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LIMITS OF SEQUENCES OF RIEMANN SURFACES REPRESENTED BY SCHOTTKY GROUPS

(To Professor Yukio Kusunoki on the occasion of his 60th birthday)

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0. Introduction. In this paper, we state an application of the interchange operators introduced in the previous paper [8]. We consider the following problem. We give a point τ in an augmented Schottky space $\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$ associated with $\tilde{\Sigma}_0$, which represents a compact Riemann surface S with nodes. Then for any sequence of points $\{\tau_n\}$ in the Schottky space $\mathfrak{S}_g(\tilde{\Sigma}_0)$ associated with $\tilde{\Sigma}_0$ tending to the point τ, does the Riemann surfaces $S(\tau_n)$ represented by τ_n converge to S as marked surfaces as $n \to \infty$?

The answer to this problem is negative in the general case, namely in the case where $\tilde{\Sigma}_0$ is a basic system of Jordan curves (see § 1.2 for the definition). However the answer is affirmative in a special case, namely in the case where $\tilde{\Sigma}_0$ is a standard system of Jordan curves (see § 1.2 for the definition). Now the following question arises: To what Riemann surfaces does the sequence of Riemann surfaces $\{S(\tau_n)\}$ converge as marked surfaces as $n \to \infty$ in the general case? The answer is the main result (Theorem 2 in § 6) in this paper.

We use the same notation and terminologies as in [8]. In § 1, we will define convergence of Riemann surfaces, and in § 2, we will show the following: For any point τ in an augmented Schottky space, there exists a sequence of points $\{\tau_n\}$ in the Schottky space tending to τ such that the sequence of Riemann surfaces $\{S(\tau_n)\}$ represented by τ_n converges to the Riemann surface $S(\tau)$ represented by τ as marked surfaces as $n \to \infty$. In § 3, we will construct a new surface from a given surface. From § 4 through § 6, we will state and prove the main theorem. In § 7, we will explain the result by an example.

1. Definitions and terminologies

1.1. We use the same notation and terminologies as in the previous papers [7, 8].

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DEFINITION 1. Let S be a compact Riemann surface of genus g without (resp. with) nodes. We call the set $\Sigma = \{\alpha_1, \alpha_2, \cdots, \alpha_g; \gamma_1, \gamma_2, \cdots, \gamma_{2g-3}\}$ of loops (resp. loops and nodes) on S having the following property *a basic system of loops* (resp. *a basic system of loops and nodes*) *on* S: Each component of $S - \bigcup_{i=1}^{g} \alpha_i - \bigcup_{j=1}^{2g-3} \gamma_j$ is a planar and triply connected region of type [3, 0] (resp. [3, 0], [2, 1], [1, 2] or [0, 3]), where a surface of type $[m, n]$ means the sphere with m disks removed and n points deleted. If, in particular, the number of nondividing loops (resp. the number of nondividing loops and nondividing nodes) is equal to g, we call Σ *a standard system of loops* (resp. *a standard system of loops and nodes*) *on* S.

Let $\langle G_0 \rangle$ be a marked Schottky group generated by $A_{0,1}, A_{0,2}, \cdots, A_{0,g}$: $\langle G_0 \rangle = \langle A_{0,1}, A_{0,2}, \cdots, A_{0,g} \rangle$.

DEFINITION 2. If mutually disjoint Jordan curves $C_{0,1}, C_{0,2}, \cdots, C_{0,2g}, C_{0,2g+1}, C_{0,2g+2}, \cdots, C_{0,4g-3}$ on $\hat{C} = C \cup \{\infty\}$ have the following properties (i)-(iii), then we call $\tilde{\Sigma}_0 = \{C_{0,1}, \cdots, C_{0,2g}; C_{0,2g+1}, \cdots, C_{0,4g-3}\}$ *a basic system of Jordan curves for* $\langle G_0 \rangle$: (i) $C_{0,1}, C_{0,g+1}; C_{0,2}, C_{0,g+2}; \cdots, C_{0,g}, C_{0,2g}$ are defining curves of $A_{0,1}, A_{0,2}, \cdots, A_{0,g}$, respectively. Namely they comprize the boundary of $2g$-ply connected region ω_0, and $A_{0,i}$ maps $C_{0,i}$ onto $C_{0,g+i}$ and $A_{0,i}(\omega_0) \cap \omega_0 = \emptyset$ for each $i = 1, 2, \cdots, g$. (ii) $C_{0,2g+j}$ ($j = 1, 2, \cdots, 2g-3$) lie in ω_0. (iii) Each component of $\omega_0 - \bigcup_{j=1}^{2g-3} C_{0,2g+j}$ is a triply connected planar region. If, in particular, a basic system of Jordan curves $\tilde{\Sigma}_0$ has the following property (iv), we call $\tilde{\Sigma}_0$ *a standard system of Jordan curves for* $\langle G_0 \rangle$: (iv) For each $i = 1, 2, \cdots, g$ and $j = 1, 2, \cdots, 2g-3$, $C_{0,i}$ and $C_{0,g+i}$ lie on the same side of $C_{0,2g+j}$.

We let $C_{0,i(1)}, C_{0,i(2)}, \cdots, C_{0,i(k)}, C_{0,g+i'(1)}, \cdots, C_{0,g+i'(l)}$ and $C_{0,j(1)}, C_{0,j(2)}, \cdots, C_{0,j(m)}, C_{0,g+j'(1)}, \cdots, C_{0,g+j'(n)}$ be the defining curves in $\tilde{\Sigma}_0$ in the interior and to the exterior to $C_{0,2g+j}$, respectively, where $i(1) < \cdots < i(k) \leqq g$, $i'(1) < \cdots < i'(l) \leqq g$; $j(1) < \cdots < j(m) \leqq g$, $j'(1) < \cdots < j'(n) \leqq g$. Then we say that the curve $C_{0,2g+j}$ gives a *partition* $\{i(1), \cdots, i(k), g + i'(1), \cdots, g + i'(l)\} \cup \{j(1), \cdots, j(m), g + j'(1), \cdots, g + j'(n)\}$ of the set $\{1, 2, \cdots, 2g\}$.

Let S be a compact Riemann surface of genus g with or without nodes and let $\Sigma = \{\alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3}\}$ a basic system of loops and nodes on S. Cut the surface S along the loops and nodes α_i ($i=1, 2, \cdots, g$). We denote by $\alpha'_{0,i}$ and $\alpha'_{0,g+i}$ the resulting two topological circles or two points for each i. We call $\Sigma' = \{\alpha'_1, \alpha'_2, \cdots, \alpha'_{2g}; \gamma_1, \cdots, \gamma_{2g-3}\}$ *the set of Jordan curves and points induced from* Σ, or simply *the induced set from* Σ. Each γ_j devides the set $\{\alpha'_1, \alpha'_2, \cdots, \alpha'_{2g}\}$ into two parts $\{\alpha'_{i(1)}, \cdots, \alpha'_{i(k)},$

$\alpha'_{g+i'(1)}, \cdots, \alpha'_{g+i'(l)}\}$ and $\{\alpha'_{j(1)}, \cdots, \alpha'_{j(m)}, \alpha'_{g+j'(1)}, \cdots, \alpha'_{g+j'(n)}\}$, where $i(1) < \cdots < i(k) \leqq g$, $i'(1) < \cdots < i'(l) \leqq g$; $j(1) < \cdots < j(m) \leqq g$, $j'(1) < \cdots < j'(n) \leqq g$. Then we say that γ_j gives a *partition* $\{i(1), \cdots, i(k), g+i'(1), \cdots, g+i'(l)\} \cup \{j(1), \cdots, j(m), g+j'(1), \cdots, g+j'(n)\}$ of the set $\{1, 2, \cdots, 2g\}$. If each γ_j $(j = 1, 2, \cdots, 2g-3)$ gives the same partition as $C_{0,2g+j}$, we say Σ' is *compatible* with $\tilde{\Sigma}_0$.

Let S_1 and S_2 be compact Riemann surfaces of genus g with or without nodes. Let $\Sigma_1 = \{\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1,g}; \gamma_{11}, \gamma_{12}, \cdots, \gamma_{1,2g-3}\}$ and $\Sigma_2 = \{\alpha_{21}, \alpha_{22}, \cdots, \alpha_{2,g}; \gamma_{21}, \gamma_{22}, \cdots, \gamma_{2,2g-3}\}$ be basic systems of loops and nodes on S_1 and S_2, respectively. Let Σ'_1 and Σ'_2 be the induced sets from Σ_1 and Σ_2, respectively. If each $\gamma_{1,j}$ $(j = 1, 2, \cdots, 2g-3)$ gives the same partition as $\gamma_{2,j}$, we say Σ'_1 is *compatible* with Σ'_2.

1.2. Let S be a compact Riemann surface of genus g with or without nodes. We denote by $N(S)$ the set of all nodes on S. From now on, we assume that $g \geqq 2$ and that each component of $S \setminus N(S)$ has the hyperbolic metric, that is, the Poincaré metric. The Poincaré metric $\lambda(z)|dz|$ on S is defined as the Poincaré metric on each component of $S \setminus N(S)$.

DEFINITION 3 (Abikoff [1, p. 30]). Let S_1 and S_2 be compact Riemann surfaces of genus g with or without nodes. If the following (i) and (ii) are satisfied, we call a continuous surjection $f: S_1 \to S_2$ a *deformation*, and denote it by $\langle S_1, S_2, f\rangle$:

(i) $f^{-1}|S'_2$ is a homeomorphism, where $S'_2 = S_2 \setminus N(S_2)$.

(ii) $f^{-1}(\text{node})$ is a node or a simple loop.

Let $\Sigma_1 = \{\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1,g}; \gamma_{11}, \gamma_{12}, \cdots, \gamma_{1,2g-3}\}$ and $\Sigma_2 = \{\alpha_{21}, \alpha_{22}, \cdots, \alpha_{2g}; \gamma_{21}, \gamma_{22}, \cdots, \gamma_{2,2g-3}\}$ be basic systems of loops and nodes on S_1 and S_2, respectively. We assume that Σ_1 and Σ_2 have the induced sets Σ'_1 and Σ'_2, respectively such that Σ'_1 is compatible with Σ'_2, and we write $\Sigma_1 \sim \Sigma_2$ for the fact. From now on, we consider a deformation $\langle S_1, S_2, f\rangle$ satisfying the following (i) and (ii): (i) If α_{2i} (resp. γ_{2j}) is a loop, then $f^{-1}(\alpha_{2i})$ (resp. $f^{-1}(\gamma_{2j})$) is homotopic to α_{1i} (resp. γ_{1j}). (ii) If α_{2i} (resp. γ_{2j}) is a node, then $f^{-1}(\alpha_{2i}) = \alpha_{1i}$ (resp. $f^{-1}(\gamma_{2j}) = \gamma_{1j}$) in the case where α_{1i} (resp. γ_{1j}) is a node, and $f^{-1}(\alpha_{2i})$ (resp. $f^{-1}(\gamma_{2j})$) is homotopic to α_{1i} (resp. γ_{1j}) in the case where α_{1i} (resp. γ_{1j}) is a loop. Set $P(S_1) = f^{-1}(N(S_2))$. We note that $P(S_1) \supset N(S_1)$.

