Tôhoku Math. Journ.
37 (1985), 107-123.

THE NIELSEN DEVELOPMENT AND TRANSITIVE POINTS UNDER A CERTAIN FUCHSIAN GROUP

(Dedicated to Professor Yukio Kusunoki on his sixtieth birthday)

SHUNSUKE MOROSAWA AND MASAMI NAKADA

1. Preliminaries. Let D be the unit disk in the complex plane and let ∂D be its boundary. We think of D as endowed with the Poincaré metric $ds = (1 - |z|^2)^{-1}|dz|$, $z \in D$. In this paper, we consider a Fuchsian group Γ acting on D whose elements are all hyperbolic transformations and whose Dirichlet fundamental region F with the center at the origin 0 is a noneuclidean regular $4g$-sided polygon ($g \geqq 2$). Moreover, the action of Γ on D is given by identifying the sides of F as in Fig. 1. We denote by $\{\alpha_i, \beta_i\}_{i=1}^g$ the generators of Γ.

We label the directed geodesic segment from 0 to $\alpha_i(0)$ (or $\beta_i(0)$) the letter a_i (or b_i) and the directed geodesic segment from 0 to $\alpha_i^{-1}(0)$ (or $\beta_i^{-1}(0)$) the letter a_i^{-1} (or b_i^{-1}). Similarly, for every $\gamma \in \Gamma$, we label the directed geodesic segment from $\gamma(0)$ to $\gamma(\alpha_i(0))$ the letter a_i and so on. In this way, we have the net in D consisting of geodesic segments labeled as a_i, a_i^{-1}, b_i and b_i^{-1} ($1 \leqq i \leqq g$). Every mesh of the net is also a noneuclidean regular $4g$-sided polygon. We denote by O_1 (or O_2) the order of the letters corresponding to the sides on the mesh located succeedingly in the clockwise (or anticlockwise) sense. At every vertex of the net, there are $4g$ directed geodesic segments. We denote by O_3 (or O_4) the order of the letters corresponding to the sides located succeedingly around the vertex in the clockwise (or anticlockwise) sense. Hence every consecutive subsequence of $a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}a_1b_1a_1^{-1}b_1^{-1} \cdots$ is of order O_1. Similarly, every consecutive subsequence of $b_ga_gb_g^{-1}a_g^{-1} \cdots b_1a_1b_1^{-1}a_1^{-1}b_ga_gb_g^{-1}a_g^{-1} \cdots$, $b_ga_g^{-1}b_g^{-1}a_g \cdots b_1a_1^{-1}b_1^{-1}a_1b_ga_g^{-1}b_g^{-1}a_g \cdots$ or $a_1b_1^{-1}a_1^{-1}b_1 \cdots a_gb_g^{-1}a_g^{-1}b_ga_1b_1^{-1}a_1^{-1}b_1 \cdots$ is of order O_2, O_3 or O_4, respectively.

In the following argument, we set $a_i = c_{4i-4}$, $b_i^{-1} = c_{4i-3}$, $a_i^{-1} = c_{4i-2}$ and $b_i = c_{4i-1}$ ($1 \leqq i \leqq g$). Further, we set $\alpha_i = \gamma_{4i-4}$, $\beta_i^{-1} = \gamma_{4i-3}$, $\alpha_i^{-1} = \gamma_{4i-2}$ and $\beta_i = \gamma_{4i-1}$ ($1 \leqq i \leqq g$).

We denote by c_i^{-1} the directed geodesic segment of the inverse direction of c_i. Since c_i^{-1} is the directed geodesic segment from $\gamma_i(0)$ to $0 = \gamma_i(\gamma_i^{-1}(0))$, it is by its definition the letter corresponding to the directed geodesic segment from 0 to $\gamma_i^{-1}(0)$. Clearly, c_i^{-1} is c_{i+2}, if $i = 4j - 4$ or $4j - 3$ for some j, and c_i^{-1} is c_{i-2}, if $i = 4j - 2$ or $4j - 1$ for some j.

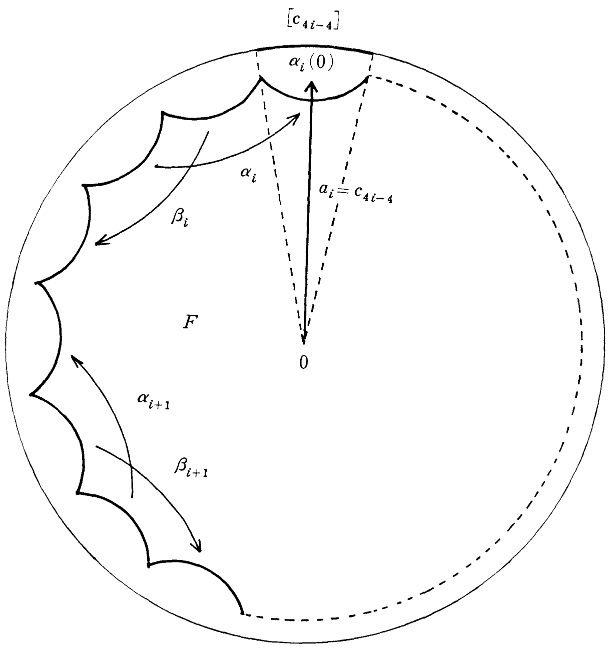

FIGURE 1

For a finite sequence $c_{i_1}c_{i_2}\cdots c_{i_n}$, we consider a directed broken geodesic path $L(c_{i_1}c_{i_2}\cdots c_{i_n})$ in D initiated at the origin 0 as follows: $L(c_{i_1})$ is the directed geodesic segment from 0 to $\gamma_{i_1}(0)$ and $L(c_{i_1}c_{i_2}\cdots c_{i_{n-1}}c_{i_n})$ consists of $L(c_{i_1}c_{i_2}\cdots c_{i_{n-1}})$ with the terminal point $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{n-1}}(0)$ and the directed geodesic segment c_{i_n} from $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{n-1}}(0)$ to $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{n-1}}\gamma_{i_n}(0)$. For an infinite sequence $c_{i_1}c_{i_2}\cdots$ we set $L(c_{i_1}c_{i_2}\cdots) = \bigcup_{n=1}^{\infty} L(c_{i_1}c_{i_2}\cdots c_{i_n})$.

Assume that a finite sequence $c_{i_1}c_{i_2}\cdots c_{i_n}$ satisfies the following three conditions:

(C.1) For every j $(2 \leqq j \leqq n)$, c_{i_j} is not $c_{i_{j-1}}^{-1}$.

(C.2) There are no more than $2g$ consecutive sequence in order O_1 or O_2.

(C.3) If there is a $2g$ consecutive sequence in order O_1 (or O_2), say $c_{i_{m+1}}c_{i_{m+2}}\cdots c_{i_{m+2g}}$ $(m + 2g < n)$, then $c_{i_{m+2g+1}}$ is one of the $2g - 1$ letters succeeding $c_{i_{m+2g}}^{-1}$ in order O_3 (or O_4).

In this case, the finite sequence $c_{i_1}c_{i_2}\cdots c_{i_n}$ is called a finite admissible symbol. Clearly, if $c_{i_1}c_{i_2}\cdots c_{i_n}$ is a finite admissible symbol, then $c_{i_1}c_{i_2}\cdots c_{i_r}$ $(1 \leqq r \leqq n)$ is also a finite admissible symbol. An infinite sequence $c_{i_1}c_{i_2}\cdots$ is called an infinite admissible symbol, if, for every n, $c_{i_1}c_{i_2}\cdots c_{i_n}$ is a finite admissible symbol. Nielsen [2] associated a finite admissible symbol $c_{i_1}c_{i_2}\cdots c_{i_n}$ with a unique closed arc $[c_{i_1}c_{i_2}\cdots c_{i_n}]$ on ∂D determined as follows: $[c_{i_1}]$ is the minor closed subarc on ∂D which is the projection of the side of F from the origin, where the side of F is orthogonal to the geodesic segment c_{i_1} (see Fig. 1), and $[c_{i_1}c_{i_2}\cdots c_{i_{n-1}}c_{i_n}]$ is the closed subarc $[c_{i_1}c_{i_2}\cdots c_{i_{n-1}}]\cap\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{n-1}}([c_{i_n}])$. He showed that, for an arbitrary point ζ of ∂D, there exists an infinite admissible symbol $c_{i_1}c_{i_2}\cdots$ satisfying $\bigcap_{n=1}^{\infty}[c_{i_1}c_{i_2}\cdots c_{i_n}]=\{\zeta\}$ and that this infinite admissible symbol $c_{i_1}c_{i_2}\cdots$ corresponding to ζ is uniquely determined except for a denumerable number of points on ∂D. For the point ζ of this exceptional set, there are two infinite admissible symbols $c_{i_1}c_{i_2}\cdots$ and $c_{j_1}c_{j_2}\cdots$ satisfying $\bigcap_{n=1}^{\infty}[c_{i_1}c_{i_2}\cdots c_{i_n}]=\bigcap_{m=1}^{\infty}[c_{j_1}c_{j_2}\cdots c_{j_m}]=\{\zeta\}$. He also showed conversely that every infinite admissible symbol $c_{i_1}c_{i_2}\cdots$ satisfies $\bigcap_{n=1}^{\infty}[c_{i_1}c_{i_2}\cdots c_{i_n}]=\{\zeta\}$ for some $\zeta\in\partial D$. Thus an infinite admissible symbol represents a point ζ of ∂D and is said to be the Nielsen development of ζ. Moreover, Nielsen characterized hyperbolic fixed points of Γ by proving the following.

