ring of rational integers and the field of real numbers. As already mentioned, $\boldsymbol{Q}$ denotes the rational number field. For a finite algebraic number field k, we denote by $k_A^\times$ the idele group of k.

2. Let k and k' be finite algebraic number fields such that $\zeta_k = \zeta_{k'}$. Let L be the Galois closure of k over $\boldsymbol{Q}$. It is well known that $L \supset k'$ and that the degree $(k; \boldsymbol{Q})$ is equal to $(k'; \boldsymbol{Q})$. Let G be the Galois group $G(L/\boldsymbol{Q})$ of L over $\boldsymbol{Q}$, $H = G(L/k)$ and $H' = G(L/k')$. Let $s = (k; \boldsymbol{Q})$. Let D and D' be the linear representations of G induced by the unit representations of H and H'. Let $\boldsymbol{Z}$ be the integer ring and $M_s(Z)$ the set of all integral $s \times s$ matrices. We put

$$\mathfrak{M}_0 = \{M \in M_s(\boldsymbol{Z}) \mid \det(M) \neq 0,\ D(g)M = MD'(g) \text{ for every } g \in G\}\,.$$

By [4] and [7], we see that $\mathfrak{M}_0$ is not empty. The following Lemma is also proved in [4].

LEMMA 1 (cf. [4, Theorem 1]). Let $\nu = \gcd\{\det(M) \mid M \in \mathfrak{M}_0\}$. *Then every prime number dividing* ν *divides* $(L; k)$.

Let $\rho_1, \cdots, \rho_s$ and $\rho'_1, \cdots, \rho'_s$ be representatives for left cosets of G by H and H', with $\rho_1 = \rho'_1 = 1$. Let $L^\times$ be the multiplicative group of L. For a matrix $A = (a_{ij}) \in M_s(Z)$, we now define an endomorphism μ_A of $L^\times$ by $\mu_A(x) = \prod_{i=1}^s \rho_i(x)^{a_{i1}}$ for $x \in L^\times$. We also define an endomorphism of $L^\times$ by $\mu'_A(x) = \prod_{i=1}^s \rho'_i(x)^{a_{i1}}$. Then we have the following:

LEMMA 2 (cf. [4, Lemma 5]). *For matrices* A *and* B *in* $\mathfrak{M}_0$ *and for* $a \in k^\times$, *we have*

(1) $\mu_A(k^\times) \subset k'^\times$.

(2) $\mu_{B^t}(\mu_A(a)) = \mu_{AB^t}(a)$. *Here* B^t *is the transpose of* B.

Let k^{ab} be the maximal abelian extension of k. Let M be a matrix in $\mathfrak{M}_0$. We now define a homomorphism of $G(k^{ab}/k)$ into $G(k'^{ab}/k')$ induced by μ_M.

LEMMA 3. *Let* v *be a place of* $\boldsymbol{Q}$, $\boldsymbol{Q}_v$ *the completion of* $\boldsymbol{Q}$ *at* v *and* $k \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v$ *the tensor product of* k *and* $\boldsymbol{Q}_v$. *Then there exists a continuous homomorphism* $\mu_{M,v}$ *of* $(k \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v)^\times$ *into* $(k' \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v)^\times$ *such that* $i'(\mu_M(a)) = \mu_{M,v}(i(a))$ *for any element* a *of* $k^\times$. *Here* i *is a natural injec-*

tion of k into $k \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v$, while i' is a natural injection of k' into $k' \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v$.

PROOF. Let $w_1, \cdots, w_m$ be the places of L lying above v. Let φ_j be a multiplicative valuation belonging to w_j. For positive number η, we put $V_k(\eta) = \{a \in k^\times \mid \varphi_j(a-1) < \eta \ \ j = 1, \cdots, m\}$. For any positive number ε there exists a positive number δ such that $\mu_M(V_k(\delta)) \subset V_{k'}(\varepsilon)$. Hence our assertion follows from the fact that k is dense in $k \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v$.