Let S and S_n $(n = 1, 2, \cdots)$ be compact Riemann surfaces of genus g with or without nodes. Let Σ and Σ_n be basic systems of loops and nodes on S and S_n, respectively, with $\Sigma_n \sim \Sigma$. Let $\langle S_n, S, f_n\rangle$ be a deformation satisfying the above (i) and (ii).

DEFINITION 4. If the following condition is satisfied, *a sequence of Riemann surfaces* $\{S_n\}$ *converges to a surface* S *as marked surfaces*: There exists a locally quasiconformal mapping $\phi_n: S\setminus N(S) \to S_n \setminus P(S_n)$ such that

(i) $\lambda_n(\phi_n(z))|d\phi_n(z)|$ uniformly converge to $\lambda(z)|dz|$ on every compact subset of $S\setminus N(S)$, where $\lambda_n(z)|dz|$ and $\lambda(z)|dz|$ are the Poincaré metrics on S_n and S, respectively,

(ii) ϕ_n maps a deleted neighborhood $N(\alpha_i)\setminus\{\alpha_i\}$ (resp. $N(\gamma_j)\setminus\{\gamma_j\}$) of α_i (resp. γ_j) to a deleted neighborhood $N(\alpha_{i,n})\setminus\{\alpha_{i,n}\}$ (resp. $N(\gamma_{j,n})\setminus\{\gamma_{j,n}\}$) of $\alpha_{i,n}$ (resp. $\gamma_{j,n}$) if $\alpha_i \in N(S)$ (resp. $\gamma_j \in N(S)$), and

(iii) ϕ_n maps a neighborhood $N(\alpha_i)$ (resp. $N(\gamma_j)$) of α_i (resp. γ_j) to a neighborhood $N(\alpha_{i,n})$ (resp. $N(\gamma_{j,n})$) of $\alpha_{i,n}$ (resp. $\gamma_{j,n}$) if $\alpha_i \notin N(S)$ (resp. $\gamma_j \notin N(S)$).

When S_n converges to S as marked surfaces, we write $(S_n, \Sigma_n) \to (S, \Sigma)$.

1.3. From now on, we fix a marked Schottky group $\langle G_0\rangle = \langle A_{0,1}, A_{0,2}, \cdots, A_{0,2g}\rangle$ and a basic system of Jordan curves $\tilde{\Sigma}_0 = \{C_{0,1}, \cdots, C_{0,2g}; C_{0,2g+1}, \cdots, C_{0,4g-3}\}$ for $\langle G_0\rangle$. We denote by $\Omega(G_0)$ the region of discontinuity of $\langle G_0\rangle$. Then $S_0 = \Omega(G_0)/\langle G_0\rangle$ is a compact Riemann surface of genus g without nodes. Let $\Pi_0: \Omega(G_0) \to S_0$ be the natural projection. Set $\alpha_{0,i} = \Pi_0(C_{0,i})$ $(i = 1, 2, \cdots, g)$ and $\gamma_{0,j} = \Pi_0(C_{0,2g+j})$ $(j = 1, 2, \cdots, 2g-3)$. Then $\Sigma_0 = \{\alpha_{0,1}, \alpha_{0,2}, \cdots, \alpha_{0,g}; \gamma_{0,1}, \gamma_{0,2}, \cdots, \gamma_{0,2g-3}\}$ is a basic system of loops on S_0.

We denote by $\mathfrak{S}_g(\tilde{\Sigma}_0)$ and $\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$ the Schottky space and the augmented Schottky space associated with $\tilde{\Sigma}_0$, respectively (see [7, p. 28] and [7, p. 32] for the definitions). Let $\tau \in \hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$. Let $S(\tau)$ be the compact Riemann surface with or without nodes represented by τ (see [7, p. 33] for the definition). Let $\langle G_j(\tau)\rangle$ $(j = 0, 1, \cdots, 2g-3)$ be the j-th marked Schottky groups associated with τ, which are defined in [6, pp. 73–75]. In particular, if $\tau \in \mathfrak{S}_g(\tilde{\Sigma}_0)$, then $\langle G_j(\tau)\rangle = T_j\langle G(\tau)\rangle T_j^{-1}$ for some $T_j \in M\ddot{o}b$. Let $\Omega(G_j(\tau))$ be the region of discontinuity of $\langle G_j(\tau)\rangle$. Let $\Omega'(G_j(\tau))$ be the set $\Omega(G_j(\tau))$ deleted the set of all images of the distinguished points under $\langle G_j(\tau)\rangle$ (see [7, p. 31] for the definition of distinguished points). We denote by $\lambda^{(j)}(\tau, z)|dz|$ the Poincaré metric on $\Omega'(G_j(\tau))$.

Let I and J be subsets of $\{1, 2, \cdots, g\}$ and $\{1, 2, \cdots, 2g-3\}$, respectively. We define the set $I(J)$ as in [7, p. 30]. We assume that $I \supset I(J)$ throughout this paper. We define subsets $\delta^I\mathfrak{S}_g(\tilde{\Sigma}_0)$, $\delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$, $\cdots$ of the augmented Schottky space $\hat{\mathfrak{S}}_g^*(\Sigma_0)$ as in [7].

PROPOSITION 1. (1) *Let* $\tau \in \delta^I\mathfrak{S}_g(\tilde{\Sigma}_0)$. *Suppose that* $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ *is a sequence of points tending to the point* τ. *Then* $\Omega(G(\tau_n))$ *tends to* $\Omega'(G(\tau))$.

Furthermore, $\lambda(\tau_n, z)$ *uniformly converges to* $\lambda(\tau, z)$ *on every compact subset of* $\Omega'(G(\tau))$.

(2) *Let* $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$. *Suppose that* $\{\tau_n\} \subset \delta^I\mathfrak{S}_g(\tilde{\Sigma}_0)$ *is a sequence of points tending to* τ. *Then* $\Omega'(G_j(\tau_n))$ *tends to* $\Omega'(G_j(\tau))$ *for each* $j = 0, 1, 2, \cdots, 2g-3$. *Futhermore,* $\lambda^{(j)}(\tau_n, z)$ *uniformly converges to* $\lambda^{(j)}(\tau, z)$ *on every compact subset of* $\Omega'(G_j(\tau))$.

This proposition is shown by similar method as in Bers [3] and Sato [5]. From Proposition 1, we easily see the following.

PROPOSITION 2. *Given* $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$. *Then there exists a sequence* $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ *tending to* τ *such that for each* $j = 0, 1, \cdots, 2g-3$, $\lambda^{(j)}(\tau_n, z)$ *uniformly converges to* $\lambda^{(j)}(\tau, z)$ *on every compact subset of* $\Omega'(G_j(\tau))$.

2. Construction of locally quasiconformal mappings. We use the same notations as in § 1. Here we will construct locally quasiconformal mappings ϕ_n of $\Omega'(G_j(\tau))$ into $\Omega'(G_j(\tau_n))$ in three cases, Case I in § 2.1, Cases II and III in § 2.2.

2.1. Case I. Let $\tau \in \delta^I\mathfrak{S}_g(\tilde{\Sigma}_0)$ and let $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ be a sequence of points tending to τ.

Let $\langle G(\tau_n)\rangle = \langle A_1(\tau_n, z), A_2(\tau_n, z), \cdots, A_g(\tau_n, z)\rangle$ and $\langle G(\tau)\rangle = \langle A_i(\tau, z) | i \notin I\rangle$, where the latter represents a marked Schottky group generated by $A_i(\tau, z)$ $(i \notin I)$ to the number of $g - |I|$ and $|I|$ is the cardinality of I. Let $C_i(\tau_n)$, $C_{g+i}(\tau_n)$ $(i = 1, 2, \cdots, g)$ be defining curves of $\langle G(\tau_n)\rangle$. We denote by $\omega(G(\tau_n))$ the fundamental domain for $\langle G(\tau_n)\rangle$ bounded by the $2g$ Jordan curves $C_i(\tau_n)$ and $C_{g+i}(\tau_n)$ $(i = 1, 2, \cdots, g)$. Let $C_i(\tau)$ $C_{g+i}(\tau)$ $(i \notin I)$ be defining curves for $\langle G(\tau)\rangle$. We denote by $\omega(G(\tau))$ the fundamental domain for $\langle G(\tau)\rangle$ bounded by the $2g - 2|I|$ defining curves. For simplicity, we write ω for $\omega(G(\tau))$. We may assume that $C_i(\tau_n)$ (resp. $C_{g+i}(\tau_n)$) converge to $C_i(\tau)$ (resp. $C_{g+i}(\tau)$) for $i \notin I$. Let $p_{i,n}$ and $p_{g+i,n}$ be the repelling and the attracting fixed points of $A_i(\tau_n, z)$, respectively. We write p_i, p_{g+i} $(i \in I)$ for the distinguished points of the first kind (see [7, p. 31] for the definition). We set $\omega' = \omega - \{p_i, p_{g+i} | i \in I\}$. We may assume that for $i \in I$, $C_i(\tau_n)$ and $C_{g+i}(\tau_n)$ converge to p_i and p_{g+i}, respectively, and that $\omega(G(\tau_n))$ converges to ω'.

For $i \in I$, we define deleted $r(n)$-neighborhoods $N_n(p_i)$ and $N_n(p_{g+i})$ $(n = 1, 2, \cdots)$ of p_i and p_{g+i}, respectively, as follows, where $r(n)$ are positive numbers: If $p_i \neq \infty$ and $p_{g+i} \neq \infty$,

$$N_n(p_i) = \{z \in \omega' \,|\, |z - p_i| < r(n)\}$$

and

$$N_n(p_{g+i}) = \{z \in \omega' \,|\, |z - p_{g+i}| < r(n)\} ,$$

if $p_i = \infty$ or $p_{g+i} = \infty$,

$$N_n(p_i) = \{z \in \omega' \,|\, |z| > 1/r(n)\}$$

or

$$N_n(p_{g+i}) = \{z \in \omega' \,|\, |z| > 1/r(n)\} .$$

For simplicity, we write C_i and C_{g+i} for $C_i(\tau)$ and $C_{g+i}(\tau)$, respectively. Similarly, we define $r(n)$-neighborhood $N_n(C_i)$ and $N_n(C_{g+i})$ of C_i and C_{g+i}, respectively:

$$N_n(C_i) = \{z \in \omega' \,|\, d_E(z, C_i) < r(n)\}$$

and

$$N_n(C_{g+i}) = \{z \in \omega' \,|\, d_E(z, C_{g+i}) < r(n)\} ,$$

where $d_E(z, C)$ denotes the Euclidean distance from the point z to the curves C.

We denote by $\partial N_n(p_i), \partial N_n(C_i), \cdots$ the boundaries of $N_n(p_i)$, $N_n(C_i), \cdots$. Set $B_n(p_i) = \partial N_n(p_i) \cap \omega'$. $B_n(p_{g+i}) = \partial N_n(p_{g+i}) \cap \omega'$, $B_n(C_i) = \partial N_n(C_i) \cap \omega'$ and $B_n(C_{g+1}) = \partial N_n(C_{g+i}) \cap \omega'$. We note that $N_n(p_i)$, $N_n(p_{g+i})$ $(i \in I)$, $N_n(C_k)$ and $N_n(C_{g+k})$ $(k \notin I)$ are mutually disjoint if $r(n)$ is sufficiently small. We choose a sequence $\{r(n)\}$ $(n = 1, 2, \cdots)$ as follows:

(i) $r(1) > r(2) > \cdots > r(n) > r(n+1) > \cdots$ and $\lim_{n\to\infty} r(n) = 0$.