THEOREM A. *Let Γ be a Fuchsian group mentioned above. Let ζ be a point of ∂D and let $c_{i_1}c_{i_2}\cdots$ be its Nielsen development. Then ζ is a hyperbolic fixed point of Γ if and only if there exists a finite admissible symbol $c_{j_1}c_{j_2}\cdots c_{j_n}$ and an integer $m\geqq 0$ such that $c_{i_{m+kn+1}}=c_{j_1}$, $c_{i_{m+kn+2}}=c_{j_2}$, $\cdots$, $c_{i_{m+kn+n}}=c_{j_n}$ for all $k=0,1,2,\cdots$.*

We call the point $\zeta\in\partial D$ a transitive point under Γ if, for all ordered pair (ζ_1,ζ_2) of two distinct points of ∂D and all $z\in D$ and for all $\varepsilon>0$, there exists an element $\gamma\in\Gamma$ such that $|\zeta_1-\gamma(z)|+|\zeta_2-\gamma(\zeta)|<\varepsilon$. If a point $\zeta\in\partial D$ is not a transitive point under Γ, we call it an intransitive point under Γ. The following theorem due to Hedlund [1] gives a characterization of transitive points under Γ.

THEOREM B. *Let Γ be a Fuchsian group mentioned above. Let ζ be a point of ∂D and let $c_{i_1}c_{i_2}\cdots$ be its Nielsen development. Then ζ is a transitive point under Γ if and only if, for every finite admissible symbol $c_{j_1}c_{j_2}\cdots c_{j_n}$, there exists an integer $m\geqq 0$ such that $c_{i_{m+1}}=c_{j_1}$, $c_{i_{m+2}}=c_{j_2}$, $\cdots$, $c_{i_{m+n}}=c_{j_n}$.*

In this paper, using these theorems, we shall prove the following.

THEOREM. *Let Γ be a Fuchsian group mentioned above. For every integer k with $1\leqq k\leqq 4g-1$, consider a mapping*

$$f_k\colon z\mapsto z\exp(\sqrt{-1}k\pi/2g)\ .$$

If a point ζ of ∂D is a transitive point under Γ, then $f_k(\zeta)$ is also a *transitive point under Γ. If a point ζ of ∂D is a hyperbolic fixed point of Γ, then $f_k(\zeta)$ is also a hyperbolic fixed point of Γ.*

The proof of this theorem for $k \equiv 0 \pmod 4$ is given in § 3 and for $k \equiv 1$, 2 or 3 (mod 4) in § 5 and § 6. Several lemmas and tables are stated in § 2 and § 4. Finally, in § 7, we give an example of the Nielsen development corresponding to an intransitive point ζ. The Nielsen development of its image $f_{2g}(\zeta)$, which is the symmetric point of ζ with respect to the origin, is also given.

2. Some lemmas and tables.

2.1. For integers $k\,(>0)$ and m, we set $[m]_k = m - kn$ with $0 \leqq [m]_k < k$, where n is an integer. For every integer i with $0 \leqq i \leqq 4g-1$, we define the integer $l(i)$ by $c_{l(i)} = c_i^{-1}$. Moreover, for any pair (c_i, c_j) with $0 \leqq i, j \leqq 4g-1$, we set $\langle c_i, c_j \rangle = [i-j]_{4g}$. As stated in Preliminaries, if $[i]_4 = 0$ or 1, then $l(i)$ is $i+2$, and if $[i]_4 = 2$ or 3, then $l(i)$ is $i-2$. Therefore we have the following Table 1.

$[i]_4$	0	1	2	3
$\langle c_{l(i)}, c_i \rangle$	2	2	$4g-2$	$4g-2$

TABLE 1

LEMMA 1. *Let $c_{i_1}c_{i_2}\cdots c_{i_n}$ be a finite admissible symbol and set $f_k(L(c_{i_1}c_{i_2}\cdots c_{i_n})) = L(c_{j_1}c_{j_2}\cdots c_{j_n})$ for a fixed k. Then $c_{j_1}c_{j_2}\cdots c_{j_n}$ is also a finite admissible symbol and*

$$j_1 = [i_1 + k]_{4g} \quad \text{and} \quad j_{r+1} = [l(j_r) + \langle c_{i_{r+1}}, c_{l(i_r)} \rangle]_{4g}\,,$$

for $1 \leqq r \leqq n-1$.

PROOF. By definition, we easily see that $c_{j_1}c_{j_2}\cdots c_{j_n}$ is a finite admissible symbol.

Since f_k is the rotation of angle $k\pi/2g$ around the origin, we have $f_k(L(c_{i_1})) = L(c_{j_1})$ for $j_1 = [i_1 + k]_{4g}$. At the terminal point of $L(c_{i_1}c_{i_2}\cdots c_{i_r})$, $r \geqq 2$, the angle θ from $c_{l(i_r)}$ to $c_{i_{r+1}}$ is $\langle c_{i_{r+1}}, c_{l(i_r)} \rangle \pi/2g$. On the other hand, at the terminal point of $L(c_{j_1}c_{j_2}\cdots c_{j_r})$, the angle from $c_{l(j_r)}$ to $c_{j_{r+1}}$ is equal to θ (see Fig. 2). In other words, we have

$$(1) \qquad \langle c_{j_{r+1}}, c_{l(j_r)} \rangle = \langle c_{i_{r+1}}, c_{l(i_r)} \rangle\,.$$

Therefore $j_{r+1} = [l(j_r) + \langle c_{i_{r+1}}, c_{l(i_r)} \rangle]_{4g}$ for $1 \leqq r \leqq n-1$. This proves the lemma.

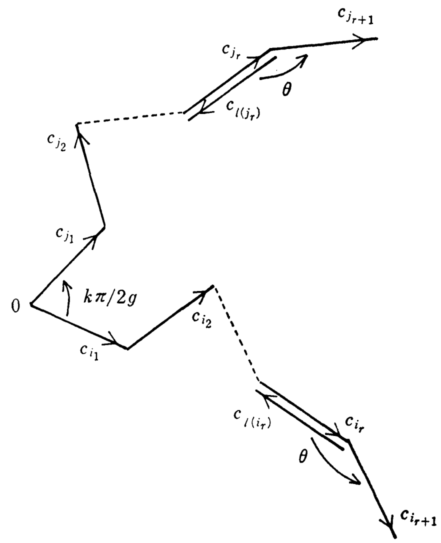

FIGURE 2

LEMMA 2. *Let $c_{i_1}c_{i_2}\cdots c_{i_n}$ and $c_{j_1}c_{j_2}\cdots c_{j_n}$ be as in Lemma 1. Then*

$$j_n = \left[i_n + k + 4\sum_{r=1}^{n-1} e_r \right]_{4g},$$

where $e_r = 0$ or ± 1.