Let $v_1, \cdots, v_{r_1}$ be the real places of k and $v_{r_1+1}, \cdots, v_{r_1+r_2}$ the imaginary places of k; $v'_1, \cdots, v'_{r'_1}$ the real places of k' and $v'_{r'_1+1}, \cdots, v'_{r'_1+r'_2}$ the imaginary places of k'. Since we have $\zeta_k = \zeta_{k'}$, we have $r_1 = r'_1$ and $r_2 = r'_2$. We put $k^\times_{v_j,+} = \{a \in k_{v_j} \mid a > 0\}$ for $j = 1, \cdots, r_1$; $k^\times_{\infty,+} = \prod_{j=1}^{r_1} k^\times_{v_j,+} \times \prod_{j=r_1+1}^{r_1+r_2} k^\times_{v_j}$ and $k'^\times_{\infty,+} = \prod_{j=1}^{r_1} k'^\times_{v'_j,+} \times \prod_{j=r_1+1}^{r_1+r_2} k'^\times_{v'_j}$. Let u be the infinite place of $\boldsymbol{Q}$. Since $\mu_{M,u}$ is continuous, we have $\mu_{M,u}(k^\times_{\infty,+}) \subset k'^\times_{\infty,+}$. Let $a = (a_v)$ be an element of $k^\times_A$ such that $a_v \in (k \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_v)^\times$. We can define a continuous homomorphism $\bar{\mu}_M$ of $k^\times_A$ into $k'^\times_A$ by $\bar{\mu}_M(a) = (\mu_{M,v}(a_v))$. Let $U_k = \overline{k^\times k^\times_{\infty,+}/k^\times}$ be the topological closure of $k^\times k^\times_{\infty,+}/k^\times$ in the idele class group $C_k = k^\times_A/k^\times$. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be the Artin mappings of C_k/U_k onto $G(k^{ab}/k)$ and of $C_{k'}/U_{k'}$ onto $G(k'^{ab}/k')$. Since $\bar{\mu}_M(k^\times) \subset k'^\times$ and $\mu_{M,u}(k^\times_{\infty,+}) \subset k'^\times_{\infty,+}$, we can define a continuous homomorphism $\tilde{\mu}_M; G(k^{ab}/k) \to G(k'^{ab}/k')$ making the diagram

$$
\begin{array}{ccc}
k^\times_A & \xrightarrow{\ \bar{\mu}_M\ } & k'^\times_A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
C_k/U_k & & C_{k'}/U_{k'} \\
{\scriptstyle \mathfrak{A}}\downarrow & & \downarrow{\scriptstyle \mathfrak{A}'} \\
C(k^{ab}/k) & \xrightarrow[\ \tilde{\mu}_M\ ]{} & G(k'^{ab}/k')
\end{array}
$$

commutative. Here f and f' are canonical homomorphisms of $k^\times_A$ into C_k/U_k and of $k'^\times_A$ into $C_{k'}/U_{k'}$. For simplicity, μ_M will denote $\tilde{\mu}_M$ in the following;

THEOREM 1. *Let k and k' be finite algebraic extensions of $\boldsymbol{Q}$ such that $\zeta_k = \zeta_{k'}$. Let k^{ab} be the maximal abelian extension of k. Let G be the Galois group $G(k^{ab}/k)$ and G' the Galois group $G(k'^{ab}/k')$. For a prime number p, we denote by $G(p)$ the pro-p-sylow subgroup of G. Then there exists a continuous homomorphism μ of G into G' such that the restriction of μ to $G(p)$ is an isomorphism of $G(p)$ onto $G'(p)$ for almost all p.*

PROOF. Let M be a matrix in $\mathfrak{M}_0$. Let B be the matrix $(\det(M)M^{-1})^t$, which belongs to $\mathfrak{M}_0$. We have defined the continuous homomorphism μ_M of G into G'. In a similar way, we can define a continuous homomorphism μ'_{Bt} of G' into G. From Lemma 2, we have $\mu'_{Bt}(\mu_M(g)) = g^{\det(M)}$ for all $g \in G$. In a similar way, we have $\mu_M(\mu'_{Bt}(g')) = g'^{\det(M)}$ for all $g' \in G'$. Let p be a prime number such that p does not divide $\det(M)$. Then we have

$$\mu_M(G(p)) \supset \mu_M(\mu'_{Bt}(G'(p))) = \{g'^{\det(M)} \mid g' \in G'(p)\} = G'(p) .$$

Suppose that $\mu_M(g) = 1$ for $g \in G(p)$. We have $g^{\det(M)} = 1$. Since p is prime to $\det(M)$, we have $g = 1$.