(ii) $B_n(p_i)$, $B_n(p_{g+i})$ $(i \in I)$ and $B_n(C_k)$, $B_n(C_{g+k})$ $(k \notin I)$ bound a $2g$-ply connected region ω_n contained in ω.

(iii) $B_n(p_i) \subset \omega(\tau_n)$, $B_n(p_{g+i}) \subset \omega(\tau_n)$ $(i \in I)$, $B_n(C_k) \subset \omega(\tau_n)$ and $B_n(C_{g+k}) \subset \omega(\tau_n)$ $(k \notin I)$.

We denote by $D_{i,n}$ (resp. $D_{g+i,n}$) the annulus bounded by $B_n(p_i)$ (resp. $B_n(p_{g+i})$) and $C_i(\tau_n)$ (resp. $C_{g+i}(\tau_n)$) for $i \in I$. Similarly, we denote by $D_{k,n}$ (resp. $D_{g+k,n}$) the annulus bounded by $B_n(C_k)$ (resp. $B_n(C_{g+k})$) and $C_k(\tau_n)$ (resp. $C_{g+k}(\tau_n)$).

We construct a mapping ϕ_n of $\Omega'(G(\tau))$ into $\Omega(G(\tau_n))$ in Case I as follows.

First step. (1) $\phi_n =$ id. in ω_n, where id. means the identity mapping.

(2) In $N_n(p_i)$ (resp. $N(p_{g+i})$) for $i \in I$, ϕ_n is a locally quasiconformal mapping of $N_n(p_i)$ (resp. $N_n(p_{g+i})$) onto $D_{i,n}$ (resp. $D_{g+i,n}$) such that $\phi_n =$ id. on $B_n(p_i)$ (resp. $B_n(p_{g+i})$).

(3) In $N_n(C_k)$ (resp. $N_n(C_{g+k})$) for $k \in I$, ϕ_n is a locally quasiconformal mapping of the closure of $N_n(C_k)$ (resp. $N_n(C_{g+k})$) onto the closure of $D_{k,n}$ (resp. $D_{g+k,n}$) such that $\phi_n =$ id. on $B_n(C_k)$ (resp. $B_n(C_{g+k})$) and that ϕ_n satisfies a relation

$$A_k(\tau_n, \phi_n(z)) = \phi_n(A_k(\tau, z)) \quad \text{for} \quad z \in C_k .$$

Second step. ϕ_n is exteded to the domain $\Omega'(G(\tau))$ as follows. For

$z \in \Omega'(G(\tau))$, there exists an element $A(\tau, z)$ of $G(\tau)$ with $A(\tau, z) \in \omega'$, which is represented as a word in $A_1(\tau, z), \cdots, A_g(\tau, z)$:

$$A(\tau, z) = W(A_1(\tau, z), \cdots, A_g(\tau, z)) . \tag{1}$$

Let $A(\tau_n, z)$ be the word obtained by replacing $A_i(\tau, z)$ in (1) with $A_i(\tau_n, z)$ for all $i = 1, 2, \cdots, g$. By setting

$$\tilde{\phi}_n(z) = A^{-1}(\tau_n, \phi_n(A(\tau, z))) ,$$

we define a mapping $\tilde{\phi}_n$ of $\Omega'(G(\tau))$ into $\Omega(G(\tau_n))$. We write again ϕ_n for $\tilde{\phi}_n$.

2.2. Case II. Let $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$ and let $\{\tau_n\} \subset \delta^I\mathfrak{S}_g(\tilde{\Sigma}_0)$ be a sequence of points tending to τ.

We similarly define $\omega_j = \omega(G_j(\tau))$ and $\omega(G_j(\tau_n))$ as in Case I. Set $\omega'_j = \omega_j \cap \Omega'(G_j(\tau))$. We set

$$I_j = \{i \mid p_i \text{ are the distinguished points of the first kind in } \omega_j\}$$

and

$$I'_j = \{i \mid C_i \text{ are defining curves for } \langle G_j(\tau)\rangle \text{ in } \omega_j\} .$$

Set

$$J_j = \{l \in J \mid p_l^{\pm}(\tau) \text{ are the distinguished points of the second kind in } \omega_j\}$$

(see [7, p. 31] for the definition of the distinguished points of the second kind). See [6, pp. 16–18] for the definitions of I_j, I'_j and J_j. We set $|I_j| + |I'_j| = g_j$. Then g_j is the genus of the Riemann surface $S_j(\tau) = \Omega(G_j(\tau))/\langle G_j(\tau)\rangle$.

The sets $N_n(p_i)$, $N_n(p_{g+i})$ $(i \in I_j)$, $N_n(C_k)$, $N_n(C_{g+k})$ $(k \in I'_j)$, $B_n(p_i)$, $B_n(p_{g+i})$, $B_n(C_k)$ and $B_n(C_{g+k})$ are similarly defined as in Case I. Let $p_i(\tau_n)$ and $p_{g+i}(\tau_n)$ $(i \in I_j)$ be the distinguished points of the first kind for τ_n in ω_j. Let $N_n(p_i(\tau_n))$ (resp. $N_n(p_{g+i}(\tau_n))$ be the set $N_n(p_i) \cup \{p_i\} \setminus \{p_i(\tau_n)\}$ (resp. $N_n(p_{g+i}) \cup \{p_{g+i}\} \setminus \{p_{g+i}(\tau_n)\}$).

For $l \in J_j$, we define deleted $r(n)$-neighborhood $N_n(p_l^{\pm})$ as follows: If $p_l^{\pm} \neq \infty$,

$$N_n(p_l^{\pm}) = \{z \in \omega'_j \mid |z - p_l^{\pm}| < r(n)\} ;$$

if $p_l^{\pm} = \infty$,

$$N_n(p_l^{\pm}) = \{z \in \omega'_j \mid |z| > 1/r(n)\} .$$

We set $B_n(p_l^{\pm}) = \partial N_n(p_l^{\pm}) \cap \omega'_j$.

Let $C_{2g+l}(\tau_n)$ $(l \in J_j)$ be Jordan curves in $\omega(G_j(\tau_n))$ which give the same partitions of the set $\{1, 2, \cdots, 2g\}$ as $C_{0,2g+l}$ (see [7, p. 33] for partition). We choose a sequence $\{r(n)\}$ $(n = 1, 2, \cdots)$ as follows:

(i) $r(1) > r(2) > \cdots > r(n) > r(n+1) > \cdots$ and $\lim_{n\to\infty} r(n) = 0$,

(ii) $B_n(p_i)$, $B_n(p_{g+i})$ $(i \in I_j)$, $B_n(C_k)$, $B_n(C_{g+k})$ $(k \in I'_j)$ and $B_n(p_l^{\pm})$ $(l \in J_j)$ bound a $2g_j + |J_j|$-ply connected region ω_n contained in ω, and

(iii) $B_n(p_i)$, $B_n(p_{g+i})$ $(i \in I_j)$, $B_n(C_k)$, $B_n(C_{g+k})$ $(k \in I'_j)$ are contained in $\omega(G_j(\tau_n))$ and $C_{2g+l}(\tau_n)$ $(l \in J_j)$ are contained in $N_n(p_l^{\pm})$.

Let $D_{k,n}$, $D_{g+k,n}$ $(k \in I'_j)$ be the same annuli as in § 2.1. We denote by $D'_{l,n}$ $(l \in J_j)$ the annuli bounded by $C_{2g+l}(\tau_n)$ and $B_n(p_l^{\pm})$.

A mapping ϕ_n of $\Omega'(G_j(\tau))$ into $\Omega'(G_j(\tau_n))$ in Case II is defined as follows.

First step. (1) $\phi_n = \text{id.}$ in ω'_n.

(2) For each $i \in I_j$, ϕ_n is a locally quasiconformal mapping of $N_n(p_i)$ (resp. $N_n(p_{g+i})$) onto $N_n(p_i(\tau_n))$ (resp. $N_n(p_{g+i}(\tau_n))$ such that $\phi_n = \text{id.}$ on $B_n(p_i)$ (resp. $B_n(p_{g+i})$).

(3) For each $k \in I'_j$, ϕ_n is similarly defined as in Case I, (3) in $N_n(C_k)$ and $N_n(C_{g+k})$.

(4) For each $l \in J_j$, ϕ_n is a locally quasiconformal mapping of $N_n(p_l^{\pm})$ onto $D'_{l,n}$ such that $\phi_n = \text{id.}$ on $B_n(p_l^{\pm})$.

Second step. ϕ_n is extended to the domain $\Omega'(G_j(\tau))$ by the same method as in the second step of Case I.

Case III. Let $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$ and let $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ be a sequence of points tending to τ.

In this case, a mapping ϕ_n of $\Omega'(G_j(\tau))$ into $\Omega(G(\tau_n))$ is defined by combining the methods of Cases I and II.

2.3. Let S be a compact Riemann surface of genus g with or without nodes. When Σ is a basic system of loops (or loops and nodes) on S such that Σ', one of the set induced from Σ, is compatible with $\tilde{\Sigma}_0$, we write $\Sigma \sim \tilde{\Sigma}_0$ for the fact.

PROPOSITION 3. *Given $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0) \subset \hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$. Suppose that $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ is a sequence of points tending to the point τ so that $\lambda^{(j)}(\tau_n, z)$ uniformly converges to $\lambda^{(j)}(\tau, z)$ on every compact subset of $\Omega'(G_j(\tau))$ for each $j = 0, 1, 2, \cdots, 2g-3$. Let Σ_n and Σ be a basic system of loops on $S(\tau_n)$ and a basic system of loops and nodes on $S(\tau)$, respectively, with $\Sigma_n \sim \tilde{\Sigma}_0 \sim \Sigma$. Then $S(\tau_n)$ converges to $S(\tau)$ as marked surfaces, that is, $(S(\tau_n), \Sigma_n) \to (S(\tau), \Sigma)$ as $n \to \infty$.*

PROOF. Let ϕ_n be the quasiconformal mapping of $\Omega'(G_j(\tau))$ into $\Omega(G_j(\tau_n))$ as defined in §§ 2.1 and 2.2. We define a function $\lambda_n^{*(j)}(\tau, z)$ on $\Omega'(G_j(\tau))$ by setting

$$\lambda_n^{*(j)}(\tau, z) = \lambda^{(j)}(\tau_n, \phi_n(z))|d\phi_n(z)/dz| .$$

By the above construction, $\lambda_n^{*(j)}(\tau, z)$ uniformly converges to $\lambda^{(j)}(\tau, z)$ on every compact subset K of $\Omega'(G_j(\tau))$, since for sufficiently large n, $\phi_n | K =$ id. and so $\lambda_n^{*(j)}(\tau, z) = \lambda^{(j)}(\tau_n, z)$ for $z \in K$, and $\lambda^{(j)}(\tau_n, z)$ uniformly converges to $\lambda^{(j)}(\tau, z)$ on K by the assumption.