PROOF. For integers h, m, n with $0 \leqq h, m, n \leqq 4g - 1$, we have

$$\langle c_h, c_n \rangle = [\langle c_h, c_m \rangle + \langle c_m, c_n \rangle]_{4g} . \tag{2}$$

Hence $\langle c_{i_n}, c_{i_1} \rangle = [\sum_{r=1}^{n-1} \langle c_{i_{r+1}}, c_{i_r} \rangle]_{4g}$ and $\langle c_{i_{r+1}}, c_{i_r} \rangle = [\langle c_{i_{r+1}}, c_{l(i_r)} \rangle + \langle c_{l(i_r)}, c_{i_r} \rangle]_{4g}$. Therefore we have

$$\langle c_{i_n}, c_{i_1} \rangle = \left[\sum_{r=1}^{n-1} \langle c_{i_{r+1}}, c_{l(i_r)} \rangle + \sum_{r=1}^{n-1} \langle c_{l(i_r)}, c_{i_r} \rangle \right]_{4g} . \tag{3}$$

Similarly we have

$$\langle c_{j_n}, c_{j_1} \rangle = \left[\sum_{r=1}^{n-1} \langle c_{j_{r+1}}, c_{l(j_r)} \rangle + \sum_{r=1}^{n-1} \langle c_{l(j_r)}, c_{j_r} \rangle \right]_{4g} .$$

By (1), we see

$$\langle c_{j_n}, c_{j_1} \rangle = \left[\sum_{r=1}^{n-1} \langle c_{i_{r+1}}, c_{l(i_r)} \rangle + \sum_{r=1}^{n-1} \langle c_{l(j_r)}, c_{j_r} \rangle \right]_{4g} . \tag{4}$$

As $\langle c_{l(i_r)}, c_{i_r} \rangle$ is equal to 2 or $4g - 2$ by Table 1, we see

$$[i_n - i_1]_{4g} = \langle c_{i_n}, c_{i_1} \rangle = \left[\sum_{r=1}^{n-1} \langle c_{i_{r+1}}, c_{l(i_r)} \rangle + 2\sum_{r=1}^{n-1} e'_r \right]_{4g} ,$$

where $e'_r = \langle c_{l(i_r)}, c_{i_r}\rangle/2 = 1$ for $[i_r]_4 = 0$ or 1 and $e'_r = \{\langle c_{l(i_r)}, c_{i_r}\rangle - 4g\}/2 = -1$ for $[i_r]_4 = 2$ or 3. Similarly we obtain

$$[j_n - j_1]_{4g} = \langle c_{j_n}, c_{j_1}\rangle = \left[\sum_{r=1}^{n-1} \langle c_{i_{r+1}}, c_{l(i_r)}\rangle + 2\sum_{r=1}^{n-1} e''_r\right]_{4g},$$

where $e''_r = \langle c_{l(j_r)}, c_{j_r}\rangle/2 = 1$ for $[j_r]_4 = 0$ or 1 and $e''_r = \{\langle c_{l(j_r)}, c_{j_r}\rangle - 4g\}/2 = -1$ for $[j_r]_4 = 2$ or 3. Therefore we have

$$j_n = \left[i_n + j_1 - i_1 + 2\sum_{r=1}^{n-1} (e''_r - e'_r)\right]_{4g} = \left[i_n + k + 4\sum_{r=1}^{n-1} e_r\right]_{4g},$$

where $e_r = (e''_r - e'_r)/2 = 0$ or ± 1. This proves the lemma.

This lemma shows the equality

$$(5) \qquad [j_n]_4 = [i_n + k]_4,$$

from which we have the following Table 2.

Value of $[j_n]_4$

$[k]_4$ \ $[i_n]_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

TABLE 2

COROLLARY. *Let* $c_{i_1}c_{i_2}\cdots c_{i_n}$ *and* $c_{j_1}c_{j_2}\cdots c_{j_n}$ *be as in Lemma* 1 *and assume* $[k]_4 = 0$. *Then* $j_r = [i_r + k]_{4g}$ *for any* r $(1 \leqq r \leqq n)$.

PROOF. By (5), we have $[j_r]_4 = [i_r]_4$ for any r $(1 \leqq r \leqq n)$. So Table 1 implies $\langle c_{l(i_r)}, c_{i_r}\rangle = \langle c_{l(j_r)}, c_{j_r}\rangle$. Hence we see $e'_r = e''_r$ in the proof of Lemma 2. Therefore we have $j_r = [i_r + k]_{4g}$ for $1 \leqq r \leqq n$.

2.2. Let $A = c_{i_1}c_{i_2}\cdots c_{i_n}$ be an arbitrary finite admissible symbol and let $c_{s_1}c_{s_2}\cdots c_{s_N}c_{i_1}c_{i_2}\cdots c_{i_n} = c_{s_1}c_{s_2}\cdots c_{s_N}A$ also be a finite admissible symbol. Set

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}A)) = L(c_{t_1}c_{t_2}\cdots c_{t_N}c_{j_1}c_{j_2}\cdots c_{j_n}).$$

By Lemma 2, there exists a p $(0 \leqq p \leqq g-1)$ uniquely determined by $c_{s_1}c_{s_2}\cdots c_{s_N}$ and k such that $j_1 = [i_1 + k + 4p]_{4g}$. So we may write $j_1 = j_1(p)$. Since j_r $(2 \leqq r \leqq n)$ is determined by $j_1(p)$ and A, we may also write $j_r = j_r(p)$ and set $A_p = c_{j_1(p)}c_{j_2(p)}\cdots c_{j_n(p)}$. Thus we can write as

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}A)) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_p) \quad \text{for some} \quad p\ (0 \leqq p \leqq g-1)\,.$$

LEMMA 3. *Let $A = c_{i_1}c_{i_2}\cdots c_{i_n}$ be an arbitrary finite admissible symbol. Let $c_{s_1}c_{s_2}\cdots c_{s_N}A$ and $c_{\nu_1}c_{\nu_2}\cdots c_{\nu_M}A$ be both finite admissible symbols and assume*

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}A)) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_p)$$
$$f_k(L(c_{\nu_1}c_{\nu_2}\cdots c_{\nu_M}A)) = L(c_{\mu_1}c_{\mu_2}\cdots c_{\mu_M}A_q)$$

for some p and q $(0 \leqq p, q \leqq g-1)$, where $A_p = c_{j_1(p)}c_{j_2(p)}\cdots c_{j_n(p)}$. Then $\langle c_{l(j_r(p))}, c_{j_r(p)}\rangle = \langle c_{l(j_r(q))}, c_{j_r(q)}\rangle$ for $1 \leqq r \leqq n$. Furthermore, if $p \neq q$, then $j_r(p) \neq j_r(q)$ for $1 \leqq r \leqq n$.

PROOF. By (5), we see $[j_r(p)]_4 = [j_r(q)]_4 = [i_r + k]_4$ for $1 \leqq r \leqq n$, so by Table 1 we have

$$\langle c_{l(j_r(p))}, c_{j_r(p)}\rangle = \langle c_{l(j_r(q))}, c_{j_r(q)}\rangle \quad \text{for} \quad 1 \leqq r \leqq n\,.$$

By Lemma 2, we have $j_1(p) = [i_1 + k + 4p]_{4g}$ and $j_1(q) = [i_1 + k + 4q]_{4g}$. Hence $j_1(p) \neq j_1(q)$ if $p \neq q$. The formula (4) shows

$$[j_m(p) - j_1(p)]_{4g} = \left[\sum_{r=1}^{m-1}\langle c_{i_{r+1}}, c_{l(i_r)}\rangle + \sum_{r=1}^{m-1}\langle c_{l(j_r(p))}, c_{j_r(p)}\rangle\right]_{4g}$$

for $2 \leqq m \leqq n$, so we have

$$j_m(p) = \left[j_1(p) + \sum_{r=1}^{m-1}\langle c_{i_{r+1}}, c_{l(i_r)}\rangle + \sum_{r=1}^{m-1}\langle c_{l(j_r(p))}, c_{j_r(p)}\rangle\right]_{4g}$$

for $2 \leqq m \leqq n$. Similarly,

$$j_m(q) = \left[j_1(q) + \sum_{r=1}^{m-1}\langle c_{i_{r+1}}, c_{l(i_r)}\rangle + \sum_{r=1}^{m-1}\langle c_{l(j_r(q))}, c_{j_r(q)}\rangle\right]_{4g}$$

for $2 \leqq m \leqq n$. Since $\langle c_{l(j_r(q))}, c_{j_r(q)}\rangle = \langle c_{l(j_r(p))}, c_{j(p)}\rangle$ as proved already, we see $j_m(p) \neq j_m(q)$ for $1 \leqq m \leqq n$, if $p \neq q$.

LEMMA 4. *Let $c_{j_1(p)}c_{j_2(p)}\cdots c_{j_n(p)}$ and $c_{j_1(q)}c_{j_2(q)}\cdots c_{j_n(q)}$ be those in Lemma 3. Then*

$$\langle c_{j_n(q)}, c_{j_1(q)}\rangle = \langle c_{j_n(p)}, c_{j_1(p)}\rangle\,.$$

In particular, if $i_1 = i_n$, then $\langle c_{j_n(p)}, c_{j_1(p)}\rangle$ is a multiple of 4.