3. Let k and k' be finite algebraic number fields such that $\zeta_k = \zeta_{k'}$. We put $s = (k; \boldsymbol{Q})$. Let L be as before the Galois closure of k over $\boldsymbol{Q}$ and p a prime number such that p does not divide $(L; \boldsymbol{Q})$. Let $\boldsymbol{Z}_p$ be the p-adic integer ring and $\boldsymbol{Q}^{(\infty,p)}$ the cyclotomic $\boldsymbol{Z}_p$-extension of $\boldsymbol{Q}$. Then there exists a sequence of fields $\boldsymbol{Q} = \boldsymbol{Q}^{(0,p)} \subset \boldsymbol{Q}^{(1,p)} \subset \cdots \subset \boldsymbol{Q}^{(n,p)} \subset \cdots \subset \boldsymbol{Q}^{(\infty,p)}$ such that $\boldsymbol{Q}^{(n,p)}/\boldsymbol{Q}$ is a cyclic extension of degree p^n, $n \geqq 0$. We put $k_n = k\boldsymbol{Q}^{n,p}$, $k'_n = k'\boldsymbol{Q}^{(n,p)}$, $L_n = L\boldsymbol{Q}^{(n,p)}$ and $L_\infty = L\boldsymbol{Q}^{(\infty,p)}$. We put furthermore $G = G(L_\infty/\boldsymbol{Q})$, $H_n = G(L_n/k_n)$, $H'_n = G(L_\infty/k'_n)$, $N_n = G(L_\infty/L_n)$ and $S = G(L_\infty/\boldsymbol{Q}^{(\infty,p)})$. Then we have $G = S \times N_0$. Let γ be a topological generator of N_0. We have the following:

LEMMA 4 (cf. [8, Lemma 1]). *Let k and k' be finite algebraic number fields such that $\zeta_k = \zeta_{k'}$. Let K be a finite Galois extension of $\boldsymbol{Q}$. Then we have $\zeta_{Kk} = \zeta_{Kk'}$.*

We have $\zeta_{k_n} = \zeta_{k'_n}$ from this Lemma 4. Let D_n and D'_n be the linear representations of G induced by the unit representations of H_n and H'_n. We should notice that we can regard D_0 and D'_0 as representations of S. Let R_n be the linear representation of N_0 induced by the unit representation of N_n. Let $D_0 \otimes R_n$ be the tensor product of D_0 and R_n. Then we have $D_n = D_0 \otimes R_n$ and $D'_n = D'_0 \otimes R_n$. We put

$$\mathfrak{M}_n = \{M \in M_{sp^n}(\boldsymbol{Z}) \mid \det(M) \neq 0,\ D_n(g)M = MD'_n(g) \text{ for every } g \in G\} .$$

We can easily show the following:

LEMMA 5. *Let M be a matrix in $\mathfrak{M}_0$ and I_{p^n} the unit matrix of degree p^n. Let $M \otimes I_{p^n}$ be the Kronecker product of M and I_{p^n}. Then we have $M \otimes I_{p^n} \in \mathfrak{M}_n$.*

We put $M_n = M \otimes I_{p^n}$. We see easily the following:

LEMMA 6. *Let M be a matrix in $\mathfrak{M}_0$. Let m and n be non-negative integers such that $m \leqq n$. Let μ_{M_m} and μ_{M_n} be the above endomorphisms of $L_m^\times$ and $L_n^\times$. Let N_{k_n/k_m} and $N_{k'_n/k'_m}$ be the norms of k_n/k_m and k'_n/k'_m. Then we have $\mu_{M_m}(N_{k_n/k_m}(x)) = N_{k'_n/k'_m}(\mu_{M_n}(x))$ for all $x \in k_n^\times$.*

By Lemma 1, there exists a matrix $M \in \mathfrak{M}_0$ such that p does not divide $\det(M)$. We have $\det(M_n) = \pm(\det(M))^{p^n}$. Hence Theorem 1, Lemma 6 and class field theory yield the following:

THEOREM 2. *Let k and k' be finite algebraic number fields such that $\zeta_k = \zeta_{k'}$. Let L be the Galois closure of $k/\boldsymbol{Q}$ and p a prime number which does not divide $(L; \boldsymbol{Q})$. Let k_∞ and k'_∞ be the cyclotomic $\boldsymbol{Z}_p$-extensions of k and k'. Let $\hat{k}_\infty$ and $\hat{k}'_\infty$ be the maximal abelian pro-p-extensions of k_∞ and k'_∞. Then the Galois group $G(\hat{k}_\infty/k_\infty)$ and $G(\hat{k}'_\infty/k'_\infty)$ are isomorphic as topological groups.*