Let $\Pi_n\colon \Omega(G_j(\tau_n)) \to S(\tau_n)$ and $\Pi\colon \Omega'(G_j(\tau)) \to S_j'(\tau)$ be the natural projections, where $S_j'(\tau) = S_j(\tau) \setminus (S_j(\tau) \cap N(S(\tau)))$ if we set $S_j(\tau) = \Omega(G_j(\tau))/\langle G_j(\tau)\rangle$. We define $\lambda_n^{*(j)}(\hat{z})|d\hat{z}|$ and $\lambda^{(j)}(\hat{z})|d\hat{z}|$ on $S_j'(\tau)$ by setting

$$\lambda_n^{*(j)}(\hat{z})|d\hat{z}| = \lambda_n^{*(j)}(\tau, z)|dz|$$

and

$$\lambda^{(j)}(\hat{z})|d\hat{z}| = \lambda^{(j)}(\tau, z)|dz| ,$$

respectively, where $\hat{z} = \Pi(z)$. Since $\lambda^{(j)}(\tau, z)|dz|$ and $\lambda_n^{*(j)}(\tau, z)|dz|$ are invariant under $\langle G_j(\tau)\rangle$, $\lambda_n^{*(j)}(\hat{z})|d\hat{z}|$ and $\lambda^{(j)}(\hat{z})|d\hat{z}|$ are well-defined. Furthermore, we define $\lambda_n^{(j)}(\hat{z})|d\hat{z}|$ on $S(\tau_n)$ by setting

$$\lambda_n^{(j)}(\hat{z})|d\hat{z}| = \lambda^{(j)}(\tau_n, z)|dz| ,$$

where $\hat{z} = \Pi_n(z)$. This is also well-defined.

We easily see that

$$\lambda_n^{*(j)}(\hat{z})|d\hat{z}| = \lambda_n^{(j)}(\hat{z}_n)|d\hat{z}_n| ,$$

where $\hat{z} = \Pi(z)$ and $\hat{z}_n = \Pi_n\phi_n(z)$ for $z \in \Omega'(G_j(\tau))$. By the above, we easily see that $\lambda_n^{*(j)}(\hat{z})|d\hat{z}|$ uniformly converges to $\lambda^{(j)}(\hat{z})|d\hat{z}|$ on every compact subset K_j of $S_j'(\tau)$ for each $j = 0, 1, 2, \cdots, 2g-3$. If we denote by $\hat{\phi}_n$ the projection of ϕ_n onto $S_j'(\tau)$, we have that

$$\lambda_n^{*(j)}(\hat{z})|d\hat{z}| = \lambda_n^{(j)}(\hat{\phi}_n(\hat{z}))|d\hat{\phi}_n(\hat{z})| .$$

Therefore $\lambda_n^{(j)}(\hat{\phi}_n(\hat{z}))|d\hat{\phi}_n(\hat{z})|$ uniformly converges to $\lambda^{(j)}(\hat{z})|d\hat{z}|$ on every compact subset of $S_j'(\tau)$ for each $j = 0, 1, 2, \cdots, 2g-3$. Hence $(S(\tau_n), \Sigma_n) \to (S(\tau), \Sigma)$. Our proof is now complete.

From Propositions 2 and 3, we have the following.

THEOREM 1. *Given a point $\tau \in \hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$. Then there exists a sequence of points $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ tending to τ such that $S(\tau_n)$ converges to $S(\tau)$ as marked surfaces.*

3. Constuction of new surfaces.

3.1. Let $\langle G_0\rangle$, $\tilde{\Sigma}_0$, Σ_0 and S_0 be as in § 1. Let I and J be subsets of $\{1, 2, \cdots, g\}$ and $\{1, 2, \cdots, 2g-3\}$, respectively. Assume that $I(J) \subset I$.

Given $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$, there exists a compact Riemann surface $S(\tau)$ of genus g with $|I| + |J|$ nodes represented by τ. We will construct a new surface from $S(\tau)$ as follows.

We denote by J_1 the subset of J consisting of all j such that $\gamma_{0,j}$ are dividing loops on S_0. Let $J_2 = \{j_1, j_2, \cdots, j_m\}$ be any subset of $J \setminus J_1$. Set $I(J_2) = \{i_1, i_2, \cdots, i_n\}$. We denote by $\tilde{\Sigma}_1$ and Σ_1 the images of $\tilde{\Sigma}_0$ and Σ_0, respectively, under the interchange operator $I_g(i_{k(1)}, j_{l(1)})$ where $i_{k(1)} \in I(\{j_{l(1)}\})$ (see [8] for the interchange operator). We set $J_{21} = J_2 \setminus \{j_{l(1)}\}$. We denote by $I_1(J_{21})$ the set $I(J_{21})$ defined for cycles in Σ_1 (see [8]). We note that $I_1(J_{21}) \subset I(J_2)$.

Choose $j_{l(2)} \in J_{21}$ such that $I_1(\{j_{l(2)}\}) \cap (I(J_2) \setminus \{i_{k(1)}\}) \neq \emptyset$. We apply the interchange operator $I_g(i_{k(2)}, j_{l(2)})$ to $\tilde{\Sigma}_1$ and Σ_1, where $i_{k(2)} \in I_1(\{j_{l(2)}\})$ and $i_{k(2)} \neq i_{k(1)}$. We denote by $\tilde{\Sigma}_2$ and Σ_2 the images of $\tilde{\Sigma}_1$ and Σ_1, respectively. We set $J_{22} = J_{21} \setminus \{j_{l(2)}\} = J_2 \setminus \{j_{l(1)}, j_{l(2)}\}$. We write $I_2(J_{22})$ for $I(J_{22})$ defined for cycles in Σ_2. Then $I_2(J_{22}) \subset I_1(J_{21})$. We choose $j_{l(3)} \in J_{22}$ such that $I_2(\{j_{l(3)}\}) \cap (I(J_2) \setminus \{i_{k(1)}, i_{k(2)}\}) \neq \emptyset$. We apply the interchange operator $I_g(i_{k(3)}, j_{l(3)})$ to $\tilde{\Sigma}_2$ and Σ_2, where $i_{k(3)} \in I_2(\{j_{l(3)}\})$ and $i_{k(3)} \neq i_{k(1)}, i_{k(2)}$. We denote by $\tilde{\Sigma}_3$ and Σ_3 the images of $\tilde{\Sigma}_2$ and Σ_2, respectively.

By the same method as above, we determine the following: $j_{l(4)}, i_{k(4)}, J_{24}, \tilde{\Sigma}_4, \Sigma_4, I_4(J_{24}); \cdots; j_{l(s)}, i_{k(s)}, J_{2,s}, \tilde{\Sigma}_s, \Sigma_s, I_s(J_{2,s})$. Here s is the integer satisfying the following (i) and (ii):

(i) $I_{s-1}(\{j_{l(s)}\}) \cap (I(J_2) \setminus \{i_{k(1)}, i_{k(2)}, \cdots, i_{k(s-1)}\}) \neq \emptyset$.

(ii) $I_s(\{j\}) \subseteqq \{i_{k(1)}, i_{k(2)}, \cdots, i_{k(s)}\}$ for any $j \in J_2 \setminus \{j_{l(1)}, j_{l(2)}, \cdots, j_{l(s)}\}$.

We set $J_3 = J \setminus (J_1 \cup J_2)$, $J_4 = \{j_{l(1)}, j_{l(2)}, \cdots, j_{l(s)}\}$ and $J_5 = J_2 \setminus J_4$. Set $I_1 = I \setminus I(J)$ and $I_4 = \{i_{k(1)}, i_{k(2)}, \cdots, i_{k(s)}\}$. We note that $I_4 \subset I(J_2)$. Set $I_3 = I_s(J_3)$ and $I_5 = I \setminus (I_1 \cup I_3 \cup I_4)$. Let I_6 be a subset of I_5. Set $I_7 = I_5 \setminus I_6$, $I^* = I \setminus I_7$, and $J^* = J \setminus J_4$.

3.2. In § 3.1, we obtained a basic system of Jordan curves $\tilde{\Sigma}_s$ from $\tilde{\Sigma}_0$ by applying interchange operators in succession. We write $\tilde{\Sigma}_0^*$ for $\tilde{\Sigma}_s$. Suppose that S^* and $\Sigma^* = \{\alpha_1^*, \cdots, \alpha_g^*; \gamma_1^*, \cdots, \gamma_{2g-3}^*\}$ are a compact Riemann surface of genus g with nodes and a basic system of loops and nodes on S^* such that one of the sets induced from Σ_0^* is compatible with $\tilde{\Sigma}_0^*$, and that α_i^* $(i \in I^*)$, γ_j^* $(j \in J^*)$ are nodes and α_i^* $(i \notin I^*)$, γ_j^* $(j \notin J^*)$ are loops, where I^* and J^* are as defined in § 3.1.

From the construction in § 3.1, we see that the pair (S^*, Σ^*) has Property (A) (see [8] for the definition). Therefore, by Theorem 2 in [7], there exists a point $\tau^* \in \delta^{I^*, J^*}\mathfrak{S}_g(\tilde{\Sigma}_0^*)$ with $S(\tau^*) = S^*$.

4. Main theorem—The first step. From this section through section 6, we will prove the following: For a given point $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$, there exists a sequence of points $\{\tau_n\}$ in $\mathfrak{S}_g(\tilde{\Sigma}_0)$ such that $\tau_n \to \tau$ and $S(\tau_n)$ does not converge to $S(\tau)$ as marked surfaces as n tends to ∞. We consider it in the case of $J = \{j\}$ and $I(J) \neq \emptyset$ in § 4, in the case of $J=\{j(1), j(2)\}$ and $I(J)\neq\emptyset$ in § 5, and in the general case in § 6.

4.1. The first step: The case of $J=\{j\}$ and $I(J)\neq\emptyset$.

We have the following two cases.

Case I. There are at least three elements k of the set $\{1, 2, \cdots, 2g\}$ such that $C_{0,k}$ is behind $C_{0,2g+j}$, which is denoted by $C_{0,2g+j}<C_{0,k}$ (see [8] for the definition).

Case II. There are two elements k of $\{1, 2, \cdots, 2g\}$ with $C_{0,2g+j}<C_{0,k}$.

Fix an element i of $I(J)$. Both Cases I and II are divided into the following six cases. Here δ_j means the direction of $\gamma_{0,j}$ in the ordered cycle $L_{0,i}$ (see [8]).

Case I-1 (Case II-1). $C_{0,2g+j}<C_{0,i}$, $C_{0,2g+j}\not< C_{0,g+i}$, $\delta_j=-1$ $(i\neq 1)$, where $C_{0,2g+j}\not< C_{0,g+i}$ means that $C_{0,g+i}$ is not behind $C_{0,2g+j}$ (see [8]).

Case I-2 (Case II-2). $C_{0,2g+j}\not< C_{0,i}$, $C_{0,2g+j}<C_{0,g+i}$, $\delta_j=+1$ $(i\neq 1)$.

Case I-3 (Case II-3). $C_{0,2g+j}<C_{0,i}$, $C_{0,2g+j}\not< C_{0,g+i}$, $\delta_j=+1$ $(i\neq 1)$.

Case I-4 (Case II-4). $C_{0,2g+j}\not< C_{0,i}$, $C_{0,2g+j}<C_{0,g+i}$, $\delta_j=-1$ $(i\neq 1)$.