PROOF. Using (4), we see that the first statement of the lemma is obvious from Lemma 3. If $i_1 = i_n$, the formula (3) shows $0 = [\sum_{r=1}^{n-1}\langle c_{i_{r+1}}, c_{l(i_r)}\rangle + \sum_{r=1}^{n-1}\langle c_{l(i_r)}, c_{i_r}\rangle]_{4g}$, from which we have $[\sum_{r=1}^{n-1}\langle c_{i_{r+1}}, c_{l(i_r)}\rangle]_{4g} = [-\sum_{r=1}^{n-1}\langle c_{l(i_r)}, c_{i_r}\rangle]_{4g}$. Hence (4) implies

$$\langle c_{j_n(p)}, c_{j_1(p)}\rangle = \left[\sum_{r=1}^{n-1}\{\langle c_{l(j_r(p))}, c_{j_r(p)}\rangle - \langle c_{l(i_r)}, c_{i_r}\rangle\}\right]_{4g}\,.$$

By Table 1, the right hand side is a multiple of 4. Therefore the second statement of the lemma is obtained.

2.3. Now we take a finite admissible symbol $A = c_i c_i$ $(0 \leqq i \leqq 4g-1)$ and assume that $c_{s_1} c_{s_2} \cdots c_{s_N} A$ is also a finite admissible symbol. Set

$$f_k(L(c_{s_1} c_{s_2} \cdots c_{s_N} A)) = L(c_{t_1} c_{t_2} \cdots c_{t_N} c_{j_1} c_{j_2}) .$$

Then, using (1) and (2), we see

$$\begin{aligned}[j_2 - j_1]_{4g} &= \langle c_{j_2}, c_{j_1} \rangle \\ &= [\langle c_{j_2}, c_{l(j_1)} \rangle + \langle c_{l(j_1)}, c_{j_1} \rangle]_{4g} = [\langle c_i, c_{l(i)} \rangle + \langle c_{l(j_1)}, c_{j_1} \rangle]_{4g} .\end{aligned}$$

For all integers m, n with $0 \leqq m, n \leqq 4g-1$, we have

$$(6) \qquad \langle c_m, c_n \rangle + \langle c_n, c_m \rangle = 4g .$$

Hence $[j_2 - j_1]_{4g} = [4g - \langle c_{l(i)}, c_i \rangle + \langle c_{l(j_1)}, c_{j_1} \rangle]_{4g}$. Therefore

$$j_2 = [j_1 - \langle c_{l(i)}, c_i \rangle + \langle c_{l(j_1)}, c_{j_1} \rangle]_{4g} .$$

The value $[j_1]_4$ is determined by Table 2 and $\langle c_{l(i)}, c_i \rangle$ and $\langle c_{l(j_1)}, c_{j_1} \rangle$ are obtained by Table 1. So we have the following Table 3.

Value of j_2

$[k]_4$ \ $[i]_4$	0	1	2	3
0	j_1	j_1	j_1	j_1
1	j_1	$[j_1 - 4]_{4g}$	j_1	$[j_1 + 4]_{4g}$
2	$[j_1 - 4]_{4g}$	$[j_1 - 4]_{4g}$	$[j_1 + 4]_{4g}$	$[j_1 + 4]_{4g}$
3	$[j_1 - 4]_{4g}$	j_1	$[j_1 + 4]_{4g}$	j_1

TABLE 3

Next we consider another finite admissible symbol $A = c_i c_{[i+1]_{4g}} c_i$ $(0 \leqq i \leqq 4g-1)$. We also assume that $c_{s_1} c_{s_2} \cdots c_{s_N} A$ is a finite admissible symbol and set

$$f_k(L(c_{s_1} c_{s_2} \cdots c_{s_N} A)) = L(c_{t_1} c_{t_2} \cdots c_{t_N} c_{j_1} c_{j_2} c_{j_3}) .$$

Then, by (1) and (2), we have

$$\begin{aligned}[j_3 - j_1]_{4g} &= \langle c_{j_3}, c_{j_1} \rangle \\ &= [\langle c_{j_3}, c_{l(j_2)} \rangle + \langle c_{l(j_2)}, c_{j_2} \rangle + \langle c_{j_2}, c_{l(j_1)} \rangle + \langle c_{l(j_1)}, c_{j_1} \rangle]_{4g} \\ &= [\langle c_i, c_{l([i+1]_{4g})} \rangle + \langle c_{l(j_2)}, c_{j_2} \rangle + \langle c_{[i+1]_{4g}}, c_{l(i)} \rangle + \langle c_{l(j_1)}, c_{j_1} \rangle]_{4g} .\end{aligned}$$

Using (2) and (6), we see

$$\begin{aligned}[\langle c_i, c_{l([i+1]_{4g})}\rangle + \langle c_{[i+1]_{4g}}, c_{l(i)}\rangle]_{4g} \\ = [\langle c_i, c_{[i+1]_{4g}}\rangle + \langle c_{[i+1]_{4g}}, c_{l([i+1]_{4g})}\rangle + \langle c_{[i+1]_{4g}}, c_i\rangle + \langle c_i, c_{l(i)}\rangle]_{4g} \\ = [-\langle c_{l([i+1]_{4g})}, c_{[i+1]_{4g}}\rangle - \langle c_{l(i)}, c_i\rangle]_{4g} .\end{aligned}$$

Hence we have

(7) $j_3 = [j_1 - \langle c_{l([i+1]_{4g})}, c_{[i+1]_{4g}}\rangle + \langle c_{l(j_2)}, c_{j_2}\rangle - \langle c_{l(i)}, c_i\rangle + \langle c_{l(j_1)}, c_{j_1}\rangle]_{4g}$.

By the use of Tables 1 and 2, we have the following Table 4.

Value of j_3

$[k]_4$ \ $[i]_4$	0	1	2	3
1	$[j_1 - 4]_{4g}$	*	$[j_1 + 4]_{4g}$	*
2	*	*	*	*
3	*	$[j_1 + 4]_{4g}$	*	$[j_1 - 4]_{4g}$

TABLE 4

For instance, in the case where $[i]_4 = 0$ and $[k]_4 = 1$, first we see $[i + 1]_4 = 1$ and Table 2 shows $[j_1]_4 = 1$ and $[j_2]_4 = 2$ and next we see $\langle c_{l(j_1)}, c_{j_1}\rangle = 2$ and $\langle c_{l(j_2)}, c_{j_2}\rangle = 4g - 2$ from Table 1. Hence (7) gives $j_3 = [j_1 - 4]_{4g}$. In other cases, similar arguments give the values of j_3 in the above table.

LEMMA 5. *Let $c_{i_1}c_{i_2}\cdots$ be the Nielsen development of ζ on ∂D and set*

$$L(c_{j_1}c_{j_2}\cdots) = f_k(L(c_{i_1}c_{i_2}\cdots)) .$$

Then $c_{j_1}c_{j_2}\cdots$ is also the Nielsen development of $f_k(\zeta)$ on ∂D.

PROOF. As is easily seen from Lemma 1, $c_{j_1}c_{j_2}\cdots$ is an infinite admissible symbol. From the construction of the interval $[c_{i_1}c_{i_2}\cdots c_{i_n}]$ we have

$$f_k([c_{i_1}c_{i_2}\cdots c_{i_n}]) = [c_{j_1}c_{j_2}\cdots c_{j_n}] .$$

Hence $f_k(\zeta) = f_k(\bigcap_{n=1}^{\infty}[c_{i_1}c_{i_2}\cdots c_{i_n}]) = \bigcap_{n=1}^{\infty} f_k([c_{i_1}c_{i_2}\cdots c_{i_n}]) = \bigcap_{n=1}^{\infty}[c_{j_1}c_{j_2}\cdots c_{j_n}]$. Therefore $c_{j_1}c_{j_2}\cdots$ is the Nielsen development of $f_k(\zeta)$ on ∂D.