Let p be an odd prime number which does not divide $(L; \boldsymbol{Q})$. Let A_n and A'_n be the Sylow p-subgroups of the ideal class groups of k_n and of k'_n, respectively. For $0 \leqq m \leqq n$, there exists a natural homomorphism $f_{m,n}: A_m \to A_n$ induced by the imbedding of the ideal group of k_m in that of k_n. Let A and A' denote the direct limits of A_n, $n \geqq 0$ and of A'_n, $n \geqq 0$, with respect to the above homomorphisms. Let Λ denote the ring of power series in an indeterminate T with coefficients of $\boldsymbol{Z}_p$: $\Lambda = \boldsymbol{Z}_p[[T]]$. Let $X_k(p)$ and $X_{k'}(p)$ be the duals of the discrete abelian group A and of A'. We can consider $X_k(p)$ and $X_{k'}(p)$ as Λ-modules in the usual manner (cf. [3]). Let M be a matrix in $\mathfrak{M}_0$ such that p does not divide $\det(M)$. We put $M_n = M \otimes I_{p^n}$. For a finite place v of k, we denote by r_v the integer ring of $(k_n)_v$ and by $r_v^\times$ the unit group of r_v. Since we have

$$\bar{\mu}_{M_n}\Big(k_n^\times\big((k_n \otimes_{\boldsymbol{Q}} \boldsymbol{R})^\times \times \prod_{v;\ \text{the finite places of } k_n} r_v^\times\big)\Big) \subset k_n'^\times\Big((k'_n \otimes_{\boldsymbol{Q}} \boldsymbol{R})^\times \times \prod_{v';\ \text{the finite places of } k'_n} r_v'^\times\Big)$$

and since p does not divide $\det(M_n)$, we can induce the isomorphism μ_n of A_n onto A'_n by $\bar{\mu}_{M_n}$. Then, for $0 \leqq m \leqq n$, we can show that $\mu_n(f_{m,n}(a)) = f'_{m,n}(\mu_m(a))$ for all $a \in A_m$. Hence we have the following:

THEOREM 3. *Let k and k' be finite algebraic number fields such that $\zeta_k = \zeta_{k'}$. Let L be the Galois closure of $k/\boldsymbol{Q}$ and p an odd prime number which does not divide $(L; \boldsymbol{Q})$. Let $X_k(p)$ and $X_{k'}(p)$ be as above. Then $X_k(p)$ and $X_{k'}(p)$ are isomorphic as topological Λ-modules.*

Lemma 4 and Theorem 3 yield the following:

COROLLARY. *Notations and assumptions being as above, let α_p be a primitive p-th root of 1. Then we have $X_{k(\alpha_p)}(p) \cong X_{k'(\alpha_p)}(p)$.*

Let $\tilde{k}_\infty$ be the maximal unramified abelian pro-p-extension of k_∞. We put $Y_k(p) = G(\tilde{k}_\infty/k_\infty)$. We can consider $Y_k(p)$ as Λ-module in the usual manner (cf. [3]). Lemma 6 and class field theory yield the following:

THEOREM 4. *Let k and k' be finite algebraic number fields such that $\zeta_k = \zeta_{k'}$. Let L be the Galois closure of $k/\boldsymbol{Q}$ and p a prime number which does not divide $(L; \boldsymbol{Q})$. Let k_∞ and k'_∞ be the cyclotomic $\boldsymbol{Z}_p$-extensions of k and of k', respectively. Let $\tilde{k}_\infty$ and $\tilde{k}'_\infty$ be the maximal unramified abelian pro-p-extensions of k_∞ and of k'_∞, respectively. Then the Galois group $Y_k(p) = G(\tilde{k}_\infty/k_\infty)$ and $Y_{k'}(p) = G(\tilde{k}'_\infty/k'_\infty)$ are isomorphic as topological Λ-modules.*

COROLLARY. *Notations and assumptions being as above, let α_p be a primitive p-th root of 1. Then we have $Y_{k(\alpha_p)}(p) \cong Y_{k'(\alpha_p)}(p)$.*

4. It would be interesting to examine whether $Y_k(p) \cong Y_{k'}(p)$ for almost all prime numbers p implies $\zeta_k = \zeta_{k'}$. We shall examine now whether $Y_k(p) = 0$ for any prime number p implies $\zeta_k = \zeta_{\boldsymbol{Q}}$. We notice that $Y_{\boldsymbol{Q}}(p) = 0$ for any prime number p follows from Iwasawa [1] and that $\zeta_k = \zeta_{\boldsymbol{Q}}$ implies $k = \boldsymbol{Q}$. For a finite algebraic number field F, we denote by h_F the class number of F and by E_F the group of units in F. Let K be a cyclic extension of F and a_K the number of ambiguous ideal classes with respect to K/F. The following Lemma is well known:

LEMMA 7 (cf. [9]). *Let K be a cyclic extension of a number field F. Then we have*

$$a_K = h_F \times \prod_v e(v) \times ((K; F)(E_F; E_F \cap N_{K/F}(K)))^{-1} ,$$

where $\prod_v e(v)$ is the product of the ramification indices of all the finite and infinite places in F with respect to K/F.