Case I-5 (Case II-5). $C_{0,2g+j}<C_{0,g+i}$, $\delta_j=+1$ $(i=1)$.

Case I-6 (Case II-6). $C_{0,2g+j}<C_{0,g+i}$, $\delta_j=-1$ $(i=1)$.

REMARK. Cases I-1, I-2, $\cdots$, I-6 are Cases II, I, I′, II′, III, III′ in [8], respectively.

4.2. We only consider Case I-1. The other cases are treated similarly and so omitted. Given $\tau\in\delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$. Then we have two marked Schottky groups $\langle G_0(\tau)\rangle=\langle A_{0(1)}(\tau, z), \cdots, A_{0(g_0)}(\tau, z)\rangle$ and $\langle \check{G}_j(\tau)\rangle=\langle \check{A}_{j(1)}(\tau, z), \cdots, \check{A}_{j(g_j)}(\tau, z)\rangle$ and defining curves $C_{0(k)}(\tau)$, $C_{g+0(k)}(\tau)$ $(k=1, 2, \cdots, g_0)$ and $\check{C}_{j(l)}(\tau)$, $\check{C}_{g+j(l)}(\tau)$ $(l=1, 2, \cdots, g_j)$ as in [6, pp. 73–75]. Furthermore, we have the fixed points $p_{0(k)}(\tau)$, $p_{g+0(k)}(\tau)$ of $A_{0(k)}(\tau, z)$ (resp. $\check{p}_{j(l)}(\tau)$, $\check{p}_{g+j(l)}(\tau)$ of $\check{A}_{j(l)}(\tau, z)$), the distinguished points of the first kind $p_{0(2g_0+1)}(\tau), \cdots, p_{0(2g_0+m_0)}(\tau)$ (resp. $\check{p}_{j(2g_j+1)}(\tau), \cdots, \check{p}_{j(2g_j+m_j)}(\tau)$), and the distinguished point of the second kind $p_j^+(\tau)$ (resp. $\check{p}_j^-(\tau)$).

Let $S(\tau)$ be the Riemann surface with nodes represented by τ. Let $\alpha_k(\tau)$ ($k\notin I$, i.e., $k=0(1), \cdots, 0(g_0), j(1), \cdots, j(g_j)$) be the projections of $C_k(\tau)$, and $\alpha_l(\tau)$ $(l\in I)$ (resp. $\gamma_j(\tau)$) the projections of the distinguished points of the first kind $p_l(\tau)$ (resp. $p_j^{\pm}(\tau)$). Let $\gamma_l(\tau)$ $(1\leqq l\leqq j-1, j+1\leqq l\leqq 2g-3)$ be loops on $S(\tau)$ such that $\Sigma=\{\alpha_1(\tau), \cdots, \alpha_g(\tau); \gamma_1(\tau), \cdots, \gamma_{2g-3}(\tau)\}$ is a basic system of loops and nodes on $S(\tau)$ with $\Sigma\sim\tilde{\Sigma}_0$. Let $C_{2g+l}(\tau)$ for l with $\gamma_l\subset S_0(\tau)=\Omega(G_0(\tau))/\langle G_0(\tau)\rangle$ (resp. $\check{C}_{2g+l}(\tau)$ for l with $\gamma_l\subset S_j(\tau)=\Omega(G_j(\tau))/\langle G_j(\tau)\rangle$) be the liftings of $\gamma_l(\tau)$ to $\omega_0(\tau)$ (resp. $\check{\omega}_j(\tau)$), where $\omega_0(\tau)$ (resp. $\check{\omega}_j(\tau)$) is the fundamental region bounded by $C_{0(k)}(\tau)$ and

$C_{0(g+k)}(\tau)$ $(k = 1, 2, \cdots, g_0)$ for $\langle G_0(\tau)\rangle$ (resp. $\check{C}_{j(m)}(\tau)$ and $\check{C}_{j(g+j(m))}(\tau)$ $(m = 1, 2, \cdots, g_j)$ for $\langle \check{G}_j(\tau)\rangle$).

4.3. From § 4.3 through § 4.5, we will construct a Riemann surface S^* from $S(\tau)$, a basic system of loops and nodes $\Sigma^* = \{\alpha_1^*, \cdots, \alpha_g^*; \gamma_1^*, \cdots, \gamma_{2g-3}^*\}$ from Σ and a point $\tau^* \in \hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0^*)$ from τ, where $\tilde{\Sigma}_0^*$ is the image of $\tilde{\Sigma}_0$ under the interchange operator $I_g(i, j)$.

(1) We will define points $p_{0(k)}^*$, $p_{g+0(k)}^*$ $(k = 1, 2, \cdots, g_0)$, $p_{0(2g_0+l)}^*$ $(l = 1, 2, \cdots, m_0)$ except p_i^* and p_{g+i}^* and Jordan curves $C_{0(k)}^*$, $C_{g+0(k)}^*$ $(k = 1, 2, \cdots, g_0)$ by $p_{0(k)}^* = p_{0(k)}(\tau)$, $p_{g+0(k)}^* = p_{g+0(k)}(\tau)$; $p_{0(2g_0+l)}^* = p_{0(2g_0+l)}(\tau)$; $C_{0(k)}^* = C_{0(k)}(\tau)$, $C_{g+0(k)}^* = C_{g+0(k)}(\tau)$. We set $p_i^* = p_{g+i}(\tau)$ and $p_{g+i}^* = p_j^+(\tau)$ and set $C_{2g+l}^* = C_{2g+l}(\tau)$ for l with $C_{0,2g+j} \not< C_{0,2g+l}$, namely for l with $\gamma_l \subset S_0(\tau)$.

(2) We will define points $\check{p}_{j(k)}^*$, $\check{p}_{g+j(k)}^*$ $(k = 1, 2, \cdots, g_j)$, $\check{p}_{j(2g_j+l)}^*$ $(l = 1, 2, \cdots, m_j)$ except $\check{p}_i^*$ and $\check{p}_{g+i}^*$ and Jordan curves $\check{C}_{j(k)}^*$, $\check{C}_{g+j(k)}^*$ $(k = 1, 2, \cdots, g_j)$ by $\check{p}_{j(k)}^* = \check{p}_{j(k)}(\tau)$, $\check{p}_{g+j(k)}^* = \check{p}_{g+j(k)}(\tau)$; $\check{p}_{j(2g_j+l)}^* = \check{p}_{j(2g_j+l)}(\tau)$; $\check{C}_{j(k)}^* = \check{C}_{j(k)}(\tau)$, $\check{C}_{g+j(k)}^* = \check{C}_{g+j(k)}(\tau)$. We set $\check{p}_i^* = \check{p}_j^-(\tau)$ and $\check{p}_{g+i}^* = \check{p}_i(\tau)$, and set $\check{C}_{2g+l}^* = \check{C}_{2g+l}(\tau)$ for l with $C_{0,2g+j} < C_{0,2g+l}$, namely for l with $\gamma_l \subset S_j(\tau)$.

4.4. By using multi-suffices, we write $C_0(i_1, i_2, \cdots, i_\mu)$, $C_0(i_1, \cdots, i_\mu, \cdots, i_\nu)$ and $C_0(j_1, j_2, \cdots, j_\sigma)$ for $C_{0,2g+j}$, $C_{0,i}$ and $C_{0,g+i}$, respectively.

(1) We choose Jordan curves K_1 and $\check{K}_2$ as follows: K_1 (resp. $\check{K}_2$) forms the boundary curves of a triply connected region $\sigma^*(j_1, \cdots, j_{\sigma-1})$ (resp. $\sigma^*(i_1, \cdots, i_{\nu-1})$) together with $C^*(j_1, \cdots, j_{\sigma-1})$ and $C^*(j_1, \cdots, j_{\sigma-1}, 1 - j_\sigma)$ (resp. $C^*(i_1, \cdots, i_{\nu-1})$ and $C^*(i_1, \cdots, i_{\nu-1}, 1 - i_\nu)$), and contains the point p_i^* (resp. $\check{p}_{g+i}^*$) in the interior.

(2) We determine a Möbius transformation T as follows and fix it: $T(p_i^*) = \check{p}_i^*$, $T(p_{g+i}^*) = \check{p}_{g+i}^*$ and $K_2^* = T^{-1}(\check{K}_2)$ lies in the interior to K_1. Then we note that the outside $\check{K}_2$ is mapped to the inside K_2^* under the mapping T^{-1}. We write C_{2g+j}^* for K_2^*.

(3) We set $C_{j(k)}^* = T^{-1}(\check{C}_{j(k)}^*)$, $C_{g+j(k)}^* = T^{-1}(\check{C}_{g+j(k)}^*)$, $p_{j(k)}^* = T^{-1}(\check{p}_{j(k)}^*)$ and $p_{g+j(k)}^* = T^{-1}(\check{p}_{g+j(k)}^*)$ $(k = 1, 2, \cdots, g_j)$, and $p_{j(2g_j+l)}^* = T^{-1}(\check{p}_{j(2g_j+l)}^*)$ $(l = 1, 2, \cdots, m_j)$. We set $C_{2g+l}^* = T^{-1}(\check{C}_{2g+l}^*)$ for l with $C_{0,2g+j} < C_{0,2g+l}$. We note that all these points and curves are contained in the interior to C_{2g+j}^*.

4.5. For each $k = 0(1), \cdots, 0(g_0)$ (resp. $l = j(1), \cdots, j(g_j)$), we define a Möbius transformation $A_k^*(\tau, z)$ (resp. $A_l^*(\tau, z)$) by $A_k^*(\tau, z) = A_k(\tau, z)$ (resp. $A_l^*(\tau, z) = T^{-1}\check{A}_l(\tau, z)T$). Let t_k^* $(|t_k^*| < 1)$ $(k = 0(1), \cdots, 0(g_0), j(1), \cdots, j(g_j))$ be the inverse of multipliers of $A_k^*(\tau, z)$. We set $t_k^* = 0$ $(k \in \{1, 2, \cdots, g\} \setminus \{0(1), \cdots, 0(g_0), j(1), \cdots, j(g_j)\}$, i.e., $k \in I$).

By the same way as in [7], we determine ρ_l^* $(l = 1, 2, \cdots, 2g - 3)$

from $p_1^*, \cdots, p_{2g}^*$ with respect to $\tilde{\Sigma}_0^*$. We set

$$\tau^* = (t_1^*, \cdots, t_g^*, \rho_1^*, \cdots, \rho_{2g-3}^*) .$$

Then $\tau^* \in \hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0^*)$. Let $S^* = S(\tau^*)$ be the Riemann surface with nodes represented by τ^*.

Let α_k^* ($k = 0(1), \cdots, 0(g_0), j(1), \cdots, j(g_j)$) (resp. α_l^* ($l \in I$)) be the projections of C_k^* (resp. p_l^*) onto S^*. Let γ_l^* ($l = 1, 2, \cdots, 2g-3$) be the projections of C_{2g+l}^* onto S^*. Now we define a basic system of loops and nodes Σ^* on S^* by

$$\Sigma^* = \{\alpha_1^*, \cdots, \alpha_g^*; \gamma_1^*, \cdots, \gamma_{2g-3}^*\} .$$

We note that $\Sigma^* \sim \tilde{\Sigma}_0^*$.