3. Proof of Theorem in the case $[k]_4 = 0$. First assume that a point ζ on ∂D is a transitive point under Γ. Let $c_{s_1}c_{s_2}\cdots$ be the Nielsen development of ζ. Take an arbitrary finite admissible symbol $c_{j_1}c_{j_2}\cdots c_{j_n}$ and set

$$f_{4g-k}(L(c_{j_1}c_{j_2}\cdots c_{j_n})) = L(c_{i_1}c_{i_2}\cdots c_{i_n}) .$$

Lemma 1 shows that $c_{i_1}c_{i_2}\cdots c_{i_n}$ is an admissible symbol. Since $[k]_4 = 0$, we have $[4g-k]_4 = 0$ and Corollary of Lemma 2 implies $i_r = [j_r + 4g - k]_{4g} = [j_r - k]_{4g}$ $(1 \leqq r \leqq n)$. By Theorem B, there exists an N such that $c_{s_{N+1}} = c_{i_1}$, $c_{s_{N+2}} = c_{i_2}, \cdots, c_{s_{N+n}} = c_{i_n}$. Since $i_r = [j_r - k]_{4g}$, we have $j_r = [i_r + k]_{4g}$. Therefore, by Corollary of Lemma 2, we see

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}c_{i_1}c_{i_2}\cdots c_{i_n})) = L(c_{t_1}c_{t_2}\cdots c_{t_N}c_{j_1}c_{j_2}\cdots c_{j_n})$$

for some finite admissible symbol $c_{t_1}c_{t_2}\cdots c_{t_N}$. Hence $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}c_{i_1}c_{i_2}\cdots c_{i_n}c_{s_{N+n+1}}\cdots)) = L(c_{t_1}c_{t_2}\cdots c_{t_N}c_{j_1}c_{j_2}\cdots c_{j_n}c_{t_{N+n+1}}\cdots)$. Noting Lemma 5, we see that the Nielsen development of $f_k(\zeta)$ includes the sequence $c_{j_1}c_{j_2}\cdots c_{j_n}$. Theorem B implies that $f_k(\zeta)$ is a transitive point under Γ.

Next assume that ζ is a hyperbolic fixed point of Γ. Then, by Theorem A, the Nielsen development of ζ is of the form $c_{s_1}c_{s_2}\cdots c_{s_N}c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_1}c_{i_2}\cdots$ for some N and for some finite admissible symbol $c_{i_1}c_{i_2}\cdots c_{i_n}$. Corollary of Lemma 2 implies

$$\begin{aligned} &f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_1}c_{i_2}\cdots)) \\ &\qquad = L(c_{t_1}c_{t_2}\cdots c_{t_N}c_{j_1}c_{j_2}\cdots c_{j_n}c_{j_1}c_{j_2}\cdots c_{j_n}c_{j_1}c_{j_2}\cdots) , \end{aligned}$$

where $j_r = [i_r + k]_{4g}$ $(1 \leqq r \leqq n)$. This means that the Nielsen development of $f_k(\zeta)$ is of the form $c_{t_1}c_{t_2}\cdots c_{t_N}c_{j_1}c_{j_2}\cdots c_{j_n}c_{j_1}c_{j_2}\cdots c_{j_n}\cdots$. Theorem A shows that $f_k(\zeta)$ is a hyperbolic fixed point of Γ.

4. More several lemmas for the proof in the case $[k]_4 \neq 0$.

4.1. We need more several lemmas for the proof of our theorem in the case $[k]_4 \neq 0$.

LEMMA 6. *Let $c_{i_1}c_{i_2}\cdots c_{i_n}$ be a finite admissible symbol. Then there exists an i_{n+1} such that $c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_{n+1}}c_{i_1}c_{i_2}\cdots c_{i_n}$ and $c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_{n+1}}c_{i_1}c_{[i_1+1]_{4g}}c_i$ are both finite admissible symbols.*

PROOF. First assume that $c_{i_{n-2g+1}}c_{i_{n-2g+2}}\cdots c_{i_n}$ $(n-2g+1 \geqq 1)$ is arranged in order O_1 (or O_2). We choose $c_{i_{n+1}}$ out of $2g-1$ letters succeeding $c_{i_n}^{-1}$ in order O_3 (or O_4) such that $c_{i_{n+1}} \neq c_{i_1}^{-1}$. Here, if $c_{i_1}c_{i_2}\cdots c_{i_{2g}}$ is arranged in order O_1 (or O_2), then $c_{i_{n+1}}c_{i_1}$ must not be arranged in order O_1 (or O_2). There are $2g-3$ such choices of $c_{i_{n+1}}$. Since $g \geqq 2$, it follows that such a $c_{i_{n+1}}$ exists. By this choice of $c_{i_{n+1}}$, the sequences $c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_{n+1}}c_{i_1}c_{i_2}\cdots c_{i_n}$ and $c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_{n+1}}c_{i_1}c_{[i_1+1]_{4g}}c_{i_1}$ become finite admissible symbols.

Next assume that $n \leqq 2g-1$ or that $c_{i_{n-2g+1}}c_{i_{n-2g+2}}\cdots c_{i_n}$ is not arranged in order O_1 or O_2. There are $4g-3$ choices of $c_{i_{n+1}}$ such that $c_{i_{n+1}} \neq c_{i_n}^{-1}$ and $c_{i_n}c_{i_{n+1}}$ is not arranged in order O_1 or O_2. Among these $4g-3$ ones, we choose a $c_{i_{n+1}}$ such that $c_{i_{n+1}}c_{i_1}$ is not arranged in order O_1 or O_2. Since there are $4g-5$ choices of $c_{i_{n+1}}$ satisfying these conditions and since $g \geqq 2$, we can choose a desired $c_{i_{n+1}}$.

4.2. Let $c_{j_1}c_{j_2}\cdots c_{j_n}$ be an arbitrary finite admissible symbol and set

$$f_{4g-k}(L(c_{j_1}c_{j_2}\cdots c_{j_n})) = L(c_{i_1}c_{i_2}\cdots c_{i_n})\ .$$

For this $c_{i_1}c_{i_2}\cdots c_{i_n}$, we choose $c_{i_{n+1}}$ as in Lemma 6 and set $A = c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_{n+1}}$. We see that $AA\cdots A$, $Ac_{i_1}Ac_{i_1}\cdots Ac_{i_1}A$ and $Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}\cdots Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}A$ are all finite admissible symbols. Evidently, we have $f_k(L(c_{i_1}c_{i_2}\cdots c_{i_n})) = L(c_{j_1}c_{j_2}\cdots c_{j_n})$ so that $f_k(L(A)) = L(c_{j_1}c_{j_2}\cdots c_{j_n}c_{j_{n+1}})$ for some j_{n+1}. Consider a finite admissible symbol $c_{s_1}c_{s_2}\cdots c_{s_M}A$. As was stated in § 2.2, we may write as $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_M}A)) = L(c_{t_1}c_{t_2}\cdots c_{t_M}A_p)$ for a p determined by $c_{s_1}c_{s_2}\cdots c_{s_M}$ and k, where $A_p = c_{j_1(p)}c_{j_2(p)}\cdots c_{j_{n+1}(p)}$ and $j_1(p) = [i_1+k+4p]_{4g}$. In particular, we see $j_1 = [i_1+k]_{4g} = j_1(0)$ and hence $j_1(p) = [i_1+k+4p]_{4g} = [j_1(0)+4p]_{4g} = [j_1+4p]_{4g}$. Moreover, we may set $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_M}Ac_{i_1})) = L(c_{t_1}c_{t_2}\cdots c_{t_M}A_pc_{j_1(q)})$ for a q determined by $c_{s_1}c_{s_2}\cdots c_{s_M}A$ and k $(0 \leqq q \leqq g-1)$.

LEMMA 7. *Let $A = c_{i_1}c_{i_2}\cdots c_{i_{n+1}}$ be the finite admissible symbol as stated above and let p and q be integers determined for a finite admissible symbol $c_{s_1}c_{s_2}\cdots c_{s_M}Ac_{i_1}$ as above. Suppose that $p=q$ and that $B = c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}Ac_{i_1}\cdots Ac_{i_1}A$ is a finite admissible symbol, where A appears g times. If $([i_1]_4, [k]_4) = (0,2)$, $(0,3)$, $(1,1)$ or $(1,2)$, then*

$$f_k(L(B)) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}A_{[p'-1]_g}c_{j_1([p'-1]_g)}\cdots A_{[p'-g+2]_g}c_{j_1([p'-g+2]_g)}A_{[p'-g+1]_g})$$

for some p' $(0 \leqq p' \leqq g-1)$ and for some $c_{t_1}c_{t_2}\cdots c_{t_N}$. Furthermore, if $([i_1]_4, [k]_4) = (2,2)$, $(2,3)$, $(3,1)$ or $(3,2)$, then

$$f_k(L(B)) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}A_{[p'+1]_g}c_{j_1([p'+1]_g)}\cdots A_{[p'+g-2]_g}c_{j_1([p'+g-2]_g)}A_{[p'+g-1]_g})$$

for some p' $(0 \leqq p' \leqq g-1)$ and for some $c_{t_1}c_{t_2}\cdots c_{t_N}$.