COROLLARY. *If $Y_k(p) = 0$ for all prime numbers p, then k is totally* real.

PROOF. Let p be a prime number which splits completely in $k/\boldsymbol{Q}$. We put $k_n = k\boldsymbol{Q}^{(n,p)}$. If k is not totally real, it follows from Lemma 7 that p^n divides h_{k_n}. This shows that $Y_k(p)$ is not trivial.

In the rest of this section, we shall give examples of real quadratic fields F and prime numbers p such that $Y_F(p) \neq 0$. Since the center of p-groups are non-trivial, we have the following:

LEMMA 8. *Let K be a cyclic p-extension of F. Then the prime number $p \mid h_K$ if and only if $p \mid a_K$.*

Now, we put $1 + p^n\boldsymbol{Z}_p = \{x \in \boldsymbol{Z}_p \mid x \equiv 1 \pmod{p^n}\}$. Let α_{p-1} be a primitive $(p-1)$-th root of 1. Then local class field theory yields the following:

LEMMA 9. *Let $\boldsymbol{Q}_p$ be the p-adic number field and $\boldsymbol{Q}_{p,n} = \boldsymbol{Q}_p\boldsymbol{Q}^{(n,p)}$. Then we have $N_{\boldsymbol{Q}_{p,n}/\boldsymbol{Q}_p}(\boldsymbol{Q}_{p,n}^\times) = \langle p \rangle \times \langle \alpha_{p-1} \rangle \times (1 + p^{n+1}\boldsymbol{Z}_p)$, where $\langle p \rangle$ and $\langle \alpha_{p-1} \rangle$ are the subgroups generated by p and by α_{p-1} in $\boldsymbol{Q}_p^\times$, respectively.*

PROPOSITION. *Let F be a real quadratic field and ε a fundamental unit of F. We assume that an odd prime number p splits completely in F and that p does not divide h_F. We put $F_n = F\boldsymbol{Q}^{(n,p)}$. Then the following conditions are equivalent:*

(1) *The prime number p divides h_{F_1}.*

(2) $\varepsilon^{p-1} \equiv 1 \pmod{p^2\boldsymbol{Z}_p}$.

(3) $\varepsilon^{p^n-p^{n-1}} \equiv 1 \pmod{p^{n+1}\boldsymbol{Z}_p}$ *for all positive integers n.*

(4) *The prime number p divides h_{F_n} for all positive integers n.*

PROOF. Since p is an odd prime, it is clear that (2) and (3) are equivalent. Let us show the equivalence of (1) and (2). Assume that $\varepsilon^{p-1} \equiv 1 \pmod{p^2\boldsymbol{Z}_p}$. Then from Lemma 9 and Hasse's norm theorem follows that there exists an element η of F_1 such that $N_{F_1/F}(\eta) = \varepsilon$. Hence it follows from Lemma 7 that p divides h_{F_1}. Now, assume $p \mid h_{F_1}$. It follows from Lemma 8 that $p \mid a_{F_1}$. Since $p \nmid h_F$, Lemma 7 yields that $E_F \subset N_{F_1/F}(F_1)$. Hence, from Lemma 9 follows that $\varepsilon \equiv 1 \pmod{p^2\boldsymbol{Z}_p}$. We can similarly prove that (3) and (4) are equivalent.

According to this Proposition, we have only to examine whether (2) holds for F and p to know whether $Y_F(p) \neq 0$ holds. We have examined this for $F = \boldsymbol{Q}(\sqrt{d})$ and found the following pairs (d, p) for which we have $Y_{\boldsymbol{Q}(\sqrt{d})}(p) \neq 0$:

d	2	6	19	23	31	33	37	41	43	57	62
p	31	523	79	7	157	29	7	7221	3	59	263

The author would like to thank Mr. T. Fukuda at the Tokyo Institute of Technology who helped him in making this table using a computer of the institute and Prof. H. Wada at Sophia University who kindly made a program for this calculation. He is also thankful to Profs. S. Iyanaga and T. Kanno for their encouragement and advises.

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