4.6. Here we will construct basic systems of loops Σ_n^* with $\Sigma_n^* \sim \tilde{\Sigma}_0^*$, and a sequence of points $\{\tau_n^*\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0^*)$ such that $\tau_n^* \to \tau^*$ and $(S(\tau_n^*), \Sigma_n^*) \to (S(\tau^*), \Sigma^*)$ as n tends to ∞, where $S(\tau_n^*)$ are the Riemann surfaces represented by τ_n^*.

For $l = 1, 2, \cdots, 2g-3$, we set $C_{2g+l,n}^* = C_{2g+l}^*$ ($n = 1, 2, \cdots$). For $k \notin I$, we set $C_{k,n}^* = C_k^*$, $C_{g+k,n}^* = C_{g+k}^*$, $p_{k,n}^* = p_k^*$ and $p_{g+k,n}^* = p_{g+k}^*$ ($n = 1, 2, \cdots$). We set $A_{k,n}^*(z) = A_k^*(\tau, z)$. For $l \in I$, we choose $C_{l,n}^*$ and $C_{g+l,n}^*$ ($n = 1, 2, \cdots$) as follows:

(i) Each $C_{l,n}^*$ (resp. $C_{g+l,n}^*$) is a circle of the radius $r(l, n)$ (resp. $r(g+l, n)$) about p_l^* (resp. p_{g+l}^*) such that $\lim_{n\to\infty} r(l, n) = 0$ (resp. $\lim_{n\to\infty} r(g+l, n) = 0$).

(ii) For each $l \in I$, let $A_{l,n}^*(z)$ be a Möbius transformation satisfying $A_{l,n}^*(p_{l,n}^*) = p_{l,n}^*$, $A_{l,n}^*(p_{g+l,n}^*) = p_{g+l,n}^*$ and $A_{l,n}^*(C_{l,n}^*) = C_{g+l,n}^*$. Then $\langle G_n^* \rangle = \langle A_{1,n}^*(z), \cdots, A_{g,n}^*(z) \rangle$ is a Schottky group.

(iii) If we set

$$\tilde{\Sigma}_n^* = \{C_{1,n}^*, \cdots, C_{2g,n}^*; C_{2g+1,n}^*, \cdots, C_{4g-3,n}^*\} ,$$

then $\tilde{\Sigma}_n^*$ is a basic system of Jordan curves for $\langle G_n^* \rangle$ with $\tilde{\Sigma}_n^* \sim \tilde{\Sigma}_0^*$, where $\tilde{\Sigma}_n^* \sim \tilde{\Sigma}_0^*$ means that for each $l = 1, 2, \cdots, 2g-3$, $C_{2g+l,n}^*$ gives the same partition of $\{1, 2, \cdots, 2g\}$ as C_{2g+l}^*.

REMARK. We may choose $p_{k,n}^*$, $p_{g+k,n}^*$, $C_{k,n}^*$, $C_{g+k,n}^*$ and $C_{2g+l,n}^*$ as follows:

(i) $p_{k,n}^* \to p_k^*$ and $p_{g+k,n}^* \to p_{g+k}^*$ ($k = 1, 2, \cdots, g$) as $n \to \infty$.

(ii) For $k \notin I$, $C_{k,n}^* \to C_k^*$ and $C_{g+k,n}^* \to C_{g+k}^*$ as $n \to \infty$.

(iii) For each $k \in I$, $C_{k,n}^*$ (resp. $C_{g+k,n}^*$) is a Jordan curve with the diameter $r(k, n)$ (resp. $r(g+k, n)$) such that $r(k, n) \to 0$ (resp. $r(g+k, n) \to 0$) as $n \to \infty$ and $p_{k,n}^*$ (resp. $p_{g+k,n}^*$) is contained in the interior to $C_{k,n}^*$ (resp. $C_{g+k,n}^*$).

(iv) Let $A^*_{k,n}(z)$ ($k=1, 2, \cdots, g$; $n=1, 2, \cdots$) be Möbius transformations satisfying $A^*_{k,n}(p^*_{k,n})=p^*_{k,n}$, $A^*_{k,n}(p^*_{g+k,n})=p^*_{g+k,n}$, $A^*_{k,n}(C^*_{k,n})=C^*_{g+k,n}$, and $\lim_{n\to\infty}\lambda^*_{k,n}=\lambda^*_k$ (resp. ∞) for $k\notin I$ (resp. $k\in I$), where $\lambda^*_{k,n}$ and λ^*_k are the multipliers of $A^*_{k,n}$ and A^*_k, respectively. Then $\langle G^*_n\rangle=\langle A^*_{1,n}(z), \cdots, A^*_{g,n}(z)\rangle$ is a Schottky group.

(v) If we set

$$\tilde{\Sigma}^*_n=\{C^*_{1,n}, \cdots, C^*_{2g,n}; C^*_{2g+1}, \cdots, C^*_{4g-3}\},$$

then $\tilde{\Sigma}^*_n$ is a basic system of Jordan curves for $\langle G^*_n\rangle$ with $\tilde{\Sigma}^*_n\sim\tilde{\Sigma}^*_0$.

Let $\tau^*_n\in\mathfrak{S}_g(\tilde{\Sigma}^*_0)$ be the point corresponding to $\langle G^*_n\rangle$ (cf. Theorem 1 in [7]), that is, $\langle G^*_n\rangle=\langle G(\tau^*_n)\rangle$. Let $\Pi_n: \Omega(G(\tau^*_n))\to\Omega(G(\tau^*_n))/\langle G(\tau^*_n)\rangle=S(\tau^*_n)$ be the natural projection. We set $\alpha^*_{k,n}=\Pi_n(C^*_{k,n})$ ($k=1, 2, \cdots, g$; $n=1, 2, \cdots$) and $\gamma^*_{l,n}=\Pi_n(C^*_{2g+l,n})$ ($l=1, 2, \cdots, 2g-3$; $n=1, 2, \cdots$). Then $\Sigma^*_n=\{\alpha^*_{1,n}, \cdots, \alpha^*_{g,n}; \gamma^*_{1,n}, \cdots, \gamma^*_{2g-3,n}\}$ is a basic system of loops on $S(\tau^*_n)$. By the same way as in § 2, we see that $\tau^*_n\to\tau^*$ and $(S(\tau^*_n), \Sigma^*_n)\to(S(\tau^*), \Sigma^*)$ as $n\to\infty$.

4.7. Let $\Sigma_n=\{\alpha_{1,n}, \cdots, \alpha_{g,n}; \gamma_{1,n}, \cdots, \gamma_{2g-3,n}\}$, τ_n and $\langle G(\tau_n)\rangle$ be the images of Σ^*_n, τ^*_n and $\langle G(\tau^*_n)\rangle$ under the interchange operator $I_g(i, j)$, respectively. Then we see that $\tau_n\in\mathfrak{S}_g(\tilde{\Sigma}_0)$ and that Σ_n is a basic system of loops on $S_n=\Omega(G(\tau_n))/\langle G(\tau_n)\rangle$ with $\Sigma_n\sim\tilde{\Sigma}_0$. Let $\hat{\Sigma}^*=\{\hat{\alpha}^*_1, \cdots, \hat{\alpha}^*_g; \hat{\gamma}^*_1, \cdots, \hat{\gamma}^*_{2g-3}\}$ be the following basic system of loops and nodes on $S^*=S(\tau^*)$: $\hat{\alpha}^*_k=\alpha^*_k$ ($k\neq i$), $\hat{\alpha}^*_i=\gamma^*_j$, $\hat{\gamma}^*_l=\gamma^*_l$ ($l\neq j$) and $\hat{\gamma}^*_j=\alpha^*_i$. Then we note that $\hat{\Sigma}^*\sim\tilde{\Sigma}_0$. From § 4.6, we have that $\tau_n\to\tau$ and $(S(\tau_n), \Sigma_n)\to(S(\tau^*), \hat{\Sigma}^*)$ $(\neq(S(\tau), \Sigma))$ as $n\to\infty$.

5. Main theorem—The second step.

5.1. The second step. The case of $J=\{\hat{j}(1), \hat{j}(2)\}$ and $I(J)\neq\emptyset$.

Let $i(1)\in I(\{\hat{j}(1)\})$. Let $\tilde{\Sigma}_1$ be the image of $\tilde{\Sigma}_0$ under the interchange operator $I_g(i(1), \hat{j}(1))$. We set $J_1=\{\hat{j}(2)\}$. We consider the case of $I(J_1)\setminus\{i(1)\}\neq\emptyset$ with respect to $\tilde{\Sigma}_1$. Let $i(2)\in I(J_1)$. We write $\tilde{\Sigma}_2$ for the image of $\tilde{\Sigma}_1$ under the interchange operator $I_g(i(2), \hat{j}(2))$.

The second step is divided into the following three cases: Case 1. $C_{2g+\hat{j}(1)}<C_{2g+\hat{j}(2)}$; Case 2. $C_{2g+\hat{j}(2)}<C_{2g+\hat{j}(1)}$; Case 3. There is no relation between $C_{2g+\hat{j}(1)}$ and $C_{2g+\hat{j}(2)}$, that is, $C_{2g+\hat{j}(1)}\not< C_{2g+\hat{j}(2)}$ and $C_{2g+\hat{j}(2)}\not< C_{2g+\hat{j}(1)}$. For $C_{i(1)}$ and $C_{g+i(1)}$ (resp. $C_{i(2)}$ and $C_{g+i(2)}$), we have either $C_{2g+\hat{j}(1)}<C_{i(1)}$ or $C_{2g+\hat{j}(1)}<C_{g+i(1)}$ (resp. $C_{2g+\hat{j}(2))}<C_{i(2)}$ or $C_{2g+\hat{j}(2)}<C_{g+i(2)}$). We only consider the following case:

$$C_{2g+\hat{j}(1)}<C_{i(1)} \quad\text{and}\quad C_{2g+\hat{j}(2)}<C_{i(2)}.$$

Other cases are similarly treated.

In the above case, there may be the following twelve cases:

Case 1. $C_{2g+\hat{j}(1)} < C_{2g+\hat{j}(2)}$, therefore in this case $C_{2g+\hat{j}(1)} < C_{i(2)}$ and $C_{2g+\hat{j}(2)} \not< C_{g+i(1)}$.

Case 1-1. $C_{2g+\hat{j}(1)} < C_{g+i(2)}$, $C_{2g+\hat{j}(2)} \not< C_{i(1)}$.
Case 1-2. $C_{2g+\hat{j}(1)} \not< C_{g+i(2)}$, $C_{2g+\hat{j}(2)} \not< C_{i(1)}$.
Case 1-3. $C_{2g+\hat{j}(1)} < C_{g+i(2)}$, $C_{2g+\hat{j}(2)} < C_{i(1)}$.
Case 1-4. $C_{2g+\hat{j}(1)} \not< C_{g+i(2)}$, $C_{2g+\hat{j}(2)} < C_{i(1)}$.

Case 2. $C_{2g+\hat{j}(2)} < C_{2g+\hat{j}(1)}$, therefore in this case $C_{2g+\hat{j}(2)} < C_{i(1)}$ and $C_{2g+\hat{j}(1)} \not< C_{g+i(2)}$.