PROOF. First we consider the cases $([i_1]_4, [k]_4) = (0,2)$, $(0,3)$, $(1,1)$ or $(1,2)$. Set $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N})) = L(c_{t_1}c_{t_2}\cdots c_{t_N})$. By Lemma 2, there exists a p' determined by $c_{s_1}c_{s_2}\cdots c_{s_N}$ and k such that $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}A)) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'})$. We set $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1})) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(q')})$. Then, Lemma 4 shows $\langle c_{j_1(q)}, c_{j_1(p)}\rangle = \langle c_{j_1(q')}, c_{j_1(p')}\rangle$. The assumption $p=q$ implies $p'=q'$. Hence we see $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1})) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')})$.

By Table 3, we have $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}c_{i_1}))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{[j_1(p')-4]_{4g}})$. Since $j_1(p')=[i_1+k+4p']_{4g}$, we see $j_1([p'-1]_g)=[i_1+k+4[p'-1]_g]_{4g}=[i_1+k+4(p'-1)]_{4g}=[j_1(p')-4]_{4g}$. So we have $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}c_{i_1}))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{j_1([p'-1]_g)})$ and hence $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}A))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}A_{[p'-1]_g})$. Now assume $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}Ac_{i_1}))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}A_{[p'-1]}c_{j_1(q'')})$. Then, using Lemma 4 and the assumption $p=q$ again, we see $j_1([p'-1]_g)=j_1(q'')$ and hence $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}Ac_{i_1}))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}A_{[p'-1]_g}c_{j_1([p'-1]_g)})$. Continuing this procedure, we have the desired formula.

In the cases $([i_1]_4, [k]_4)=(2,2)$, $(2,3)$, $(3,1)$ or $(3,2)$, we have the desired by the argument similar to the above.

LEMMA 8. *Let the finite admissible symbol* $A=c_{i_1}c_{i_2}\cdots c_{i_{n+1}}$ *and integers* p *and* q *be those in Lemma 7. Suppose that* $p=q$ *and that* $B=c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}\cdots Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}A$ *is admissible, where* A *appears* g *times. If* $([i_1]_4, [k]_4)=(0,1)$ *or* $(3,3)$, *then*

$$f_k(L(B))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{m_1}c_{j_1([p'-1]_g)}A_{[p'-1]_g}\cdots A_{[p'-g+2]_g}c_{j_1([p'-g+2]_g)}c_{m_{g-1}}c_{j_1([p'-g+1]_g)}A_{[p'-g+1]_g})$$

for some p' $(0\leqq p'\leqq g-1)$ *and for some* $c_{t_1}c_{t_2}\cdots c_{t_N}$, $c_{m_1},\cdots,c_{m_{g-1}}$. *If* $([i_1]_4, [k]_4)=(1,3)$ *or* $(2,1)$, *then*

$$f_k(L(B))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{m_1}c_{j_1([p'+1]_g)}A_{[p'+1]_g}\cdots A_{[p'+g-2]_g}c_{j_1([p'+g-2]_g)}c_{m_{g-1}}c_{j_1([p'+g-1]_g)}A_{[p'+g-1]_g})$$

for some p' $(0\leqq p'\leqq g-1)$.

PROOF. We assume $([i_1]_4, [k]_4)=(0,1)$ or $(3,3)$. Set $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}))=L(c_{t_1}c_{t_2}\cdots c_{t_N})$. By Lemma 2, there exists a p' determind by $c_{s_1}c_{s_2}\cdots c_{s_N}$ and k such that $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}A))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'})$. From the proof of Lemma 7, we see $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')})$. Table 4 implies $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{m_1}c_{[j_1(p')-4]_{4g}})$, where m_1 is determined by $c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}$ and k. As was seen in the proof of Lemma 7, we have $[j_1(p')-4]_{4g}=j_1([p'-1]_g)$ and hence $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{m_1}c_{j_1([p'-1]_g)})$. By Table 3, we see $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}c_{i_1}))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{m_1}c_{j_1([p'-1]_g)}c_{j_1([p'-1]_g)})$ and hence $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}A))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{m_1}c_{j_1([p'-1]_g)}A_{[p'-1]_g})$. By using Lemma 4 and the assumption $p=q$ again, we have $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}Ac_{i_1}))=L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{m_1}c_{j_1([p'-1]_g)}A_{[p'-1]_g}c_{j_1([p'-1]_g)})$. Repeating this procedure, we have the desired.

The similar argument gives also the desired in the cases $([i_1]_4, [k]_4)=(1,3)$ or $(2,1)$.

Let a finite admissible symbol $A = c_{i_1}c_{i_2} \cdots c_{i_{n+1}}$ and integers p and q be those in Lemma 7. If $p \neq q$, then Lemma 4 gives $\langle c_{j_1(q)}, c_{j_1(p)} \rangle = 4r$ for some r $(1 \leqq r \leqq g-1)$, where r is independent of p and q. Let m be the smallest natural number satisfying $[rm]_g = 0$. Obviously we see $1 \leqq m \leqq g$.

LEMMA 9. *Let a finite admissible symbol $A = c_{i_1}c_{i_2} \cdots c_{i_{n+1}}$ and integers p and q be those in Lemma 7 and let $B = c_{s_1}c_{s_2} \cdots c_{s_N}AA \cdots A$ be a finite admissible symbol, where A appears g times. Suppose that $p \neq q$ and that m given in the above is equal to g. Then*

$$f_k(L(B)) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p'}A_{[p'+r]_g} \cdots A_{[p'+(g-1)r]_g})$$

for some p', where $[p'+ur]_g$, $u = 0, 1, \cdots, g-1$ are all distinct.

PROOF. By Lemma 2, there exists a p' determined by $c_{s_1}c_{s_2} \cdots c_{s_N}$ and k such that $f_k(L(c_{s_1}c_{s_2} \cdots c_{s_N}A)) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p'})$. Set $f_k(L(c_{s_1}c_{s_2} \cdots c_{s_N}Ac_{i_1})) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p'}c_{j_1(q')})$. Lemma 4 implies $\langle c_{j_1(q')}, c_{j_1(p')} \rangle = 4r$ for some r $(1 \leqq r \leqq g-1)$, so $j_1(q') = [j_1(p') + 4r]_{4g}$. Since $j_1(p') = [i_1 + k + 4p']_{4g}$, we have $j_1(q') = [i_1 + k + 4[p'+r]_g]_{4g} = j_1([p'+r]_g)$. Hence we see $f_k(L(c_{s_1}c_{s_2} \cdots c_{s_N}Ac_{i_1})) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p'}c_{j_1([p'+r]_g)})$ and $f_k(L(c_{s_1}c_{s_2} \cdots c_{s_N}AA)) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p'}A_{[p'+r]_g})$. Repeating this procedure, we have the desired formula. The assumption implies $[p'+u_1r]_g \neq [p'+u_2r]_g$ for u_1, u_2 $(\neq u_1)$ with $1 \leqq u_1, u_2 \leqq g-1$.

LEMMA 10. *Let a finite admissible symbol $A = c_{i_1}c_{i_2} \cdots c_{i_{n+1}}$ and integers p and q be those in Lemma 7 and let r be the one stated before Lemma 9. Suppose that $p \neq q$ and that m, the smallest natural number with $[rm]_g = 0$, is smaller than g. Let $B = c_{s_1}c_{s_2} \cdots c_{s_N}AA \cdots Ac_{i_1}$ be a finite admissible symbol, where A appears m times. Then*

$$f_k(L(B)) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p'}A_{[p'+r]_g} \cdots A_{[p'+(m-1)r]_g}c_{j_1(p')})$$

for some p' $(0 \leqq p' \leqq g-1)$, where $[p'+ur]_g$, $u = 0, 1, \cdots, m-1$, are all distinct.

PROOF. By the same manner as in the proof of Lemma 9, we have $f_k(L(c_{s_1}c_{s_2} \cdots c_{s_N}AA \cdots A)) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p'}A_{[p'+r]_g} \cdots A_{[p'+(m-1)r]_g})$ for some p' $(0 \leqq p' \leqq g-1)$. Set $f_k(L(c_{s_1}c_{s_2} \cdots c_{s_N}AA \cdots Ac_{i_1})) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p'}A_{[p'+r]_g} \cdots A_{[p'+(m-1)r]_g}c_{j_1(q')})$. Then we see $\langle c_{j_1(q')}, c_{j_1([p'+(m-1)r]_g)} \rangle = 4r$ and hence $j_1(q') = [j_1([p'+(m-1)r]_g) + 4r]_{4g}$. From $j_1([p'+(m-1)r]_g) = [i_1 + k + 4[p'+(m-1)r]_g]_{4g}$, we see $j_1(q') = [i_1 + k + 4[p'+mr]_g]_{4g}$ and hence $q' = [p'+mr]_g$. The assumption $[mr]_g = 0$ implies $p' = q'$. Since m is the smallest natural number with $[mr]_g = 0$, we see that $[p'+ur]_g$, $u = 0, 1, \cdots, m-1$, are all distinct.

5. Proof of the first half of the theorem for $[k]_4 \neq 0$. Let $\zeta \in \partial D$ be a transitive point under Γ and let $c_{s_1}c_{s_2}\cdots$ be its Nielsen development. Suppose that $c_{j_1}c_{j_2}\cdots c_{j_n}$ is an arbitrary finite admissible symbol. As in § 4, we write $f_{4g-k}(L(c_{j_1}c_{j_2}\cdots c_{j_n})) = L(c_{i_1}c_{i_2}\cdots c_{i_n})$. We also choose a $c_{i_{n+1}}$ as stated in Lemma 6 and determine $c_{j_{n+1}}$ by

$$f_k(L(c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_{n+1}})) = L(c_{j_1}c_{j_2}\cdots c_{j_n}c_{j_{n+1}}) .$$

We set $A = c_{i_1}c_{i_2}\cdots c_{i_n}c_{i_{n+1}}$. As stated in § 4, we see that $AA\cdots A$, $Ac_{i_1}Ac_{i_1}\cdots Ac_{i_1}A$ and $Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}A\cdots Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}A$ are all finite admissible symbols.

Theorem B implies that there exists an M such that the Nielsen development of ζ is of the form $c_{s_1}c_{s_2}\cdots c_{s_M}Ac_{i_1}c_{s_{M+(n+2)+1}}\cdots$. Hence, as was stated in § 2.2, there exist p and q $(0 \leqq p, q \leqq g-1)$ with the property

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_M}Ac_{i_1}c_{s_{M+(n+2)+1}}\cdots)) = L(c_{t_1}c_{t_2}\cdots c_{t_M}A_pc_{j_1(q)}c_{t_{M+(n+2)+1}}\cdots) .$$

Here we recall $A_0 = c_{j_1}c_{j_2}\cdots c_{j_{n+1}}$.

Now we prove the first half of our theorem in the case $[k]_4 \neq 0$ by dividing the case into the following six cases (i)$\sim$(vi).

(i) $p = q$ and $([i_1]_4, [k]_4) = (0, 2)$, $(0, 3)$, $(1, 1)$ or $(1, 2)$.

We denote the admissible symbol $Ac_{i_1}Ac_{i_1}\cdots Ac_{i_1}A$ by $\bar{A}$, where A appears g times. Since ζ is a transitive point under Γ, there exists an N such that its Nielsen development is of the form $c_{s_1}c_{s_2}\cdots c_{s_N}\bar{A}c_{s_{N+(n+1)g+(g-1)+1}}\cdots$. By Lemma 7, we have

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}\bar{A})) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}A_{[p'-1]_g}c_{j_1([p'-1]_g)}\cdots A_{[p'-g+2]_g}c_{j_1([p'-g+2]_g)}A_{[p'-g+1]_g})$$

for some p' $(0 \leqq p' \leqq g-1)$. Since the integers $\{[p'-u]_g\}_{u=0}^{g-1}$ are all distinct, we have $[p'-u]_g = 0$ for some u with $0 \leqq u \leqq g-1$. Hence the Nielsen development $c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}A_{[p'-1]_g}c_{j_1([p'-1]_g)}\cdots A_{[p'-g+2]_g}c_{j_1([p'-g+2]_g)}A_{[p'-g+1]_g}c_{t_{N+(n+1)g+(g-1)+1}}\cdots$ of $f_k(\zeta)$ includes $A_0 = c_{j_1}c_{j_2}\cdots c_{j_n}c_{j_{n+1}}$.

(ii) $p = q$ and $([i_1]_4, [k]_4) = (2, 2)$, $(2, 3)$, $(3, 1)$ or $(3, 2)$.

Using Lemma 7 and the fact that $\{[p'+u]_g\}_{u=0}^{g-1}$ are all distinct, we see similarly to the case (i) that the Nielsen development of $f_k(\zeta)$ includes A_0.

(iii) $p = q$ and $([i_1]_4, [k]_4) = (0, 1)$ or $(3, 3)$.

Set $\bar{A} = Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}\cdots Ac_{i_1}c_{[i_1+1]_{4g}}c_{i_1}A$, where A appears g times. Since ζ is a transitive point under Γ, there exists an N such that its Nielsen development is of the form $c_{s_1}c_{s_2}\cdots c_{s_N}\bar{A}c_{s_{N+(n+1)g+3(g-1)+1}}\cdots$.

By Lemma 8, we have

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}\bar{A})) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{m_1}c_{j_1([p'-1]_g)}A_{[p'-1]_g}\cdots A_{[p'-g+2]_g}c_{j_1([p'-g+2]_g)}c_{m_{g-1}}c_{j_1([p'-g+1]_g)}A_{[p'-g+1]_g}) .$$

The integers $\{[p'-u]_g\}_{u=0}^{g-1}$ are all distinct so that the Nielsen development $c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}c_{j_1(p')}c_{m_1}c_{j_1([p'-1]_g)}A_{[p'-1]_g}\cdots A_{[p'-g+2]_g}c_{j_1([p'-g+2]_g)}c_{m_{g-1}}c_{j_1([p'-g+1]_g)}$ $A_{[p'-g+1]_g}c_{t_{N+(n+1)g+3(g-1)+1}}\cdots$ of $f_k(\zeta)$ includes $A_0 = c_{j_1}c_{j_2}\cdots c_{j_n}c_{j_{n+1}}$.

(iv) $p = q$ and $([i_1]_4, [k]_4) = (1, 3)$ or $(2, 1)$.

Using Lemma 8 and the fact that $\{[p'+u]_g\}_{u=0}^{g-1}$ are all distinct, we see similarly to the case (iii) that the Nielsen development of $f_k(\zeta)$ includes A_0.

Next we consider the remained two cases with $p \neq q$. In these cases, we set $\langle c_{j_1(q)}, c_{j_1(p)}\rangle = 4r$ $(1 \leqq r \leqq g-1)$. Let m be the smallest natural number satisfying $[rm]_g = 0$.

(v) $p \neq q$ and $m = g$.

Set $\bar{A} = AA\cdots A$, where A appears g times. Since ζ is a transitive point under Γ, there exists an N such that the Nielsen development of ζ is of the form $c_{s_1}c_{s_2}\cdots c_{s_N}\bar{A}c_{s_{N+g(n+1)+1}}\cdots$. By Lemma 9, we have

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}\bar{A})) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}A_{[p'+r]_g}\cdots A_{[p'+(g-1)r]_g})$$

for some p', where $\{[p'+ru]_g\}_{u=0}^{g-1}$ are all distinct. Therefore $[p'+ru]_g = 0$ for some u $(0 \leqq u \leqq g-1)$. Hence the Nielsen development $c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}A_{[p'+r]_g}\cdots A_{[p'+(m-1)r]_g}c_{t_{N+g(n+1)+1}}\cdots$ of $f_k(\zeta)$ includes A_0.

The final case (vi) $p \neq q$ and $1 \leqq m \leqq g-1$ is further divided into four cases (vi)-(1)$\sim$(vi)-(4). Set $\bar{A} = AA\cdots A$, where A appears m times. Since ζ is a transitive point under Γ, there exists an N such that its Nielsen development is of the form $c_{s_1}c_{s_2}\cdots c_{s_N}\bar{A}c_{i_1}c_{s_{N+m(n+1)+2}}\cdots$. Then Lemma 10 implies

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}\bar{A}c_{i_1})) = L(c_{t_1}c_{t_2}\cdots c_{t_N}A_{p'}A_{[p'+r]_g}\cdots A_{[p'+(m-1)r]_g}c_{j_1(p')})$$

for some p' $(0 \leqq p' \leqq g-1)$. Now we set $B = \bar{A}$ and have $f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}Bc_{i_1})) = L(c_{t_1}c_{t_2}\cdots c_{t_N}B_{p'}c_{j_1(p')})$, where $B_{p'} = A_{p'}A_{[p'+r]_g}\cdots A_{[p'+(m-1)r]_g}$.