Case 2-1. $C_{2g+\hat{j}(2)} < C_{g+i(1)}$, $C_{2g+\hat{j}(1)} \not< C_{i(2)}$.
Case 2-2. $C_{2g+\hat{j}(2)} < C_{g+i(1)}$, $C_{2g+\hat{j}(1)} < C_{i(2)}$.
Case 2-3. $C_{2g+\hat{j}(2)} \not< C_{g+i(1)}$, $C_{2g+\hat{j}(1)} \not< C_{i(2)}$.
Case 2-4. $C_{2g+\hat{j}(2)} \not< C_{g+i(1)}$, $C_{2g+\hat{j}(1)} < C_{i(2)}$.

Case 3. $C_{2g+\hat{j}(1)} \not< C_{2g+j(2)}$ and $C_{2g+\hat{j}(2)} \not< C_{2g+j(1)}$.
Case 3-1. $C_{2g+\hat{j}(1)} \not< C_{g+i(2)}$, $C_{2g+\hat{j}(2)} \not< C_{g+i(1)}$.
Case 3-2. $C_{2g+\hat{j}(1)} < C_{g+i(2)}$, $C_{2g+\hat{j}(2)} \not< C_{g+i(1)}$.
Case 3-3. $C_{2g+\hat{j}(1)} \not< C_{g+i(2)}$, $C_{2g+\hat{j}(2)} < C_{g+i(1)}$.
Case 3-4. $C_{2g+\hat{j}(1)} < C_{g+i(2)}$, $C_{2g+\hat{j}(2)} < C_{g+i(1)}$.

5.2. Here we only consider Case 1-3. Other cases are similarly treated. We use similar procedures as in § 4. First, we use $C_{2g+\hat{j}(1)}$, $C_{i(1)}$ and $C_{g+i(1)}$ instead of C_{2g+j}, $\check{C}_i$ and C_{g+i} in § 4, respectively. In this case, it is slightly different from the way in § 4. Namely, we have three Schottky groups $\langle G_0(\tau)\rangle$, $\langle \tilde{G}_{\hat{j}(1)}(\tau)\rangle$ and $\langle \check{G}_{\hat{j}(2)}(\tau)\rangle$. We set $p^*_{g+i(1)} = p^+_{\hat{j}(1)}$, $p^{*-}_{\hat{j}(2)} = p_{g+i(1)}$; $\tilde{p}^*_{i(1)} = \tilde{p}^-_{\hat{j}(1)}$, $\tilde{p}^{*+}_{\hat{j}(2)} = \tilde{p}^+_{\hat{j}(2)}$, $\tilde{p}^*_{g+i(2)} = \tilde{p}_{g+i(2)}$; $\check{p}^*_{\hat{j}(2)} = \check{p}^-_{\hat{j}(2)}$ and $\check{p}^*_{g+i(1)} = \check{p}_{i(1)}$ and then we use the same procedure as in § 4 for $\langle G_0(\tau)\rangle$ and $\langle \check{G}_{\hat{j}(2)}(\tau)\rangle$. We denote this procedure by $[C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)}]$. We denote by $(C_{2g+j}; C_i, C_{g+i})$ the procedure in § 4. Second, we use $C^*_{2g+\hat{j}(2)}$, $\tilde{C}^*_{g+i(2)}$, and $C^*_{i(2)}$ instead of C_{2g+j}, C_i and C_{g+i} in § 4, and we use the same procedure as in § 4 for $\langle G^*_0(\tau)\rangle$ and $\langle \tilde{G}^*_{\hat{j}(2)}(\tau)\rangle$. We write $[C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)}] - (C^*_{2g+\hat{j}(2)}, C^*_{g+i(2)}, C_{i(2)})$ for the above two procedures.

Given a point $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$. We get a point $\tau^* \in \hat{\mathfrak{S}}^*_g(\tilde{\Sigma}_1)$ from τ by using the procedure $[C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)}]$, and a point $\tau^{**} \in \hat{\mathfrak{S}}^*_g(\tilde{\Sigma}_2)$ from τ^* by using the procedure $(C^*_{2g+\hat{j}(2)}; C^*_{g+i(2)}, C_{i(2)})$. Let $\Sigma^{**} = \{\alpha^{**}_1, \cdots, \alpha^{**}_g; \gamma^{**}_1, \cdots, \gamma^{**}_{2g-3}\}$ be a basic system of loops and nodes of $S(\tau^{**})$ which is obtained by the same method as in § 4. We note that $\Sigma^{**} \sim \tilde{\Sigma}_2$. Next we construct the following sequence of points $\{\tau^{**}_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_2)$ by a similar method as in § 4:

$$\tau^{**}_n \to \tau^{**} \quad \text{and} \quad (S(\tau^{**}_n), \Sigma^{**}_n) \to (S(\tau^{**}), \Sigma^{**})$$

as $n\to\infty$, where Σ_n^{**} is a basic system of loops on $S(\tau_n^{**})$ with $\Sigma_n^{**}\sim\tilde{\Sigma}_2$ which are obtained by the same method as in § 4. We set $\tau_n^*=I_g^{-1}(i(2), \hat{j}(2))(\tau_n^{**})$ and $\tau_n=I_g^{-1}(i(1), \hat{j}(1))(\tau_n^*)$. Then it is easily seen that $\tau_n\in\mathfrak{S}_g(\tilde{\Sigma}_0)$ and $\tau_n\to\tau$ as $n\to\infty$. Let $\hat{\Sigma}^{**}=\{\hat{\alpha}_1^{**},\cdots,\hat{\alpha}_g^{**};\hat{\gamma}_1^{**},\cdots,\hat{\gamma}_{2g-3}^{**}\}$ be the following basic system of loops and nodes on $S(\tau^{**})$: $\hat{\alpha}_{i(1)}^{**}=\gamma_{\hat{j}(1)}^{**}$, $\hat{\alpha}_{i(2)}^{**}=\gamma_{\hat{j}(2)}^{**}$, $\hat{\alpha}_k^{**}=\alpha_k^{**}$ $(k\neq i(1), i(2))$, $\hat{\gamma}_{\hat{j}(1)}^{**}=\alpha_{i(1)}^{**}$, $\hat{\gamma}_{\hat{j}(2)}^{**}=\alpha_{i(2)}^{**}$ and $\hat{\gamma}_l^{**}=\gamma_l^{**}$ $(l\neq\hat{j}(1), \hat{j}(2))$, We set $\Sigma_n=I_g(i(1), \hat{j}(1))^{-1}\cdot I_g(i(2), \hat{j}(2))^{-1}(\Sigma_n^{**})$. Then we have that

$$(S(\tau_n), \Sigma_n)\to(S(\tau^{**}), \hat{\Sigma}^{**}) \quad \text{as} \quad n\to\infty .$$

5.3. Other cases can similarly be treated to the above. For each case, we use the following procedures:

Case 1-1. $(C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)})-(C_{2g+\hat{j}(2)}^*; C_{i(2)}^*, C_{g+i(2)}^*)$.
Case 1-2. $(C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)})-(C_{2g+\hat{j}(2)}^*; C_{i(2)}^*, C_{g+i(2)}^*)$.
Case 1-3 was already treated in § 5.2. Case 1-4 does not occur.
Case 2-1. $(C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)})-(C_{2g+\hat{j}(2)}^*; C_{i(2)}^*, C_{g+i(2)}^*)$.
Case 2-2. $(C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)})-(C_{2g+\hat{j}(2)}^*; C_{i(2)}^*, C_{g+i(2)}^*)$.
Case 2-3. $[C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)}]-(C_{2g+\hat{j}(2)}^*; C_{i(2)}^*, C_{g+i(2)}^*)$.
Case 2-4 does not occur.
Case 3-1. $(C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)})-(C_{2g+\hat{j}(2)}^*; C_{i(2)}^*, C_{g+i(2)}^*)$.
Case 3-2. $(C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)})-(C_{2g+\hat{j}(2)}^*; C_{i(2)}^*, C_{g+i(2)}^*)$.
Case 3-3. $[C_{2g+\hat{j}(1)}; C_{i(1)}, C_{g+i(1)}]-(C_{2g+\hat{j}(2)}^*; C_{i(2)}^*, C_{g+i(2)}^*)$.
Case 3-4 does not occur.

6. Main theorem—The third step. Last, we will treat the general case. Let $\tau\in\delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$ be as in § 3, where $I\supset I(J)\neq\emptyset$. Let $\tilde{\Sigma}_0^*$ be as in § 3, that is,

$$\tilde{\Sigma}_0^*=I_g(i_{k(s)}, j_{l(s)})\cdots I_g(i_{k(1)}, j_{l(1)})(\tilde{\Sigma}_0) .$$

We write Φ for $I_g(i_{k(s)}, j_{l(s)})\cdots I_g(i_{k(1)}, j_{l(1)})$. Let I^* and J^* be as in § 3. By the same methods as in §§ 4 and 5, we determine $\tau_1\in\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_1)$ from τ, $\tau_2\in\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_2)$ from $\tau_1,\cdots,\tau_s\in\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0^*)$ from τ_{s-1}, where $\tilde{\Sigma}_t=I_g(i_{k(t)}, j_{l(t)})(\tilde{\Sigma}_{t-1})$ $(t=1, 2,\cdots, s)$ and $\tilde{\Sigma}_0^*=\tilde{\Sigma}_s$.

We set $\tau^*=\tau_s$. Let $\Sigma^*=\{\alpha_1^*,\cdots,\alpha_g^*;\gamma_1^*,\cdots,\gamma_{2g-3}^*\}$ be a basic system of loops and nodes on $S(\tau^*)$ with $\Sigma^*\sim\tilde{\Sigma}_0^*$ which is obtained by the same method as in §§ 4 and 5. We note that α_k^* $(k\in I^*)$ and γ_l^* $(l\in J^*)$ are nodes, and α_k^* $(k\notin I^*)$ and γ_l^* $(l\notin J^*)$ are loops. As in §§ 4 and 5, we construct the following sequence of points $\{\tau_n^*\}\subset\mathfrak{S}_g(\tilde{\Sigma}_0^*)$: $\tau_n^*\to\tau^*$ and $(S(\tau_n^*), \Sigma_n^*)\to(S(\tau^*), \Sigma^*)$, where Σ_n^* are basic systems of loops on $S(\tau_n^*)$ with $\Sigma_n^*\sim\tilde{\Sigma}_0^*$ which are obtained as in §§ 4 and 5. We set $\tau_n=\Phi^{-1}(\tau_n^*)$. Then the sequence of points $\{\tau_n\}\subset\mathfrak{S}_g(\tilde{\Sigma}_0)$ satisfies the following:

$$\tau_n\to\tau \quad \text{and} \quad (S(\tau_n), \Sigma_n)\to(S(\tau^*), \hat{\Sigma}^*) \quad \text{as} \quad n\to\infty ,$$

where $\Sigma_n = \Phi^{-1}(\Sigma_n^*)$ and $\hat{\Sigma}^*$ is the basic system of loops and nodes on $S(\tau^*)$ with $\hat{\Sigma}^* \sim \tilde{\Sigma}_0$ which is obtained from Σ^* as in §§ 4 and 5. Then we have the following main theorem.