(vi)-(1) $([i_1]_4, [k]_4) = (0, 2)$, $(0, 3)$, $(1, 1)$ or $(1, 2)$.

Set $\bar{B} = Bc_{i_1}Bc_{i_1}\cdots Bc_{i_1}B$, where B appears g times. Since ζ is a transitive point under Γ, its Nielsen development is of the form $c_{s_1}c_{s_2}\cdots c_{s_K}\bar{B}c_{s_{K+gm(n+1)+(g-1)+1}}\cdots$ for some positive integer K. By Lemma 7, we have

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_K}\bar{B})) = L(c_{t_1}c_{t_2}\cdots c_{t_K}B_{p''}c_{j_1(p'')}B_{[p''-1]_g}c_{j_1([p''-1]_g)}\cdots B_{[p''-g+2]_g}c_{j_1([p''-g+2]_g)}B_{[p''-g+1]_g})$$

for some p'' $(0 \leqq p'' \leqq g-1)$, where

$$B_{[p''-u]_g} = A_{[p''-u]_g} A_{[[p''-u]_g+r]_g} \cdots A_{[[p''-u]_g+(m-1)r]_g} \quad \text{for} \quad 0 \leqq u \leqq g-1 .$$

Therefore the Nielsen development $c_{t_1}c_{t_2} \cdots c_{t_K} B_{p''} c_{j_1(p'')} B_{[p''-1]_g} c_{j_1([p''-1]_g)} \cdots B_{[p''-g+2]_g} c_{j_1([p''-g+2]_g)} B_{[p''-g+1]_g} c_{t_{K+gm(n+1)+(g-1)+1}} \cdots$ of $f_k(\zeta)$ includes $\{A_{[p''-u]_g}\}_{u=0}^{g-1}$. We have $[p''-u]_g = 0$ for some u $(0 \leqq u \leqq g-1)$ and the Nielsen development of $f_k(\zeta)$ includes A_0.

(vi)-(2) $([i_1]_4, [k]_4) = (2, 2), (2, 3), (3, 1)$ or $(3, 2)$.

Set $\bar{B} = Bc_{i_1}Bc_{i_1} \cdots Bc_{i_1}B$, where B appears g times. Applying the argument in (ii) to B, we see $f_k(L(c_{s_1}c_{s_2} \cdots c_{s_K}\bar{B})) = L(c_{t_1}c_{t_2} \cdots c_{t_K} B_{p''} c_{j_1(p'')} B_{[p''+1]_g} \cdots B_{[p''+g-2]_g} c_{j_1([p''+g-2]_g)} B_{[p''+g-1]_g})$. Hence the Nielsen development of $f_k(\zeta)$ includes A_0 as in the case (vi)-(1).

(vi)-(3) $([i_1]_4, [k]_4) = (0, 1)$ or $(3, 3)$.

Set $\bar{B} = Bc_{i_1}c_{[i_1+1]_{4g}}c_{i_1}Bc_{i_1}c_{[i_1+1]_{4g}}c_{i_1} \cdots Bc_{i_1}c_{[i_1+1]_{4g}}c_{i_1}B$, where B appears g times. Applying the argument in (iii) to B, we see

$$f_k(L(c_{s_1}c_{s_2} \cdots c_{s_K}\bar{B})) = L(c_{t_1}c_{t_2} \cdots c_{t_K} B_{p''} c_{j_1(p'')} c_{m_1} c_{j_1([p''-1]_g)} B_{[p''-1]_g} \cdots B_{[p''-g+2]_g} c_{j_1([p''-g+2]_g)} c_{m_{g-1}} c_{j_1([p''-g+1]_g)} B_{[p''-g+1]_g}) .$$

Hence the Nielsen development of $f_k(\zeta)$ includes A_0.

(vi)-(4) $([i_1]_4, [k]_4) = (1, 3)$ or $(2, 1)$.

We apply the argument in (iv) to B and see as in the case (vi)-(3) that the Nielsen development of $f_k(\zeta)$ includes A_0.

Thus, in all cases (i)$\sim$(vi), we see that the Nielsen development of $f_k(\zeta)$ includes an arbitrary finite admissible symbol $c_{j_1}c_{j_2} \cdots c_{j_n}$. Theorem B shows that $f_k(\zeta)$ is a transitive point under Γ.

6. Proof of the second part of the theorem for $[k]_4 \neq 0$. Let ζ be a hyperbolic fixed point of Γ. Then, by Theorem A, the Nielsen development of ζ is written as $c_{s_1}c_{s_2} \cdots c_{s_N}AA \cdots$ by some finite admissible symbol $A = c_{i_1}c_{i_2} \cdots c_{i_n}$ and some integer N. Set

$$f_k(L(c_{s_1}c_{s_2} \cdots c_{s_N}AA \cdots)) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p_1}A_{p_2} \cdots) ,$$

where $A_{p_v} = c_{j_1(p_v)}c_{j_2(p_v)} \cdots c_{j_n(p_v)}$. Lemma 4 shows $\langle c_{j_1(p_v)}, c_{j_1(p_{v-1})} \rangle = 4r$ for some r $(0 \leqq r \leqq g-1)$, which is independent of v. If $r = 0$, then $j_1(p_v) = j_1(p_{v-1})$ and the Nielsen development of $f_k(\zeta)$ is of the form $c_{t_1}c_{t_2} \cdots c_{t_N}A_{p_1}A_{p_1} \cdots$. Hence, by Theorem A, $f_k(\zeta)$ is a hyperbolic fixed point of Γ. If $r \neq 0$, we denote by m the smallest natural number satisfying $[mr]_g = 0$. Set $B = AA \cdots A$, where A appears m times. Then, by Lemma 10, we have

$$f_k(L(c_{s_1}c_{s_2} \cdots c_{s_N}Bc_{i_1})) = L(c_{t_1}c_{t_2} \cdots c_{t_N}A_{p_1}A_{[p_1+r]_g} \cdots A_{[p_1+(m-1)r]_g}c_{j_1(p_1)}) .$$

The Nielsen development of ζ can be also written as $c_{s_1}c_{s_2} \cdots c_{s_N}BB \cdots$ and Lemma 4 gives

$$f_k(L(c_{s_1}c_{s_2}\cdots c_{s_N}BB\cdots)) = L(c_{t_1}c_{t_2}\cdots c_{t_N}B_{p_1}B_{p_1}\cdots)\ ,$$

where $B_{p_1} = A_{p_1}A_{[p_1+r]_g}\cdots A_{[p_1+(m-1)r]_g}$. Hence, by Theorem A, $f_k(\zeta)$ is a hyperbolic fixed point of Γ.

7. An example. Consider the case where $g = 3$ and $k = 6$. The mapping f_6 is the rotation of angle π about the origin. We take the symbol

$$A = c_0c_3c_0c_0c_3c_0c_0c_0c_3c_0c_0c_0c_0c_3\cdots .$$

By definition, this is an infinite admissible symbol. Moreover, it does not contain any cyclic part. Hence the point $\zeta\in\partial D$, whose Nielsen development is A, is not a hyperbolic fixed point of Γ. On the other hand, this symbol does not contain the finite admissible symbol c_1. Therefore, ζ is an intransitive point under Γ. Hence our Theorem shows that $f_6(\zeta)$ is an intransitive point under Γ. Set

$$f_6(L(A)) = L(c_{i_1}c_{i_2}\cdots c_{i_n}\cdots)\ .$$

By (3), we see $[i_n]_4 = [0+6]_4$ or $[i_n]_4 = [3+6]_4$. Therefore, $i_n = 1, 2, 5, 6, 9$ or 10. This fact also implies the intransitivity of the point $f_6(\zeta)$.

References

[1] G. A. Hedlund, On the measure of the non-special geodesics on a surface of constant negative curvature, Proc. Nat. Acad. Sci. 19 (1933), 345–348.

[2] J. Nielsen, Untersuchungen zur topologie der geschlossenen zweiseitige Flächen, Acta Math. 50 (1927), 189–358.

Mathematical Institute and Department of Mathematics
Tôhoku University / Yamagata University
Sendai, 980 / Yamagata, 990
Japan / Japan