THEOREM 2. *Let $\langle G_0 \rangle$ and $\tilde{\Sigma}_0$ be a fixed marked Schottky group and a fixed basic system of Jordan curves for $\langle G_0 \rangle$, respectively. Given a point $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$. Let $\tilde{\Sigma}_0^*$, I^* and J^* be as in § 3. Let $\tau^* \in \delta^{I^*,J^*}\mathfrak{S}_g(\tilde{\Sigma}_0^*)$ be the point obtained from τ as in the above. Then there exists the following sequences of points $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$:*

$$\tau_n \to \tau \quad \textit{and} \quad (S(\tau_n), \Sigma_n) \to (S(\tau^*), \hat{\Sigma}^*) \quad \textit{as} \quad n \to \infty ,$$

where Σ_n and $\hat{\Sigma}^*$ are a basic system of loops on $S(\tau_n)$ with $\Sigma_n \sim \tilde{\Sigma}_0$ and a basic system of loops and nodes on $S(\tau^*)$ with $\hat{\Sigma}^* \sim \tilde{\Sigma}_0$, respectively, as above.

COROLLARY. *Given $\tau \in \delta^{I,J}\mathfrak{S}_g(\tilde{\Sigma}_0)$, where $I \supset I(J)$. If $I(J) \neq \emptyset$, then there exists a sequence of points $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ such that* (i) *$\tau_n \to \tau$ as $n \to \infty$ and* (ii) *$S(\tau_n)$ does not converge to $S(\tau)$ as marked surfaces.*

REMARK. By similar methods as in [5] and in the proof of Theorem 1, we easily show that if $\tilde{\Sigma}_0$ is a standard system of Jordan curves, then $S(\tau_n)$ converges to $S(\tau)$ as marked surfaces for any point $\tau \in \hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$ and for any sequence of points $\{\tau_n\} \subset \mathfrak{S}_g(\tilde{\Sigma}_0)$ with $\tau_n \to \tau$.

7. An example. Here we will give an example for Theorem 2. We write $(a, b; c, d)$ for a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For $n = 10, 11, 12, \cdots$, we set

$$\begin{aligned}
&A_{1,n} = (n, -1/n; n, 0) , \\
&A_{2,n} = (n^2 + 3, -(2n^2 + 6 + (1/n^2)); n^2, -2n^2) , \\
&C_{1,n}: |z| = 2/3 , \\
&C_{2,n}: |z - 2| = 1/n^2 , \\
&C_{3,n}: |z - 1| = 3/(2n^2) , \\
&C_{4,n}: |z - (1 + (3/n^2))| = 1/n^2 , \\
&C_{5,n}: |z - 1| = 5/n^2 .
\end{aligned}$$

In particular, we set $A_i = A_{i,10}$ $(i = 1, 2)$, $\langle G_0 \rangle = \langle A_1, A_2 \rangle$, $C_i = C_{i,10}$ $(i = 1, 2, 3, 4, 5)$ and $\tilde{\Sigma}_0 = \{C_1, C_2, C_3, C_4; C_5\}$. Then $\langle G_0 \rangle$ is a marked Schottky group and $\tilde{\Sigma}_0$ is a basic system of Jordan curves for $\langle G_0 \rangle$. We apply the interchange operator $I_g(1, 1)$ on $\tilde{\Sigma}_0$ and $\langle G_0\rangle$. If we set $\tilde{\Sigma}_0^* = I_g(1, 1)(\tilde{\Sigma}_0) = \{C_1^*, C_2^*, C_3^*, C_4^*, C_5^*\}$, then we have $C_1^* = A_1^{-1}(C_5)$, $C_2^* = C_2$, $C_3^* = C_5$, $C_4^* = A_1^{-1}(C_4)$ and $C_5^* = C_1$. If we set $\langle G_0^*\rangle = I_g(1, 1)(\langle G_0\rangle) = \langle A_1^*, A_2^*\rangle$, then we have $A_1^* = A_1$ and $A_2^* = A_1^{-1}A_2$.

We set $\langle G_n\rangle = \langle A_{1,n}, A_{2,n}\rangle$ $(n = 10, 11, 12, \cdots)$ where $\langle G_{10}\rangle = \langle G_0\rangle$. We easily see that $\langle G_n\rangle$ are marked Schottky groups $(n = 10, 11, \cdots)$. Let $\tau_n = (t_{1,n}, t_{2,n}, \rho_{1,n})$ be the points in $\mathfrak{S}_g(\tilde{\Sigma}_0)$ corresponding to $\langle G_n\rangle$ $(n = 10, 11, \cdots)$. If we set $\langle G_n^*\rangle = I_g(1, 1)(\langle G_n\rangle) = \langle A_{1,n}^*, A_{2,n}^*\rangle$, then we have $A_{1,n}^* = A_{1,n}$ and $A_{2,n}^* = A_{1,n}^{-1}A_{2,n} = (n, -2n; -3n, 6n + (1/n))$. Let $\tau_n^* = (t_{1,n}^*, t_{2,n}^*, \rho_{1,n}^*)$ be the points in $\mathfrak{S}_g(\tilde{\Sigma}_0^*)$ corresponding to $\langle G_n^*\rangle$. Set $S_n^* = \Omega(G_n^*)/\langle G_n^*\rangle$ and $S_n = \Omega(G_n)/\langle G_n\rangle$. Let Π_n (resp. Π_n^*) be the natural projections of $\Omega(G_n)$ (resp. $\Omega(G_n^*)$) onto S_n (resp. S_n^*). We set $\alpha_{i,n} = \Pi_n(C_{i,n})$ $(i = 1, 2)$, $\gamma_{1,n} = \Pi_n(C_{5,n})$, $\alpha_{i,n}^* = \Pi_n^*(C_{i,n}^*)$ $(i = 1, 2)$ and $\gamma_{1,n}^* = \Pi_n^*(C_{5,n}^*)$. Then $\Sigma_n = \{\alpha_{1,n}, \alpha_{2,n}; \gamma_{1,n}\}$ and $\Sigma_n^* = \{\alpha_{1,n}^*, \alpha_{2,n}^*; \gamma_{1,n}^*\}$ are basic systems of loops on S_n and S_n^*, respectively, and $\Sigma_n^* = I_g(1, 1)(\Sigma_n)$.

Let $\lambda_{i,n}$, $p_{i,n}$ and $p_{2+i,n}$ (resp. $\lambda_{i,n}^*$, $p_{i,n}^*$ and $p_{2+i,n}^*$) be the multipliers, the attracting and the repelling fixed points of $A_{i,n}$ (resp. $A_{i,n}^*$), respectively, for $n = 10, 11, 12, \cdots$, where $|\lambda_{i,n}| > 1$ (resp. $|\lambda_{i,n}^*| > 1$). Then we have

$$\begin{aligned}
p_{1,n} &= (n - \sqrt{n^2 - 4})/2n\,, \qquad p_{3,n} = (n + \sqrt{n^2 - 4})/2n\,,\\
p_{2,n} &= (3(n^2 + 1) + \sqrt{n^4 - 6n^2 + 5})/2n^2\,,\\
p_{4,n} &= (3(n^2 + 1) - \sqrt{n^4 - 6n^2 + 5})/2n^2\,,\\
\lambda_{1,n} &= (n^2 - 2 + n\sqrt{n^2 - 4})/2\,,\\
\lambda_{2,n} &= (n^4 - 6n^2 + 7 + \sqrt{n^8 - 12n^6 + 50n^4 - 84n^2 + 45})/2\,,\\
p_{1,n}^* &= p_{1,n}\,, \qquad p_{3,n}^* = p_{3,n}\,,\\
p_{2,n}^* &= (5n + (1/n) + \sqrt{49n^2 + 10 + (1/n^2)})/6n\,,\\
p_{4,n}^* &= (5n + (1/n) - \sqrt{49n^2 + 10 + (1/n^2)})/6n\,,\\
\lambda_{1,n}^* &= \lambda_{1,n}\,, \quad \text{and}\\
\lambda_{2,n}^* &= (49n^2 + 12 + (1/n^2)\\
&\qquad + \sqrt{2401n^4 + 1176n^2 + 238 + (24/n^2) + (1/n^4)})/2\,.
\end{aligned}$$

Let T_n be the Möbius transformations determined by

$$T_n(p_{1,n}) = 0\,, \qquad T_n(p_{3,n}) = 1 \qquad \text{and} \qquad T_n(p_{2,n}) = \infty$$

for $n = 10, 11, 12, \cdots$. Then $\rho_{1,n} = T_n(p_{4,n})$. By simple calculation, we have

$$\rho_{1,n} = \frac{(2n^2 + 3)^2 - (\sqrt{n^4 - 6n^2 + 5} - n\sqrt{n^2 - 4})^2}{4n\sqrt{n^2 - 4}\sqrt{n^4 - 6n^2 + 5}}\,.$$

Hence $\rho_{1,n} \to 1$ as $n \to \infty$.

On the other hand, let T_n^* be the Möbius transformation determined by

$$T_n^*(p_{1,n}^*) = 0\ , \qquad T_n^*(p_{3,n}^*) = 1 \qquad \text{and} \qquad T_n^*(p_{4,n}^*) = \infty$$

for $n = 10, 11, 12, \cdots$. Then we have

$$1 - (1/\rho_{1,n}^*) = \frac{32n^4 + 96n^2 + 64 + (4/n^2)}{(9n^2 - 5 + \sqrt{n^2 - 4}\sqrt{49n^2 + 10 + (1/n^2)})^2}\ .$$

Hence $\rho_{1,n}^* \to 8/7$ as $n \to \infty$.

Since $t_{i,n} = 1/\lambda_{i,n}$ and $t_{i,n}^* = 1/\lambda_{i,n}^*$ $(i = 1, 2)$, $\tau_n \to \tau = (0, 0, 1)$ and $\tau_n^* \to \tau^* = (0, 0, 8/7)$ as $n \to \infty$. τ (resp. τ^*) is a point in the augmented Schottky space $\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0)$ (resp. $\hat{\mathfrak{S}}_g^*(\tilde{\Sigma}_0^*)$). Let S and S^* be the Riemann surfaces represented by τ and τ^*, respectively. Let $\Sigma^* = \{\alpha_1^*, \alpha_2^*; \gamma_1^*\}$ be a basic system of loops and nodes on S^* with $\Sigma^* \sim \tilde{\Sigma}_0^*$ such that α_i^* $(i = 1, 2)$ are nodes and γ_1^* is a loop. Let $\hat{\Sigma}^* = \{\hat{\alpha}_1^*, \hat{\alpha}_2^*; \hat{\gamma}_1^*\}$ be a basic system of loops and nodes on S^* such that $\hat{\alpha}_1^* = \gamma_1^*$, $\hat{\alpha}_2^* = \alpha_2^*$ and $\hat{\gamma}_1^* = \alpha_1^*$. We note that $\hat{\Sigma}^* \sim \tilde{\Sigma}_0$. Then by using the method of the proof of Theorem 1, we have that

$$(S_n^*, \Sigma_n^*) \to (S^*, \Sigma^*) \quad \text{as} \quad n \to \infty\ .$$

Since $S_n = S_n^*$ except markings and $S \neq S^*$, we have that

$$(S_n, \Sigma_n) \to (S^*, \hat{\Sigma}^*)\ (\neq (S, \Sigma)) \quad \text{as} \quad n \to \infty\ .$$

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